

Improved Master Theorems for Divide-and-Conquer Recurrences

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Abstract. This paper presents new theorems to analyze divide-and-conquer recurrences, which improve other similar ones in several aspects. In particular, these theorems provide more information, free us almost completely from technicalities like floors and ceilings, and cover a wider set of toll functions and weight distributions, stochastic recurrences included.

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1. Introduction

This work presents new theorems to solve many divide-and-conquer recurrences that arise in practice. Recall that a recurrence is a definition of a function F_n in terms of the values of F at indices smaller than n ; a recurrence is divide-and-conquer—DAC, for short—if the average size of these indices is a fraction of n . Our reference source here is the Master Theorem—MT, for short—as it can be found in Sedgewick and Flajolet [1996]. Other references in this subject include the (classic) Master Theorem [Aho et al. 1974; Bentley et al. 1980; Cormen et al. 1990], several improvements [Kao 1997; Verma 1994; 1997; Wang and Fu 1996] as well as other related results [Karp 1994].

Assume that we have the recurrence $F_n = t_n + W \cdot F_{S_n}$, with $t_n > 0$ and $S_n = Z \cdot n + \mathcal{O}(1)$ for some $0 < Z < 1$. If F_n describes the cost to solve with a certain algorithm a problem of size n , then t_n —customarily called toll function—is the cost of the divide and combine steps, W is the fixed number of

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recursive calls at each step, and S_n is the size of the subproblems to be recursively solved. Let $\alpha = -\log_Z W$. Then, the MT states that the solution to the recurrence is

$$F_n = \begin{cases} \Theta(n^\alpha), & \text{if } t_n = \mathcal{O}(n^a) \text{ and } a < \alpha; \\ \Theta(t_n \log n), & \text{if } t_n = \Theta(n^\alpha \log^c n) \text{ and } c \geq 0;^1 \\ \Theta(t_n), & \text{if } t_n = \Omega(n^a) \text{ and } a > \alpha. \end{cases}$$

(To be completely rigorous, the last case above requires additional conditions of smoothness for the toll function t_n .)

Usually, a MT does not provide an explicit solution but partial information about the function under study, like bounds on its growing order. Nevertheless, a MT has two interesting properties: First, it typically provides results which rely exclusively upon the asymptotic behaviour of the toll function and of the distribution of weights; for instance, in the MT above we only use the main term of t_n and the value α , which depends on W and Z . In particular, this means that the values of F_n at small indices are irrelevant to most MTs. Second, a MT is easy and fast to use. As an example, we can solve the DAC recurrence

$$M_n = n - 1 + M_{\lfloor n/2 \rfloor} + M_{\lceil n/2 \rceil} \quad (1)$$

for $n \geq 2$, with $M_0 = M_1 = 0$, which defines the number of comparisons to sort an array of n keys with mergesort in the worst case [Flajolet and Golin 1994]. The term $\mathcal{O}(1)$ in the definition of S_n covers expressions with floors and ceilings in the argument of the recursive call. Hence, a direct application of the rules above yields $M_n = \Theta(n \log n)$.

In this paper, we will make the MT more flexible and more informative. For instance, consider the recurrence

$$B_n = 1 + \frac{\lfloor (n-1)/2 \rfloor}{n} \cdot B_{\lfloor (n-1)/2 \rfloor} + \frac{\lceil (n-1)/2 \rceil}{n} \cdot B_{\lceil (n-1)/2 \rceil} \quad (2)$$

for $n \geq 2$, with $B_0 = 0$ and $B_1 = 1$, which defines the expected number of steps of a binary search for a random key in an array of size n . This recurrence does not follow the MT pattern utterly, since we have $1 - 1/n$ expected recursive calls at each step. Therefore, to use the MT we must assume that the solution to the recurrence $F_n = 1 + F_{n/2}$ must be close to B_n —which is true—, and conclude $B_n = \Theta(\log n)$. We need a posterior reasoning to rigorously prove that this approximation does not lead to a wrong answer.

By contrast, the MT presented in Section 2 directly deals with recurrences where the number of recursive calls is not constant but tends to a constant. As a consequence, we will get rid of the annoying technicalities related to factors with floors and ceilings. And we will see that in some cases it is possible to get the multiplying factor of the dominating term of the function under study. This improvements will yield the result $B_n = \log_2 n + o(\log n)$ for the example above.

¹ Through all this work, $\log^c n$ means $(\log n)^c$.

Yet we will extend the MT further, to deal with recurrences where the asymptotic sizes of the subproblems to be recursively solved consist in several fixed fractions of the original problem,² like

$$F_n = t_n + \left(2 - \frac{1}{\sqrt{n}}\right) F_{\lfloor (n+1)/3 \rfloor} + 4F_{\lfloor (n/2) - \sqrt[3]{n} \rfloor} + \left(\frac{1}{2} + \frac{1}{n}\right) F_{\lceil (4n/5) + \ln^2 n \rceil} \quad (3)$$

for n large enough. (Since all the theorems in this work rely only upon the asymptotic properties of the toll function and the distribution of weights, we can avoid explicitly stating the initial values of the recurrence.)

It is frequent that the analysis of the cost of a given algorithm or data structure results in a stochastic recurrence, which does not follow any of the patterns mentioned above. As an example of stochastic recurrence, consider

$$Q_n = n + 1 + \frac{2}{n} \sum_{0 \leq k < n} Q_k \quad (4)$$

for $n \geq 1$, with $Q_0 = 0$. This recurrence defines the expected number of comparisons to sort an n -key random array with quicksort. Clearly, the MTs we know so far cannot solve it, since the size of the subarrays to be recursively sorted is not fixed; it can be either an insignificant part of the whole input or almost it all. In this case, though, other techniques lead us to a closed solution [Hoare 1962].

Much more difficulties presents the analysis of the stochastic recurrence

$$Q_n = 1 + \frac{4}{n(n+1)} \sum_{0 \leq k < n} (n-k)Q_k \quad (5)$$

for $n \geq 1$, with $Q_0 = 0$, where Q_n denotes the expected number of comparisons during a half-defined search in a random quad-tree of size n . An accurate asymptotic expression for Q_n can be obtained after a thorough analysis by means of generating functions [Flajolet et al. 1993], but this technique requires a deep knowledge and expertise.

Section 3 presents a new MT for stochastic recurrences, which shares the good properties of the MT in Section 2: it is simple and fast to use, and only the main term of t_n and information on the asymptotic distribution of weights will be relevant. In addition, we will see how a simple iteration of the former MT sometimes yields several of the main terms of the function under study with their corresponding multiplicative factors.

The rest of the paper is organized as follows: Sections 4 and 5 present general-purpose theorems that solve many DAC recurrences that do not have to follow any particular pattern. Those theorems will allow us to prove in Sections 6 and 7 the MTs given in Sections 2 and 3, but also have other applications. For

² To the best of the author's knowledge, Kao [1997] was the first to develop a basic MT for this kind of recurrences in 1986 (personal communications). Some additional results about these recurrences may be found in Verma [1997].

instance, they are useful for the analysis and asymptotic improvement of quicksort and quickselect [Martínez and Roura 1998].

In Section 8, we will see that it is possible to directly compute the main term of the variance of the cost of some algorithms, if the nonrecursive cost is large enough and the algorithm follows one of two typical patterns (quickselect and quicksort are prototypical examples of those patterns).

Finally, Section 9 presents some open problems.

Preliminary versions of this work appeared in Roura [1997a; 1997b].

2. The Discrete Master Theorem

In this section, we cast in the form of a Master Theorem some of the results that will be proved in latter sections. This MT deals with recurrences like (1), (2), and (3), where the problem is broken into pieces such that the asymptotic size of each one is a fixed fraction of the size of the whole problem. We call them discrete recursive definitions.

Definition 2.1. We say that

$$F_n = \begin{cases} b_n, & \text{if } 0 \leq n < N \\ t_n + \sum_{1 \leq d \leq D} R_{d,n} F_{S_{d,n}}, & \text{if } n \geq N \end{cases}$$

is a *discrete recursive definition* of F_n iff $D \geq 1$; $R_{d,n} = w_d + r_{d,n} \geq 0$, where $w_d > 0$ and $\sum_{1 \leq d \leq D} |r_{d,n}| = \mathcal{O}(n^{-\rho})$ for some $\rho > 0$; and $S_{d,n} = z_d \cdot n + s_{d,n}$, where $0 < z_d < 1$ and $\sum_{1 \leq d \leq D} |s_{d,n}|/n = \mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$.

Here, D is the finite number of subproblems to be recursively solved; $R_{d,n}$ is the number of recursive calls to deal with the d th subproblem, where w_d is the asymptotic number of calls to it; and $S_{d,n}$ is the (integer) size of the d th subproblem, where z_d is the asymptotic fraction of the original problem to be solved by the d th recursive call.

For example, (1) is a discrete recursive definition. There we have two subproblems to recursively deal with: $D = 2$; whose size is asymptotically 1/2 of the size of the original problem: $z_1 = z_2 = 1/2$, $-1 \leq s_{1,n} \leq 0$, $0 \leq s_{2,n} \leq 1$; and there is exactly one call to each one: $w_1 = w_2 = 1$, $r_{1,n} = r_{2,n} = 0$. For the bounds of $s_{n,k}$, we have used the fact that $r - 1 \leq \lfloor r \rfloor \leq r$ and $r \leq \lceil r \rceil \leq r + 1$ for every real r . Notice that $\rho = \sigma = 1$ is a possible choice in this example.

The Discrete MT (Theorem 2.3) covers the presence of sublogarithmical factors in the toll function, that is, factors whose growing order is smaller than $\log^\varepsilon n$ for any $\varepsilon > 0$. This allows us to deal with toll functions like $t_n = n^2 \ln^3 n \cdot \ln \ln n$, for instance. We define precisely this concept.

Definition 2.2. Let μ_n be a strictly positive nondecreasing function for large n . Moreover, assume that for every $\varepsilon > 0$ the function $\log^\varepsilon n / \mu_n$ is increasing as long as n is large enough. Then we say that μ_n is a *sublogarithmical* function.

Now we are ready to state the Discrete MT (examples of its use can be found in Appendix A).

THEOREM 2.3 (DISCRETE MASTER THEOREM). *Let F_n be a function defined by a discrete recursive definition, and let $B n^a \ln^c n \cdot \xi_n$ be the main term of t_n , where $B > 0$, a and c are arbitrary constants, and $\xi_n = \mu_n$ or $\xi_n = 1/\mu_n$ for some*

sublogarithmical function μ_n . Let $\Phi(x) = \sum_{1 \leq d \leq D} w_d \cdot (z_d)^x$, and $\mathcal{H} = 1 - \Phi(a)$. Then,

- (1) if $\mathcal{H} > 0$, then $F_n \sim t_n/\mathcal{H}$;
- (2) if $\mathcal{H} = 0$, then
 - (2.1) if $c > -1$, then $F_n \sim t_n \ln n/\mathcal{H}'$, where $\mathcal{H}' = -(c + 1)\sum_{1 \leq d \leq D} w_d \cdot (z_d)^a \ln z_d$;
 - (2.2) if $c = -1$, then $F_n = \mathcal{O}(n^a \log^\varepsilon n)$ for any $\varepsilon > 0$, and $F_n = \Omega(n^a)$ if $F_n \geq 0$ for every $n \geq 0$;
 - (2.2') if $c = -1$ and $\mu_n = 1$, then $F_n \sim t_n \ln n \cdot \ln \ln n/\mathcal{H}''$, where $\mathcal{H}'' = -\sum_{1 \leq d \leq D} w_d \cdot (z_d)^a \ln z_d$;
 - (2.3) if $c < -1$, then $F_n = \mathcal{O}(n^a)$ ($F_n = \Theta(n^a)$, if $F_n \geq 0$ for every $n \geq 0$);
- (3) if $\mathcal{H} < 0$, then $F_n = \mathcal{O}(n^\alpha)$ ($F_n = \Theta(n^\alpha)$, if $F_n \geq 0$ for every $n \geq 0$), where α is the unique solution of $\Phi(\alpha) = 1$.

PROOF. Section 6 is devoted to prove this theorem. \square

The Discrete MT could be trivially adapted to deal with negative toll functions by changing the conditions “ $B > 0$ ” and “ $F_n \geq 0$ for every $n \geq 0$ ” to “ $B < 0$ ” and “ $F_n \leq 0$ for every $n \geq 0$ ”, respectively.

Also, though we will not prove it in this work, we could relax Definition 2.1 to permit the value $z_d = 1$ for some indices, as long as (a) there is, at least, one index d such that $0 < z_d < 1$; and (b) the total sum of weights w_d of the indices d with $z_d = 1$ is strictly smaller than 1. Notice that neither

$$A_n = t_n + A_{n-3} \tag{6}$$

nor

$$B_n = t_n + 2B_{\lfloor n/2 \rfloor} + 2B_{n-1} \tag{7}$$

is, even under the relaxation above, a discrete recursive definition. The first recurrence does not satisfy any of the two conditions, while the second recurrence fails to fulfill the condition (b). They are out of the scope of this paper. By contrast, the recurrence

$$F_n = t_n + F_{\lfloor n/2 \rfloor} + \frac{1}{2} \cdot F_{n-2}$$

is a discrete recursive definition, and so we could analyze it exactly as stated by the Discrete MT.

The case $z_d = 0$ for some indices $1 \leq d \leq D$ is especially awkward. We avoid dealing with it.

3. The Continuous Master Theorem

This section covers the analysis of recurrences like (4) and (5), which we will call continuous. But first we need to define the concept of shape function (the reason for this name will be clear after Definition 3.2).

Definition 3.1. Let $\omega(z) \geq 0$ be a function over $[0, 1]$ such that $\int_0^1 \omega(z) dz$ exists and is at least 1. Furthermore, assume that there is some $\mu < 0$ such that $\int_0^1 \omega(z)z^\mu dz$ also converges.³ Then we say that $\omega(z)$ is a *shape function*.

Definition 3.2. We say that

$$F_n = \begin{cases} b_n, & \text{if } 0 \leq n < N \\ t_n + \sum_{0 \leq k < n} \omega_{n,k} F_k, & \text{if } n \geq N \end{cases}$$

is a *continuous recursive definition* of F_n iff there exist some shape function $\omega(z)$, some constant $0 < q \leq 1$ and some function $M_n = \Theta(n^q)$ with integer values such that, if we define $z_{n,j} = j/M_n$ for every $0 \leq j \leq M_n$, $I_{n,j} = [z_{n,j} \cdot n, z_{n,j+1} \cdot n)$ for every $0 \leq j < M_n$, and

$$\varepsilon_{n,j} = \left| \sum_{k \in I_{n,j}} \omega_{n,k} - \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz \right| \tag{8}$$

for every $0 \leq j < M_n$, then $\sum_{0 \leq j < M_n} \varepsilon_{n,j} = \mathcal{O}(n^{-\rho})$ for some $\rho > 0$.

Usually, $M_n = n$ is a possible choice, and in this case (8) reduces to

$$\varepsilon_{n,k} = \left| \omega_{n,k} - \int_{k/n}^{(k+1)/n} \omega(z) dz \right|.$$

Loosely speaking, we can use the integral in the expression above to find a good approximation to $\omega_{n,k}$. Notice that there can only be one shape function $\omega(z)$ related to a given continuous recursive definition, except for bizarre shape functions obtained from $\omega(z)$ by changing its value at a finite number of points, or by similar minor perturbations.

For instance, consider the recurrence

$$S_n = n + 1 + \frac{2}{n^2} \sum_{0 \leq k < n} k S_k \tag{9}$$

for the expected number of comparisons to select the i th out of n keys using quickselect—also known as FIND [Hoare 1961]—when i is chosen at random. Its shape function is $\omega(z) = 2z$ (see Figure 1), since

$$\int_{k/n}^{(k+1)/n} \omega(z) dz = \int_{k/n}^{(k+1)/n} 2z dz = z^2 \Big|_{k/n}^{(k+1)/n} = \frac{2k}{n^2} + \frac{1}{n^2} = \omega_{n,k} + \frac{1}{n^2},$$

and hence $\varepsilon_{n,k} = 1/n^2$ and $\sum_{0 \leq k < n} \varepsilon_{n,k} = 1/n = \mathcal{O}(n^{-\rho})$ with $\rho = 1$.

Therefore, $\omega(z)$ is nothing except the asymptotic shape of the distribution of weights, which now is very similar to a continuous probability distribution, where the area beneath the function is the asymptotic number of recursive calls. Since, by definition, $\int_0^1 \omega(z) dz \geq 1$, we are assuming that there is at least one

³ The author's conjecture is that this technical property is always true if $\int_0^1 \omega(z) dz$ exists and is at least 1. Section 7 provides more details.

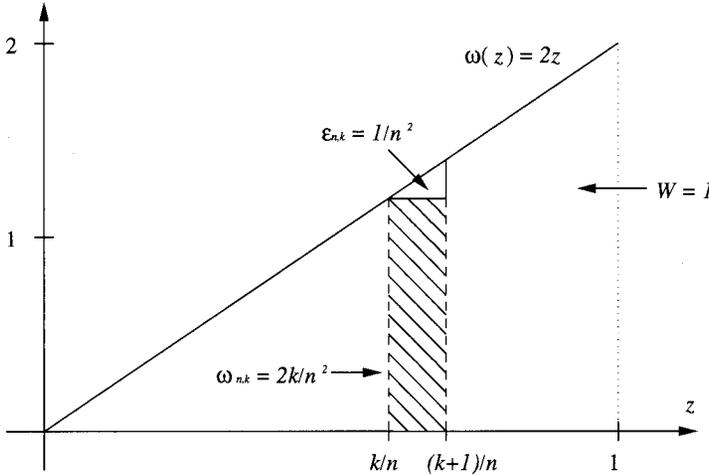


FIG. 1. Shape function for FIND.

recursive call for large n . This condition (very likely to hold in practice) simplifies the study of these recurrences.

On the other hand, we have allowed M_n to be $\Theta(n^q)$ for some $q \leq 1$, as long as $q > 0$. So it is not necessary that $\omega(z)$ fits the n weights individually, but a polynomial number of groups of weights. This relaxation could be useful when dealing with particularly difficult recurrences.

THEOREM 3.3 (CONTINUOUS MASTER THEOREM). *Let F_n be a function defined by a continuous recursive definition, and let $B n^a \ln^c n \cdot \xi_n$ be the main term of t_n , where $B > 0$, a and c are arbitrary constants, and $\xi_n = \mu_n$ or $\xi_n = 1/\mu_n$ for some sublogarithmical function μ_n . Let $\varphi(x) = \int_0^1 \omega(z)z^x dz$, and $\mathcal{H} = 1 - \varphi(a)$. Then,*

- (1) if $\mathcal{H} > 0$, then $F_n \sim t_n/\mathcal{H}$;
- (2) if $\mathcal{H} = 0$, then
 - (2.1) if $c > -1$, then $F_n \sim t_n \ln n/\mathcal{H}'$, where $\mathcal{H}' = -(c + 1) \int_0^1 \omega(z)z^a \ln z dz$;
 - (2.2) if $c = -1$, then $F_n = \mathcal{O}(n^a \log^\varepsilon n)$ for any $\varepsilon > 0$, and $F_n = \Omega(n^a)$ if $F_n \geq 0$ for every $n \geq 0$;
 - (2.2') if $c = -1$ and $\mu_n = 1$, then $F_n \sim t_n \ln n \cdot \ln \ln n/\mathcal{H}''$, where $\mathcal{H}'' = -\int_0^1 \omega(z)z^a \ln z dz$;
 - (2.3) if $c < -1$, then $F_n = \mathcal{O}(n^a)$ ($F_n = \Theta(n^a)$, if $F_n \geq 0$ for every $n \geq 0$);
- (3) if $\mathcal{H} < 0$ (including the case $\mathcal{H} = -\infty$) then $F_n = \mathcal{O}(n^\alpha)$ ($F_n = \Theta(n^\alpha)$, if $F_n \geq 0$ for every $n \geq 0$), where α is the unique solution of $\varphi(\alpha) = 1$.

PROOF. Section 7 is devoted to prove this theorem. \square

To use the Continuous Master Theorem, we must first identify the shape function $\omega(z)$ for the distribution of weights. This is equivalent to identifying the values $\{w_d\}_{1 \leq d \leq D}$ and $\{z_d\}_{1 \leq d \leq D}$ for the discrete case. One possibility is to conjecture that $\omega(z) = n \cdot \omega_{n,zn}$. For instance, for (9) this technique yields $\omega(z) = n \cdot 2/n^2 \cdot zn = 2z$, which we have already proved to be right. But for (5) we have additional problems, since the expression $n \cdot \omega_{n,zn} = 4(1 - z)/(1 +$

$1/n$) includes n . Lemma 7.2 in Section 7 provides a way to find $\omega(z)$ that works in many practical situations.

Examples of how to use the Continuous Master Theorem can be found in Appendix B.

4. The Core Theorems

In this section and Section 5, we will prove some general theorems that apply to many DAC recurrences. In Sections 6 and 7, we will see how to use these results to prove the statements of both the Discrete and the Continuous MTs.

We begin introducing the concept of divide-and-conquer recursive definition formally.

Definition 4.1. Let F_n be a function defined for every $n \geq 0$, and let

$$\mathcal{F} = [N, \{b_n\}_{0 \leq n < N}, \{t_n\}_{n \geq N}, \{w_{n,k}\}_{n \geq N}^{0 \leq k < n}]$$

be a recursive definition, with $N \geq 1$ and $w_{n,k} \geq 0$. For every $n \geq N$, let $W_n = \sum_{0 \leq k < n} w_{n,k}$ and $Z_n = \sum_{0 \leq k < n} (w_{n,k}/W_n)(k/n)$. Then we say that \mathcal{F} is a *DAC recursive definition* of F_n if and only if (a) $F_n = b_n$ for every $0 \leq n < N$; (b) for every $n \geq N$,

$$F_n = t_n + \sum_{0 \leq k < n} w_{n,k} F_k; \tag{10}$$

and (c) it exists some upper bound $U < 1$ such that $Z_n \leq U$ for all n large enough. We also define $W = \lim_{n \rightarrow \infty} W_n$ and $Z = \lim_{n \rightarrow \infty} Z_n$, if they exist.

This definition can be easily interpreted: W_n is the total number of recursive calls to solve a problem of size n ; Z_n is the average fraction of the original problem that is solved by a recursive call (it turns out that $0 \leq Z_n < 1$); and for large n , the average size of the recursive calls is, at most, a fraction of the whole problem. For example, for (9), we have $W_n = 1 - 1/n$, $W = 1$, $Z_n = 2/3 - 1/3n$ and $Z = 2/3$, while for (1) we have $W_n = 2$, $W = 2$, $Z_n = 1/2$ and $Z = 1/2$. By contrast, (6) is not a DAC recursive definition.

Most of the results from which both MTs are derived refer to canonical recursive definitions, which are defined as follows:

Definition 4.2. Let \mathcal{F} be a DAC recursive definition of a function F_n . We say that \mathcal{F} is a *canonical recursive definition* if and only if both these properties hold: (a) The constant W exists and is equal to 1. (b) If we define $m_n = W_n - W$, then $|m_n| = \mathcal{O}(n^{-\rho})$ for some $\rho > 0$.

Hence, a DAC recursive definition is canonical if the number of recursive calls tends to 1, with a minimum convergence speed. For instance, for (9), we have $W = 1$, and $|m_n| = 1/n = \mathcal{O}(n^{-\rho})$ with $\rho = 1$.

Now, assume that $F_n \geq 0$ is defined by (10). Then we have $F_n = \Omega(t_n)$, because F_n cannot grow more slowly than t_n does. So the question is identifying under which conditions F_n can grow faster than t_n . For the recursive definitions we deal with and roughly speaking, we will prove that there is a growing order $\Theta(n^\alpha)$ associated to every distribution of weights, irrespectively of how small t_n

is. This would be similar to state that $F_n = \Theta(\max\{t_n, n^\alpha\})$, which is almost true.

For instance, consider the recurrence

$$F_n = t_n + F_{\lfloor n/4 \rfloor}. \tag{11}$$

We will see in this section that $\alpha = 0$ for any canonical recurrence, such as this one. Therefore, for “large” values of t_n (case 1 of the MTs) like n, n^3 or even 2^n , we should get $F_n = \Theta(t_n)$, which is true. For “small” values of t_n (case 3 of the MTs) like $1/n, 1/n^3$ or 2^{-n} , F_n should be $\Theta(1)$, which is also true. However, things are not so easy for values of t_n close to $\Theta(1)$ (case 2 of the MTs). For example, for $t_n = 1$ the growing order of F_n turns out to be $\Theta(\log n)$ instead of $\Theta(1)$. We will see in Section 5 why this additional factor appears.

At this point, it is worth commenting on the word “Core” in the title of this section. It tries to suggest the idea that recursive definitions with a toll function small enough—and thus inside the zone dominated by the term $\Theta(n^\alpha)$ associated to the distribution of weights—are the most difficult to analyze. Indeed, there is no way to find the lower order terms of F_n nor even the multiplicative factor of the main term n^α , but to consider all the values of F_n , the values at small indices included.

For example, consider (11), and assume that it holds for $n \geq 1$. Let $t_n > 0$ be small enough. Then $F_n = \Theta(1)$, but modifying F_0 yields a significant change in F_n for every n . In fact, changing the value of the function at any index significantly affects F at an infinite number of indices. Set in terms of a recursion tree (see Cormen et al. [1990], for example), the solution to the recurrence is dominated by the values at the leaves.

Moreover, in some cases, the multiplicative factor of the main term n^α is not even asymptotically constant. Consider the following recurrence,

$$F_n = t_n + \begin{cases} F_{2\lfloor n/4 \rfloor}, & \text{if } n \text{ is even} \\ F_{2\lfloor (n-1)/4 \rfloor + 1}, & \text{if } n \text{ is odd} \end{cases}$$

for every $n \geq 2$. Let $t_n > 0$ be small enough and such that $t_{2n} = t_{2n+1}$, and let $F_0 = 0$ and $F_1 = 1$. Then it is easy to see that $F_n = \Theta(1)$, but F_n cannot tend to a constant, since it fluctuates periodically.

We end this section with two theorems which formalize one of the claims above, namely that any canonical recurrence is on the one hand $\Omega(1)$, and on the other hand $\mathcal{O}(1)$ for t_n small enough. The three technical conditions in the statement of Theorem 4.3 typically hold and, loosely speaking, avoid cases like “everything is zero” and “the negative and the positive contributions cancel out.”

THEOREM 4.3. *Let F_n be defined by a canonical recursive definition, and let $b_n \geq 0, t_n > 0$ and $\sum_{0 \leq k < N} w_{n,k} = \mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$. Then $F_n = \Omega(1)$.*

PROOF. By hypothesis, $|m_n| = |W_n - 1| = \mathcal{O}(n^{-\rho})$, where $\rho > 0$. Choose some ν such that $0 < \nu < \max\{\rho, \sigma\}$, and define $W'_n = \sum_{N \leq k < n} w_{n,k}$. Taking into account that $W_n \geq 1 - |m_n|$, we have $W'_n = W_n - \sum_{0 \leq k < N} w_{n,k} \geq 1 + \mathcal{O}(n^{-\rho}) - \mathcal{O}(n^{-\sigma}) \geq 1 - n^{-\nu}$, as long as n is large enough.

Now define

$$Z'_n = \sum_{N \leq k < n} \frac{w_{n,k}}{1 - n^{-\nu}} \cdot \frac{k}{n} \leq \frac{W_n}{1 - n^{-\nu}} \sum_{0 \leq k < n} \frac{w_{n,k}}{W_n} \cdot \frac{k}{n} = \frac{W_n}{1 - n^{-\nu}} \cdot Z_n \leq \frac{W_n}{1 - n^{-\nu}} \cdot U,$$

where $0 < U < 1$ is the upper bound for the Z_n 's that exists by definition. Choose any V such that $U < V < 1$. Then it is clear that $Z'_n \leq V$ for large n .

For the next step of the proof, choose some $N' > N$ large enough to get $W'_n \geq 1 - n^{-\nu}$, $Z'_n \leq V$, and also $n^{-\nu} \leq 1/2$ for every $n \geq N'$. Let $h(x) = \exp(2V^\nu/(1 - V^\nu)x^\nu)$. In what follows, we will use the easy-to-prove fact that $h(x) \geq 1$ is a decreasing convex function in the interval $[1, +\infty)$.

Set $M = \min\{F_n/h(n)\}_{N \leq n < N'}$. Notice that $b_n \geq 0$ and $t_n > 0$ imply $M > 0$. We can prove by induction that $F_n \geq M \cdot h(n)$ holds not only for all $N \leq n < N'$ but also for all $n \geq N'$: Assuming the induction hypothesis, we have

$$F_n = t_n + \sum_{0 \leq k < n} w_{n,k} F_k \stackrel{\text{i.h.}}{\geq} \sum_{N \leq k < n} w_{n,k} M \cdot h(k) = M \cdot W'_n \sum_{N \leq k < n} \frac{w_{n,k}}{W'_n} \cdot h(k).$$

The convexity of $h(x)$, the bound for W'_n and the fact that $h(x)$ is decreasing yields

$$F_n \geq M \cdot W'_n \cdot h\left(\sum_{N \leq k < n} \frac{w_{n,k}}{W'_n} \cdot k\right) \geq M(1 - n^{-\nu}) \cdot h\left(\sum_{N \leq k < n} \frac{w_{n,k}}{1 - n^{-\nu}} \cdot k\right).$$

From the definition of Z'_n , $F_n \geq M(1 - n^{-\nu}) \cdot h(nZ'_n)$. Taking into account that $1 - x \geq \exp(-2x)$ for all $0 \leq x \leq 1/2$, and replacing x by $n^{-\nu}$, we get $1 - n^{-\nu} \geq \exp(-2n^{-\nu})$. Furthermore, $Z'_n \leq V$ and $h(x)$ is decreasing, and thus we have $F_n \geq M \exp(-2n^{-\nu}) \cdot h(nV) = M \cdot h(n)$, the last step obtained through simple manipulations. This ends the inductive proof.

Finally, $h(n) \geq 1$ yields $F_n \geq M > 0$ for every $n \geq N$, and the statement of the theorem follows. \square

THEOREM 4.4. *Let F_n be defined by a canonical recursive definition, and let $t_n = \mathcal{O}(\log^c n)$ for some $c < -1$. Then $F_n = \mathcal{O}(1)$.*

PROOF. Let \mathcal{F} be the recursive definition for F_n . First of all, we introduce a recursive definition \mathcal{G} for a new function G_n , by modifying \mathcal{F} as follows: Let $\sigma_{n,k}$ denote the weights of \mathcal{F} , Y_n denote $\sum_{0 \leq k < n} \sigma_{n,k}$, and assume without loss of generality that $Y_n > 0$ —recall that $\lim_{n \rightarrow +\infty} Y_n = 1$. Define $w_{n,k}$, the weights of \mathcal{G} , to be equal to $\sigma_{n,k}$ if $Y_n \geq 1$, and equal to $\sigma_{n,k}/Y_n$ if $Y_n < 1$ (this is just a normalization of the weights so that they sum at least 1). Let u_n be the toll function of \mathcal{F} ; we define the toll function of \mathcal{G} as $t_n = |u_n|$. Altogether, $-G_n \leq F_n \leq G_n$ trivially holds for every $n \geq 0$, so it is enough to bound G_n by above to prove the theorem.

Let W_n, W , etc. be as in Definitions 4.1 and 4.2 but for the set of weights $w_{n,k}$ of \mathcal{G} . Then, on the one hand, we have that W is still 1, but now $W_n \geq 1$ and $m_n \geq 0$. On the other hand Z_n remains the same and hence $Z_n \leq U$ for some $0 < U < 1$. Choose some V such that $U < V < 1$, and some ν such that $0 < \nu < \rho$, where $\rho > 0$ is the constant in Definition 4.2. Choose some $a \geq 1$ large

enough to get $V^v/(1 - V^v) \leq a^v$. We introduce two auxiliary functions, $f(x) = \exp(-V^v/(1 - V^v)(x + a)^v)$, and $g(x) = \sum_{i \geq 1} g_i(x)^c$, where $g_i(x) = \ln(x + a) - i \ln V$. It is not difficult to prove that, for every $x \geq 0$, $0 < f(x) \leq 1$ is an increasing concave function, and $g(x) \geq 0$ is a well-defined decreasing convex function.

There are three bounds for the quantities introduced so far that we will find helpful through this proof (they all hold for n large enough, say for $n \geq N'$ for some constant N'). First, taking into account that $m_n = \mathcal{O}(n^{-\rho})$, we have $0 \leq m_n \leq 1/((n + a)^v + 1)$. Second, we define $Z'_n = (nZ_n + a)/(n + a) = Z_n + a(1 - Z_n)/(n + a) \leq U + a/n \leq V$. Finally, by the initial hypothesis about t_n we know that there exists some constant $K > 0$ such that $t_n \leq K \ln^c(n + a)$.

Assume $n \geq N'$, and introduce

$$S_n = \sum_{0 \leq k < n} w_{n,k} f(k) = \sum_{0 \leq k < n} \frac{w_{n,k}}{W_n} \cdot f(k) + \frac{m_n}{W_n} \sum_{0 \leq k < n} w_{n,k} f(k),$$

where for the second equality above we have used the fact that $W_n = 1 + m_n$. Then, since $f(x)$ is positive, increasing and concave, and m_n is positive, we can bound S_n to get

$$S_n \leq f\left(\sum_{0 \leq k < n} \frac{w_{n,k}}{W_n} \cdot k\right) + \frac{m_n}{W_n} \cdot W_n f(n) = f(nZ_n) + m_n f(n).$$

By the definition of Z'_n , we know that $nZ_n + a = (n + a)Z'_n \leq (n + a)V$. Therefore,

$$f(nZ_n) = \exp\left(\frac{-V^v}{(1 - V^v)(nZ_n + a)^v}\right) \leq \exp\left(\frac{-1}{(1 - V^v)(n + a)^v}\right). \quad (12)$$

For the next step we use an auxiliary function $A_n = 1 - 1/((n + a)^v + 1) = (n + a)^v/((n + a)^v + 1)$. Since $1/A_n = 1 + 1/(n + a)^v$, and $1 + x \leq \exp(x)$ for every $x \geq 0$, it follows that $1/A_n \leq \exp(1/(n + a)^v)$, and thus $A_n \geq \exp(-1/(n + a)^v)$. Hence, $A_n f(n) \geq \exp(-1/(n + a)^v) f(n) = \exp(-(1 - V^v)/(n + a)^v)$, and from (12), $m_n \leq 1/((n + a)^v + 1)$, and the definition of A_n we conclude $S_n \leq f(n)$. We will use this inequality in a moment.

Now introduce $R_n = \sum_{0 \leq k < n} w_{n,k} g(k)$, and use the convexity of $g(x)$ to get

$$R_n \geq \sum_{0 \leq k < n} \frac{w_{n,k}}{W_n} \cdot g(k) \geq g\left(\sum_{0 \leq k < n} \frac{w_{n,k}}{W_n} \cdot k\right) = g(nZ_n) = \sum_{i \geq 1} g_i(nZ_n)^c.$$

By definition, $g_i(nZ_n) = \ln(nZ_n + a) - i \ln V = \ln((n + a)Z'_n) - i \ln V$. Assume $n \geq N'$; the bound for Z'_n yields $g_i(nZ_n) \leq \ln(n + a) + \ln V - i \ln V = g_{i-1}(n)$. Therefore, $R_n \geq \sum_{i \geq 1} g_{i-1}(n)^c = \ln^c(n + a) + g(n)$.

We are ready to prove the theorem. Set $M = \max\{(G_n/K + g(n))/f(n)\}_{0 \leq n < N'}$. Then we have $G_n/K \leq M \cdot f(n) - g(n)$ for any $n < N'$. We can

prove this bound by induction when $n \geq N'$:

$$\begin{aligned} \frac{G_n}{K} &= \frac{t_n}{K} + \sum_{0 \leq k < n} w_{n,k} \cdot \frac{G_k \text{ i.h.}}{K} \geq \ln^c(n+a) + \sum_{0 \leq k < n} w_{n,k}(M \cdot f(k) - g(k)) \\ &= M \cdot S_n + \ln^c(n+a) - R_n \leq M \cdot f(n) - g(n). \end{aligned}$$

Finally, we have $F_n \leq G_n \leq K(M \cdot f(n) - g(n)) \leq K \cdot M$ for every $n \geq 0$ (and $F_n \geq -K \cdot M$), and the theorem is proved. \square

5. The Flesh Theorems

In this section, we deal with the canonical recursive definitions whose toll function is large enough to dominate the solution. These recurrences are typically easier to analyze than those in Section 4 (and thus the “Flesh” in the title), and in most cases we can get the multiplicative factor of the main term of the function under study. Moreover, sometimes it is possible to extract one by one several main terms with their multiplicative factors, until the core is reached (see Appendix B).

The reason for the name chosen in the following definition will be clear after Theorem 5.3.

Definition 5.1. Let u_n be a function over the integers. We say that γ_n is a *bounding function* of u_n if and only if there exist some constant $N_\gamma \geq 1$ and some strictly positive nonincreasing function $\beta(z)$ defined over $(0, 1)$ such that (a) $u_n \gamma_n - u_k \gamma_k \geq \beta(k/n)u_n$ for every $n > N_\gamma$ and every $N_\gamma \leq k < n$; (b) $u_n, \gamma_n > 0$ for every $n \geq N_\gamma$; (c) γ_n is a subpolynomial function; and (d) $u_n \gamma_n = \omega(1)$.

For instance, $\gamma_n = 1$ is a bounding function of $u_n = n^2$. Take $\beta(z) = 1 - z^2$. Then we have $u_n \gamma_n - u_k \gamma_k = n^2 - k^2 = (1 - (k/n)^2)n^2 = \beta(k/n)u_n$. The other conditions trivially hold.

The word “entropy” is defined below with a meaning wider than usually. Our definition reduces to the traditional one when $t_n = 1$ and $\gamma_n = \log_2 n$.

Definition 5.2. Let \mathcal{F} be a canonical recursive definition of a function F_n , and let γ_n be a bounding function of t_n . We define the *entropy* of \mathcal{F} with respect to γ_n at every $n \geq \max\{N, N_\gamma\}$ as

$$\mathcal{H}_n^{(\gamma)} = \gamma_n - \sum_{N_\gamma \leq k < n} w_{n,k} \cdot \frac{t_k}{t_n} \cdot \gamma_k. \tag{13}$$

We also define $\mathcal{H}^{(\gamma)} = \lim_{n \rightarrow \infty} \mathcal{H}_n^{(\gamma)}$, if it exists.

The next theorem, together with Lemma 5.4, will allow us to bound the solution of many canonical recursive definitions.

THEOREM 5.3. *Let F_n be defined by a canonical recursive definition, and let γ_n be a bounding function of u_n . If $t_n = \mathcal{O}(u_n)$, then $F_n = \mathcal{O}(u_n \gamma_n)$. If $t_n = o(u_n)$, then $F_n = o(u_n \gamma_n)$.*

PROOF. Let $\mathcal{H}_n^{(\gamma)}$ be defined as in (13) but using u_n instead of t_n . We first prove that $\mathcal{H}_n^{(\gamma)} = \Omega(1)$, as follows. We know that $Z_n \leq U$ for large n , where $0 < U < 1$. Choose some V such that $U < V < 1$, and let n be large enough to

get $nV > N_\gamma$. Then $t_n \gamma_n > t_n \gamma_n - t_k \gamma_k \geq \beta(k/n)t_n \geq \beta(V)t_n$ for every $N_\gamma \leq k < nV$. Alternatively, we can write $\gamma_n > \gamma_n - t_k/t_n \cdot \gamma_k \geq \beta(V)$. We now use these bounds in the definition of $\mathcal{H}_n^{(\gamma)}$:

$$\begin{aligned} \mathcal{H}_n^{(\gamma)} &= \left(\sum_{0 \leq k < n} w_{n,k} - m_n \right) \gamma_n - \sum_{N_\gamma \leq k < n} w_{n,k} \cdot \frac{t_k}{t_n} \cdot \gamma_k \\ &\geq \sum_{0 \leq k < N_\gamma} w_{n,k} \gamma_n + \sum_{N_\gamma \leq k < nV} w_{n,k} \left(\gamma_n - \frac{t_k}{t_n} \cdot \gamma_k \right) - m_n \gamma_n \\ &\geq \sum_{0 \leq k < nV} w_{n,k} \beta(V) - m_n \gamma_n. \end{aligned}$$

On the other hand, from

$$V \sum_{nV \leq k < n} w_{n,k} = \sum_{nV \leq k < n} w_{n,k} \cdot \frac{nV}{n} \leq \sum_{0 \leq k < n} w_{n,k} \cdot \frac{k}{n} = Z_n W_n \leq U \cdot W_n,$$

we have $\sum_{0 \leq k < nV} w_{n,k} = W_n - \sum_{nV \leq k < n} w_{n,k} \geq (1 - U/V)W_n$. Altogether, we deduce $\mathcal{H}_n^{(\gamma)} \geq (1 - U/V)\beta(V)W_n - m_n \gamma_n$. But $m_n \gamma_n$ vanishes for large n , because $|m_n| = \mathcal{O}(n^{-\rho})$ for some $\rho > 0$ and γ_n is a subpolynomial function. Therefore, since $1 - U/V > 0$, $\beta(V) > 0$ and W_n tends to $W = 1$, we have $\mathcal{H}_n \geq Q$ for some constant $Q > 0$ and n large enough (say, larger than some constant $N_{\mathcal{H}}$).

Now we can prove the $\mathcal{O}(\cdot)$ case. Since $t_n = \mathcal{O}(u_n)$, there exist constants $K > 0$ and M such that $|t_n| \leq Ku_n$ for every $n \geq M$. Choose M to be at least $N_{\mathcal{H}}$, and introduce a new function,

$$G_n = \begin{cases} F_n, & \text{if } 0 \leq n < M \\ Ku_n + \sum_{0 \leq k < n} w_{n,k} G_k, & \text{if } n \geq M. \end{cases}$$

A simple proof by induction yields $-G_n \leq F_n \leq G_n$ for every n . Now define

$$I_n = \begin{cases} G_n, & \text{if } 0 \leq n < N_\gamma \\ G_n - \frac{K}{Q} \cdot u_n \gamma_n, & \text{if } n \geq N_\gamma. \end{cases}$$

Let $n \geq M$. Using the definitions of I_n and G_n in the first step, and the definition of I_n back in the second step, we get

$$\begin{aligned} I_n &= Ku_n + \sum_{0 \leq k < n} w_{n,k} G_k - \frac{K}{Q} \cdot u_n \gamma_n \\ &= Ku_n - \frac{K}{Q} \left(u_n \gamma_n - \sum_{N_\gamma \leq k < n} w_{n,k} u_k \gamma_k \right) + \sum_{0 \leq k < n} w_{n,k} I_k \\ &= Ku_n - \frac{K}{Q} \cdot u_n \mathcal{H}_n + \sum_{0 \leq k < n} w_{n,k} I_k. \end{aligned}$$

From the bound for \mathcal{H}_n , the toll function above is $\leq Ku_n - K/Q \cdot u_n Q = 0$, as long as $n \geq M$. Therefore, if we define

$$J_n = \begin{cases} |I_n|, & \text{if } 0 \leq n < M \\ \sum_{0 \leq k < n} w_{n,k} J_k, & \text{if } n \geq M, \end{cases}$$

then we have $I_n \leq J_n$. But according to Theorem 4.4, $J_n = \mathcal{O}(1)$, that is, $J_n \leq S$ for some constant $S > 0$ and n large enough. This implies $I_n \leq S$ as well, and hence $F_n \leq G_n = I_n + K/Q \cdot u_n \gamma_n \leq S + K/Q \cdot u_n \gamma_n$ or alternatively, $F_n/u_n \gamma_n \leq S/u_n \gamma_n + K/Q$. Taking into account that $u_n \gamma_n = \omega(1)$, the term $S/u_n \gamma_n$ will vanish for large n , and thus we can conclude $F_n/u_n \gamma_n \leq K'$ (and $F_n/u_n \gamma_n \geq -K'$) for any constant $K' > K/Q$ as long as n is large enough. The case $\mathcal{O}()$ is proved.

The proof of the case $o()$ follows the pattern above. Therefore, we only point out the differences. We have to show that for every $\varepsilon > 0$ there is some N_ε large enough such that $|F_n|/u_n \gamma_n \leq \varepsilon$ as long as $n \geq N_\varepsilon$. Set $\nu = Q \cdot \varepsilon/2$. Then, by hypothesis, it exists some M_ν such that $|t_n| \leq \nu u_n$ for every $n \geq M_\nu$. Define G_n using νu_n as toll function for every $n \geq M_\nu$, and I_n as $G_n - \nu/Q \cdot u_n \gamma_n$ for every $n \geq N_\nu$. Then we have $I_n \leq S_\nu$ for some constant $S_\nu > 0$ and every n large enough, and thus $F_n/u_n \gamma_n \leq S_\nu/u_n \gamma_n + \nu/Q$. Since the first term vanishes for large n , we can always pick some N_ε large enough to get $F_n/u_n \gamma_n \leq \nu/Q + \nu/Q = \varepsilon$ for every $n \geq N_\varepsilon$ (and also $F_n/u_n \gamma_n \geq -\varepsilon$), thus proving the case $o()$ and the theorem. \square

The following lemma provides bounding functions for the most usual toll functions.

LEMMA 5.4

- Let $u_n = n^a \delta_n$ for every $n \geq N_u$, where $N_u \geq 1$, $a > 0$, and δ_n is a strictly positive nondecreasing function at every $n \geq N_u$. Then $\gamma_n = 1$ is a bounding function of u_n .
- Let $v_n = \ln^c n \cdot \epsilon_n$ for every $n \geq N_v$, where $N_v \geq 2$, $c > -1$, and ϵ_n is a strictly positive nondecreasing function at every $n \geq N_v$. Then $\gamma_n = \ln n$ is a bounding function of v_n .
- Let $w_n = \ln^{-1} n$ for every $n \geq N_w$, for some $N_w \geq 3$. Then $\gamma_n = \ln n \cdot \ln \ln n$ is a bounding function of w_n .

PROOF. For the first case, let $N_\gamma = N_u$ and take $\beta(z) = 1 - z^a$. Thus, $\beta(z) > 0$ is nonincreasing in $(0, 1)$. Take any $n > N_\gamma$ and any $N_\gamma \leq k < n$. Then $u_n \gamma_n - u_k \gamma_k = n^a \delta_n - k^a \delta_k \geq n^a \delta_n - k^a \delta_n = (1 - (k/n)^a) n^a \delta_n = \beta(k/n) u_n$, and the condition (a) in Definition 5.1 is satisfied. The conditions (b), (c), and (d) are trivial to prove.

For the second case, take $\beta(z) = -(c + 1) \ln z$, if $-1 < c \leq 0$, or $\beta(z) = -\ln z$, if $c > 0$. Let $N_\gamma = N_v$, and take any k and n such that $N_\gamma \leq k < n$. For the moment, we have $v_n \gamma_n - v_k \gamma_k = \ln^{c+1} n \cdot \epsilon_n - \ln^{c+1} k \cdot \epsilon_k \geq \ln^{c+1} n \cdot \epsilon_n - \ln^{c+1} k \cdot \epsilon_n = (\ln^{c+1} n - \ln^{c+1} k) \epsilon_n$.

To check that the condition (a) in Definition 5.1 holds, we consider the last expression separately for positive and negative c . For $c > 0$, it is not difficult to prove that $\ln^{c+1} n - \ln^{c+1} k \geq -\ln(k/n) \ln^c n$ when $n \geq 1$ and $1 \leq k \leq n$.

Therefore, $v_n \gamma_n - v_k \gamma_k \geq -\ln(k/n) \ln^c n \cdot \epsilon_n = \beta(k/n) v_n$. Similarly, when $-1 < c \leq 0$ it suffices to prove that $\ln^{c+1} n - \ln^{c+1} k \geq -(c + 1) \ln(k/n) \ln^c n$ when $n \geq 1$ and $1 \leq k \leq n$, and then $v_n \gamma_n - v_k \gamma_k \geq -(c + 1) \ln(k/n) \ln^c n \cdot \epsilon_n = \beta(k/n) v_n$. The rest of conditions in Definition 5.1 trivially hold.

For the third case, let $N_\gamma = N_w$, and take any $n > N_\gamma$ and any $N_\gamma \leq k < n$. Define $\beta(z) = -\ln z$, and let $y = \ln n$ and $z = k/n$. It can be shown that $\ln y - \ln(y + \ln z) \geq -\ln z/y$ for all z in $(\exp(-y), 1]$. Therefore, $w_n \gamma_n - w_k \gamma_k = \ln \ln n - \ln \ln k = \ln y - \ln(y + \ln z) \geq -\ln z/y = \beta(k/n) w_n$. The rest of conditions for a bounding function can be easily proved. \square

The following corollary is an immediate consequence of Theorem 5.3 and Lemma 5.4.

COROLLARY 5.5. *Let the functions F_n , G_n , and H_n be defined by canonical recursive definitions with toll functions $t_n = \mathcal{O}(n^a \delta_n)$, $t'_n = \mathcal{O}(\log^c n \cdot \epsilon_n)$ and $\mathcal{O}(\log^{-1} n)$, respectively, as stated in Lemma 5.4. Then $F_n = \mathcal{O}(t_n)$, $G_n = \mathcal{O}(t'_n \log n)$ and $H_n = \mathcal{O}(\log \log n)$. Moreover, $F_n = \Theta(t_n)$ if $F_n \geq 0$ and $t_n = \mathcal{O}(n^a \delta_n)$.*

The last theorem of this section allows us to compute the constant factor of the leading term of the solution of some recurrences.

THEOREM 5.6. *Let F_n be defined by a canonical recursive definition, and let γ_n be a bounding function of t_n . Furthermore, assume that $\mathcal{H}^{(\gamma)}$ exists. Then $F_n = t_n \gamma_n / \mathcal{H}^{(\gamma)} + o(t_n \gamma_n)$.*

PROOF. From the proof of Theorem 5.3 we know that $\mathcal{H}_n^{(\gamma)} = \Omega(1)$. So, if $\mathcal{H}^{(\gamma)}$ exists, it is strictly positive. Define

$$G_n = \begin{cases} F_n, & \text{if } 0 \leq n < N_\gamma \\ F_n - \frac{t_n \gamma_n}{\mathcal{H}^{(\gamma)}}, & \text{if } n \geq N_\gamma. \end{cases}$$

For $n \geq N_\gamma$, use the definition of G_n in both directions, like in the proof of Theorem 5.3. This yields

$$G_n = t_n - \frac{\mathcal{H}_n^{(\gamma)}}{\mathcal{H}^{(\gamma)}} \cdot t_n + \sum_{0 \leq k < n} w_{n,k} G_k.$$

Let t'_n denote the toll function above, and let $h_n = \mathcal{H}_n^{(\gamma)} - \mathcal{H}^{(\gamma)}$. Then $t'_n = t_n - (\mathcal{H}^{(\gamma)} + h_n) / \mathcal{H}^{(\gamma)} \cdot t_n = -h_n t_n / \mathcal{H}^{(\gamma)} = o(t_n)$, because $h_n = o(1)$. Therefore, it suffices to apply Theorem 5.3 to get $G_n = o(t_n \gamma_n)$, which implies $F_n = t_n \gamma_n / \mathcal{H}^{(\gamma)} + G_n = t_n \gamma_n / \mathcal{H}^{(\gamma)} + o(t_n \gamma_n)$. \square

As an example of application of Theorem 5.6, we can easily compute the expected number of comparisons to find a key chosen at random in a trie with n keys, when the digits are independent and equally likely to be 0 or 1. The recurrence is

$$T_n = 1 + \frac{1}{2^n} \sum_{0 \leq k < n} \binom{n}{k} T_k.$$

(It would be a simple matter to write another recurrence with range $0 \cdots n - 1$ for k .) Note that this recurrence is not covered by any of the MTs presented in this work; however, we can solve it anyway. The entropy with respect to $\ln n$ is

$$\mathcal{H}_n^{(\ln)} = \ln n - \frac{1}{2^n} \sum_{2 \leq k \leq n} \binom{n}{k} \ln k.$$

The distribution of weights gets closer to $n/2$ as long as n grows. This implies $\mathcal{H}_n^{(\ln)} \sim \ln n - \ln(n/2)$, $\mathcal{H}^{(\ln)} = \ln 2$, and $T_n \sim \ln n / \ln 2 = \log_2 n$.

More formally, we can use the fact that

$$\frac{1}{2^n} \sum_{(1+\varepsilon)n/2 < k \leq n} \binom{n}{k} \leq \exp\left(-\frac{\varepsilon^2 n}{6}\right)$$

for any $0 \leq \varepsilon \leq 1$ (see Hagerup and Rub [1990, page 306]), and set $\varepsilon = \varepsilon_n = n^{-1/3}$ to get

$$\begin{aligned} \mathcal{H}_n^{(\ln)} &= \ln n - \frac{1}{2^n} \sum_{2 \leq k \leq (1+\varepsilon_n)n/2} \binom{n}{k} \ln k - \frac{1}{2^n} \sum_{(1+\varepsilon_n)n/2 < k \leq n} \binom{n}{k} \ln k \\ &\geq \ln n - \ln\left(\frac{(1+\varepsilon_n)n}{2}\right) - \exp\left(-\frac{\varepsilon_n^2 n}{6}\right) \ln n = \ln 2 + o(1). \end{aligned}$$

On the other hand, it is easy to prove an upper bound $\ln 2$ for $\mathcal{H}^{(\ln)}$. Therefore, $\mathcal{H}^{(\ln)} = \ln 2$, and finally $T_n \sim \log_2 n$.

6. Proof of the Discrete Master Theorem

In this section, we will derive the Discrete MT. While doing this, we will repeatedly make use of the quantities introduced in Definitions 2.1, 4.1 and 4.2.

LEMMA 6.1. $\Phi(x)$ is a strictly decreasing continuous function such that $\lim_{x \rightarrow +\infty} \Phi(x) = 0$ and $\lim_{x \rightarrow -\infty} \Phi(x) = +\infty$.

PROOF. By elementary calculus. \square

LEMMA 6.2. A discrete recursive definition is canonical if and only if $\Phi(0) = 1$.

PROOF. For the “only if” proof, we observe that $W = \lim_{n \rightarrow +\infty} W_n = \sum_{1 \leq d \leq D} w_d = \Phi(0)$.

For the “if” proof, the definition of discrete recursive definition tells us that $|m_n| = |W_n - \Phi(0)| \leq \sum_{1 \leq d \leq D} |r_{d,n}| = \mathcal{O}(n^{-\rho})$. Furthermore, $Z = \lim_{n \rightarrow \infty} Z_n = \sum_{1 \leq d \leq D} w_d \cdot z_d = \Phi(1)$, which by Lemma 6.1 is strictly smaller than 1 when $W = \Phi(0) = 1$. \square

Theorem 6.3 justifies the cases 2.3 and 3 of the Discrete MT when the recurrence is canonical, that is, when $\alpha = 0$. Note that the case 2.3 reduces to $a = 0$ and $c < -1$, and the condition $\mathcal{H} < 0$ in the case 3 is equivalent to $a < 0$.

THEOREM 6.3. Let C_n be defined by a canonical discrete recursive definition, and let $t_n = \mathcal{O}(\log^c n)$, where $c < -1$. Then $C_n = \mathcal{O}(1)$. Moreover, $C_n = \Theta(1)$ if $C_n \geq 0$ for every $n \geq 0$ and $t_n > 0$ for n large enough.

PROOF. It is enough to realize that $\sum_{0 \leq k < N} w_{n,k} = 0$ for large n , and use Theorems 4.3 and 4.4. \square

For the next step, we need to define the concept of subpolynomial function formally.

Definition 6.4. Let λ_n be a strictly positive nondecreasing function for large n . Moreover, assume that for every $\varepsilon > 0$ the function n^ε/λ_n is increasing as long as n is large enough. Then we say that λ_n is a *subpolynomial function*.

Lemma 6.5 is used to prove Theorems 6.6 and 7.10.

LEMMA 6.5. Let $t_n = n^a \xi_n$, where $a > 0$ and $\xi_n = \lambda_n$ or $\xi_n = 1/\lambda_n$ for some subpolynomial function λ_n . Then 1 is a bounding function of t_n .

PROOF. We only need to prove that $n^a \xi_n$ is as stated by Lemma 5.4, which is obvious when $\xi_n = \lambda_n$. When $\xi_n = 1/\lambda_n$, it is enough to express t_n as $n^{a/2} \delta_n$, with $\delta_n = n^{a/2}/\lambda_n$. \square

Theorem 6.6 corresponds to the case 1 of the Discrete MT when $\alpha = 0$ and $a > 0$. The statement of the MT applies to toll functions with subpolynomial factors that are $\mathcal{O}(\log^c n)$ for some c , which is more restrictive than the functions considered by Theorem 6.6.

THEOREM 6.6. Let a function be defined by a canonical discrete recursive definition, and let $t_n = n^a \xi_n$, where $a > 0$ and $\xi_n = \lambda_n$ or $\xi_n = 1/\lambda_n$ for some subpolynomial function λ_n . Then $\mathcal{H}^{(1)} = 1 - \Phi(a)$.

PROOF. Let us assume $\xi_n = \lambda_n$. Then, for large n ,

$$\mathcal{H}_n^{(1)} = 1 - \sum_{1 \leq d \leq D} R_{d,n} \cdot \frac{t_{S_{d,n}}}{t_n} = 1 - \sum_{1 \leq d \leq D} (w_d + r_{d,n}) \left(z_d + \frac{S_{d,n}}{n} \right)^a \frac{\lambda_{S_{d,n}}}{\lambda_n}.$$

But $1 \geq \lambda_{S_{d,n}}/\lambda_n \geq (S_{d,n}/n)^\varepsilon = (z_d + S_{d,n}/n)^\varepsilon$ for any $1 \leq d \leq D$ and any $\varepsilon > 0$, as long as n is large enough. Moreover, the contribution to $\mathcal{H}_n^{(1)}$ of $r_{d,n}$ and $S_{d,n}$ vanishes for large n . Altogether,

$$1 - \Phi(a) = 1 - \sum_{1 \leq d \leq D} w_d \cdot z_d^a \leq \mathcal{H}^{(1)} \leq 1 - \sum_{1 \leq d \leq D} w_d \cdot z_d^{a+\varepsilon} = 1 - \Phi(a + \varepsilon).$$

This is true no matter how small ε is, and $\Phi(x)$ is a continuous function. We can thus conclude $\mathcal{H}^{(1)} = 1 - \Phi(a)$.

A similar argument proves the case $\xi_n = 1/\lambda_n$. \square

Note that the MTs do not deal with toll functions whose growing rate is larger than polynomial, like 2^n . Nevertheless, several of the results in this work also apply to superpolynomial toll functions, which are formally defined as follows:

Definition 6.7. Let u_n be a positive function for large n . For every $a > 0$, assume that u_n/n^a is increasing as long as n is large enough. Then we say that u_n is a *superpolynomial function*.

Notice that the conditions in the first case of Lemma 5.4 trivially hold for any superpolynomial function. As a consequence, the statement of Corollary 5.5 for

toll functions like $t_n = \mathcal{O}(n^a \delta_n)$ is true not only for polynomial toll functions but for superpolynomial toll functions as well.

The next theorem deals with discrete recurrences when the toll function is superpolynomial.

THEOREM 6.8. *Let a function be defined by a canonical discrete recursive definition, and let t_n be a superpolynomial function. Then $\mathcal{H}^{(1)} = 1$.*

PROOF. The proof is very similar to that of Theorem 6.6. It is enough to realize that $\mathcal{H}^{(1)} \geq 1 - \Phi(a)$ for every $a > 0$, no matter how large a is. \square

We require Lemma 6.9 for the statements of Theorems 6.10 and 7.12.

LEMMA 6.9. *Let $t_n = \ln^c n \cdot \xi_n$, where $c > -1$ and $\xi_n = \mu_n$ or $\xi_n = 1/\mu_n$ for some sublogarithmical function μ_n . Then $\ln n$ is a bounding function of t_n .*

PROOF. If $\xi_n = \mu_n$, then t_n is clearly as stated by Lemma 5.4. If $\xi_n = 1/\mu_n$, we only need to write t_n as $\ln^{(c-1)/2} n \cdot \delta_n$, with $\delta_n = \ln^{(c+1)/2} n / \lambda_n$. \square

Theorem 6.10 corresponds to the case 2.1 of the Discrete MT when $\alpha = a = 0$.

THEOREM 6.10. *Let a function be defined by a canonical discrete recursive definition, and let $t_n = \ln^c n \cdot \xi_n$, where $c > -1$ and $\xi_n = \mu_n$ or $\xi_n = 1/\mu_n$ for some sublogarithmical function μ_n . Then $\mathcal{H}^{(\ln)} = -(c + 1) \sum_{1 \leq d \leq D} w_d \cdot \ln z_d$.*

PROOF. Let us assume $\xi_n = 1/\mu_n$ (the case $\xi_n = \mu_n$ is similar). Then

$$\mathcal{H}_n^{(\ln)} = \ln n - \sum_{1 \leq d \leq D} (w_d + r_{d,n}) \frac{\ln^{c+1} S_{d,n}}{\ln^c n} \cdot \frac{\mu_n}{\mu_{S_{d,n}}}$$

for large n . Besides,

$$\frac{\ln^{c+1} S_{d,n}}{\ln^c n} \leq \frac{\ln^{c+1} S_{d,n}}{\ln^c n} \cdot \frac{\mu_n}{\mu_{S_{d,n}}} \leq \frac{\ln^{c+1-\varepsilon} S_{d,n}}{\ln^{c-\varepsilon} n}$$

for any $1 \leq d \leq D$ and any $\varepsilon > 0$, as long as n is large enough. Furthermore, $\ln(z_d + s_{d,n}/n) = \ln z_d + \mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$. Thus,

$$\begin{aligned} \ln^{c+1} S_{d,n} &= (\ln n + \ln z_d + \mathcal{O}(n^{-\sigma}))^{c+1} \\ &= \ln^{c+1} n + (c + 1) \ln^c n \ln z_d + \mathcal{O}(\log^{c-1} n) + \mathcal{O}(n^{-\sigma} \log^c n). \end{aligned}$$

Taking into account that $\mathcal{O}(\log^{c-1} n) + \mathcal{O}(n^{-\sigma} \log^c n) = \mathcal{O}(\log^{c-1} n)$, and that

$$\sum_{1 \leq d \leq D} |r_{d,n}| \cdot \frac{\ln^{c+1} S_{d,n}}{\ln^c n} \leq \sum_{1 \leq d \leq D} |r_{d,n}| \ln n = \mathcal{O}(n^{-\rho} \log n),$$

we can bound $\mathcal{H}_n^{(\ln)}$ as follows:

$$\mathcal{H}_n^{(\ln)} \leq \ln n - \sum_{1 \leq d \leq D} (w_d + r_{d,n}) \frac{\ln^{c+1} S_{d,n}}{\ln^c n}$$

$$\begin{aligned} &= \ln n - \sum_{1 \leq d \leq D} w_d (\ln n + (c + 1) \ln z_d + \mathcal{O}(\log^{-1} n)) + \mathcal{O}(n^{-\rho} \log n) \\ &= -(c + 1) \sum_{1 \leq d \leq D} w_d \cdot \ln z_d + \mathcal{O}(\log^{-1} n), \end{aligned}$$

where, for the last step, we have used the fact that $W = 1$. A similar reasoning yields the bound

$$\mathcal{H}_n^{(\ln)} \geq -(c + 1 - \varepsilon) \sum_{1 \leq d \leq D} w_d \cdot \ln z_d + \mathcal{O}(\log^{-1} n).$$

From these last two bounds, and since the one above holds for any $\varepsilon > 0$, the theorem is proved. \square

Finally, Theorem 6.11 corresponds to the case 2.2 of the Discrete MT when $\alpha = a = 0$.

THEOREM 6.11. *Let a function be defined by a canonical discrete recursive definition, and let $t_n = \ln^{-1} n$. Then $\mathcal{H}^{(\ln \cdot \ln \ln)} = -\sum_{1 \leq d \leq D} w_d \cdot \ln z_d$.*

PROOF. By the definitions of entropy and discrete recursive definition,

$$\frac{\mathcal{H}_n^{(\ln \ln \ln)}}{\ln n} = \ln \ln n - \sum_{1 \leq d \leq D} (w_d + r_{d,n}) \ln \ln \left(\left(z_d + \frac{s_{d,n}}{n} \right) n \right),$$

as long as n is large enough. Let $y = \ln n$. Recall that $|r_{d,n}| = \mathcal{O}(n^{-\rho})$ and $|s_{d,n}|/n = \mathcal{O}(n^{-\sigma})$. Then,

$$\frac{\mathcal{H}_y^{(\ln \ln \ln)}}{y} = \ln y - \sum_{1 \leq d \leq D} (w_d + \mathcal{O}(\exp(-\rho y))) \ln(y + \ln(z_d + \mathcal{O}(\exp(-\sigma y)))).$$

The contributions of the terms $\mathcal{O}(\exp(-\rho y))$ and $\mathcal{O}(\exp(-\sigma y))$ vanish for large y . Thus,

$$\mathcal{H}^{(\ln \ln \ln)} = \lim_{y \rightarrow +\infty} \mathcal{H}_y^{(\ln \ln \ln)} = \lim_{y \rightarrow +\infty} \frac{\sum_{1 \leq d \leq D} w_d (\ln y - \ln(y + \ln z_d))}{1/y}.$$

Now it suffices to use l'Hôpital's rule to get the statement of the theorem. \square

To derive both the Discrete and the Continuous MTs, we will use the fundamental property that every discrete or continuous recursive definition is either canonical or can be reduced to a canonical one by means of a definition like $C_n = F_n/n^\alpha$, for some appropriate constant α . This concept is formalized as follows:

Definition 6.12. Let \mathcal{F} be a DAC recursive definition of a function F_n . We say that \mathcal{F} is a *proper DAC recursive definition* if and only if there exists some

constant α such that, if we define $C_n = F_n/n^\alpha$, then

$$C_n = \frac{t_n}{n^\alpha} + \sum_{0 \leq k < n} w_{n,k} \cdot \frac{k^\alpha}{n^\alpha} \cdot \frac{F_k}{k^\alpha} = \frac{t_n}{n^\alpha} + \sum_{0 \leq k < n} w_{n,k} \left(\frac{k}{n}\right)^\alpha C_k$$

is a canonical recursive definition.

For instance, any canonical recursive definition is a proper DAC recursive definition with $\alpha = 0$, that is, with $C_n = F_n$. Typically, $W \geq 1$, and then α is positive. For W smaller than 1 (which is not usual in practice) α is negative, which is problematic for small indices. This is why we required W to be at least 1 for continuous recursive definitions; note that the term $(k/n)^\alpha$ is unbounded for small k and negative α . By contrast, for discrete recursive definitions we allowed W to be smaller than 1; this is a reason to avoid the case $z_d = 0$.

As an example of nonproper DAC recursive definition, we have (7); it is easy to see that it is not canonical nor can be converted into canonical through a change like $C_n = F_n/n^\alpha$. Theorems 6.13 and 7.14 state that this is never the case with discrete and continuous recursive definitions.

THEOREM 6.13. *All discrete recursive definitions are proper DAC recursive definitions whose α is the unique solution of $\Phi(\alpha) = 1$.*

PROOF. From Lemma 6.1, $\Phi(\alpha) = 1$ has always a unique solution. Define $C_n = F_n/n^\alpha$. Then, for large n ,

$$C_n = \frac{t_n}{n^\alpha} + \sum_{1 \leq d \leq D} R'_{d,n} C_{S_{d,n}},$$

where $R'_{d,n} = (w_d + r_{d,n})(z_d + s_{d,n}/n)^\alpha$. Set $w'_d = w_d \cdot z_d^\alpha$ and $r'_{d,n} = R'_{d,n} - w'_d$. Then we have $\sum_{1 \leq d \leq D} |r'_{d,n}| = \mathcal{O}(n^{-\rho} + n^{-\sigma})$.

Let W_n, W , etc. be defined over the new weights. On the one hand, $W_n = \sum_{1 \leq d \leq D} R'_{d,n} = \sum_{1 \leq d \leq D} (w'_d + r'_{d,n})$, and $W = \sum_{1 \leq d \leq D} w'_d = \Phi(\alpha) = 1$. Furthermore, $|m_n| = |W_n - W| \leq \sum_{1 \leq d \leq D} |r'_{d,n}| = \mathcal{O}(n^{-\rho} + n^{-\sigma})$. On the other hand,

$$Z_n = \sum_{1 \leq d \leq D} \frac{R'_{d,n}}{W_n} \cdot \frac{S_{d,n}}{n} = \frac{1}{W_n} \sum_{1 \leq d \leq D} (w'_d + r'_{d,n}) \left(z_d + \frac{S_{d,n}}{n} \right),$$

and hence $Z = \sum_{1 \leq d \leq D} w'_d \cdot z_d = \Phi(\alpha + 1) < 1$. The theorem follows. \square

We now make use of Theorems 5.6 and 6.13 to summarize Theorems 6.3, 6.6, 6.8, 6.10, and 6.11 into Corollary 6.14. This corollary also includes the corresponding results for continuous recurrences (Theorems 7.9, 7.10, 7.11, 7.12, and 7.13), which will be presented and proved in Section 7.

COROLLARY 6.14. *Let F_n be defined by a discrete (respectively, continuous) recursive definition, and let α be the unique solution to $\Phi(\alpha) = 1$ (respectively, $\varphi(\alpha) = 1$).*

—If $t_n = \mathcal{O}(n^\alpha \log^c n)$, where $c < -1$, then $F_n = \mathcal{O}(n^\alpha)$. Moreover, $F_n = \Theta(n^\alpha)$ if $F_n \geq 0$ for every $n \geq 0$ and $t_n > 0$ for large n .

- If $t_n = n^a \xi_n$, where $a > \alpha$ and $\xi_n = \lambda_n$ or $\xi_n = 1/\lambda_n$ for some subpolynomial function λ_n , then $F_n \sim t_n/(1 - \Phi(a))$ (respectively, $F_n \sim t_n/(1 - \varphi(a))$).
- If t_n is a superpolynomial function, then $F_n \sim t_n$.
- If $t_n = n^\alpha \ln^c n \cdot \xi_n$, where $c > -1$ and $\xi_n = \mu_n$ or $\xi_n = 1/\mu_n$ for some sublogarithmical function μ_n , then $F_n \sim t_n \ln n/\mathfrak{H}$, where $\mathfrak{H} = -(c + 1)\sum_{1 \leq d \leq D} w_d \cdot (z_d)^\alpha \ln z_d$ (respectively, $\mathfrak{H} = -(c + 1) \int_0^1 \omega(z)z^\alpha \ln z dz$).
- If $t_n = n^\alpha \ln^{-1} n$, then $F_n \sim t_n \ln n \cdot \ln \ln n/\mathfrak{H}$, where $\mathfrak{H} = -\sum_{1 \leq d \leq D} w_d \cdot (z_d)^\alpha \ln z_d$ (respectively, $\mathfrak{H} = -\int_0^1 \omega(z)z^\alpha \ln z dz$).

7. Proof of the Continuous Master Theorem

We begin with the conjecture included in Definition 3.1.

CONJECTURE 7.1. For every positive function $\omega(z)$ such that $\int_0^1 \omega(z) dz$ converges, there exists some $\mu < 0$ such that $\int_0^1 \omega(z)z^\mu dz$ also converges.

The author of this paper could not prove that, irrespective of $\omega(z)$, the integral $\varphi(x)$ converges for every $x \in [\mu, 0]$ for some $\mu < 0$, though no counterexample of this statement was found. However, this property clearly holds for the shape functions we usually deal with. For instance, if $\omega(z)$ is bounded near 0, then any $\mu > -1$ fulfills the condition above. To the best of the author’s knowledge, it is an open problem to determine if the statement of Conjecture 7.1 is always true.

The following lemma provides an easy way to find the shape function of many continuous recurrences.

LEMMA 7.2. Let $\omega_{n,k} = A \cdot f_1 \cdots f_m/(g_1 \cdots g_{m+1})$ be the weights of a given recurrence, where $A > 0$ is an arbitrary constant, $m \geq 0$ is any integer constant, $f_i = (a_i n + b_i k + c_i)$ for all $1 \leq i \leq m$ for some constants a_i, b_i , and c_i such that at least one of a_i or b_i is not 0, and $g_i = (n + d_i)$ for all $1 \leq i \leq m + 1$, where the d_i ’s are arbitrary constants. Set $\sigma_{n,k} = A \cdot (a_1 n + b_1 k) \cdots (a_m n + b_m k)/n^{m+1}$ and $\omega(z) = n \cdot \sigma_{n,zn}$, and suppose that $\int_0^1 \omega(z) dz$ is at least 1. Then the given recurrence is continuous and $\omega(z)$ is its shape function.

PROOF. First, we prove that there exists some $\mu < 0$ such that $\int_0^1 \omega(z)z^\mu dz$ converges. Since $\omega'(z)$ exists and is bounded in $[0, 1]$, $\omega(z)$ is also bounded in $[0, 1]$. Set $h = \max\{\omega(z)\}_{0 \leq z \leq 1}$, and choose any $-1 < \mu < 0$. Then $\int_0^1 \omega(z)z^\mu dz \leq h \int_0^1 z^\mu dz < +\infty$.

To prove the conditions in Definition 3.2, let $B \geq 0$ be such that $|\omega'(z)| \leq B$ for every $0 \leq z \leq 1$. It is easy to see that this bound produces

$$\int_{k/n}^{(k+1)/n} \omega(z) dz \leq \int_{k/n}^{(k+1)/n} \left(\omega\left(\frac{k}{n}\right) + \left(z - \frac{k}{n}\right)B \right) dz = \frac{\omega(k/n)}{n} + \frac{B}{2n^2},$$

and its symmetric $\int_{k/n}^{(k+1)/n} \omega(z) dz \geq \omega(k/n)/n - B/2n^2$. Now, we set $M_n = n$ to get $\varepsilon_{n,k} \leq B/2n^2$ and $\sum_{0 \leq k < n} \varepsilon_{n,k} = \mathcal{O}(n^{-1})$. Finally, it is enough to notice

that $|\omega_{n,k} - \sigma_{n,k}| = \mathcal{O}(n^{-2})$ to obtain

$$\sum_{0 \leq k < n} \varepsilon_{n,k} \leq \sum_{0 \leq k < n} \left(\left| \omega_{n,k} - \sigma_{n,k} \right| + \left| \sigma_{n,k} - \int_{k/n}^{(k+1)/n} \omega(z) dz \right| \right) = \mathcal{O}(n^{-1}). \quad \square$$

Hereafter, we will present results for continuous recurrences that are equivalent to those for discrete recurrences in Section 6, together with some technical propositions.

LEMMA 7.3. $\varphi(x)$ is a strictly decreasing continuous function in the interval $[\mu, +\infty)$, such that $\lim_{x \rightarrow +\infty} \varphi(x) = 0$.

PROOF. By hypothesis, $\varphi(0) \geq 1$. This implies that there are z_1 and z_2 such that $0 < z_1 < z_2 < 1$ and $\int_{z_1}^{z_2} \omega(z) dz > 0$. For every $x < y$ such that $\varphi(x)$ and $\varphi(y)$ exist,

$$\varphi(x) - \varphi(y) \geq \int_{z_1}^{z_2} \omega(z)(z^x - z^y) dz \geq \int_{z_1}^{z_2} \omega(z) dz \cdot \min\{z^x - z^y\}_{z_1 \leq z \leq z_2} > 0.$$

Hence, $\varphi(x)$ is a strictly decreasing function.

Proving that $\varphi(x)$ tends to 0 as long as x grows is not more difficult. For every $\varepsilon > 0$, let $0 < z_\varepsilon < 1$ be large enough to get $\int_{z_\varepsilon}^1 \omega(z) dz \leq \varepsilon/2$. Assume $x \geq 0$. Then,

$$\begin{aligned} \varphi(x) &= \int_0^{z_\varepsilon} \omega(z) z^x dz + \int_{z_\varepsilon}^1 \omega(z) z^x dz \\ &\leq (z_\varepsilon)^x \int_0^{z_\varepsilon} \omega(z) dz + \int_{z_\varepsilon}^1 \omega(z) dz \leq (z_\varepsilon)^x \varphi(0) + \frac{\varepsilon}{2} \leq \varepsilon \end{aligned}$$

for x large enough.

It is left to prove that $\varphi(x)$ is a continuous function. Recall that $\mu < 0$ is the constant such that, according to Conjecture 7.1, $\int_0^1 \omega(z) z^\mu dz$ converges. Choose any a such that $a > \mu$. Then,

$$\begin{aligned} \lim_{x \rightarrow a} |\varphi(x) - \varphi(a)| &= \lim_{x \rightarrow a} \left| \int_0^1 \omega(z) z^\mu (z^{x-\mu} - z^{a-\mu}) dz \right| \\ &\leq \int_0^1 \omega(z) z^\mu dz \cdot \lim_{x \rightarrow a} (\max\{|z^{x-\mu} - z^{a-\mu}|\}_{0 \leq z \leq 1}) = \varphi(\mu) \cdot 0 = 0. \quad \square \end{aligned}$$

The following two technical propositions will be used in several proofs in this section.

PROPOSITION 7.4. Let a function be defined by a continuous recursive definition. For every $x > 0$, define

$$\varepsilon_{n,j}(x) = \left| \sum_{k \in I_{n,j}} \omega_{n,k} \left(\frac{k}{n} \right)^x - \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) z^x dz \right|$$

for all $0 \leq j < M_n$. Then for every $x > 0$, we have $\sum_{0 \leq j < M_n} \varepsilon_{n,j}(x) = \mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$.

PROOF. First of all, notice that $\varepsilon_{n,j}(0)$ equals $\varepsilon_{n,j}$. Take any $0 \leq j < M_n$, and assume that the first term in the definition of $\varepsilon_{n,j}(x)$ is larger than the second term. Then

$$\begin{aligned} \varepsilon_{n,j}(x) &\leq \sum_{k \in I_{n,j}} \omega_{n,k} \left(\frac{z_{n,j+1} \cdot n}{n} \right)^x - \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) (z_{n,j})^x dz \\ &= \left(\frac{j+1}{M_n} \right)^x \sum_{k \in I_{n,j}} \omega_{n,k} - \left(\frac{j}{M_n} \right)^x \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz \\ &= \frac{(j+1)^x - j^x}{(M_n)^x} \sum_{k \in I_{n,j}} \omega_{n,k} + \left(\frac{j}{M_n} \right)^x \left(\sum_{k \in I_{n,j}} \omega_{n,k} - \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz \right). \end{aligned}$$

Now, if we set

$$\psi_n = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ (M_n)^x - (M_n - 1)^x, & \text{if } x > 1 \end{cases},$$

then it is easy to see that $(j+1)^x - j^x \leq \psi_n$. Moreover, $j < M_n$. Hence, $\varepsilon_{n,j}(x) \leq \psi_n / (M_n)^x \sum_{k \in I_{n,j}} \omega_{n,k} + \varepsilon_{n,j}$. If the second term in the definition of $\varepsilon_{n,j}(x)$ is larger than the first one, a similar reasoning yields $\varepsilon_{n,j}(x) \leq \psi_n / (M_n)^x \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz + \varepsilon_{n,j}$. Altogether,

$$\begin{aligned} \sum_{0 \leq j < M_n} \varepsilon_{n,j}(x) &\leq \sum_{0 \leq j < M_n} \left(\frac{\psi_n}{(M_n)^x} \left(\sum_{k \in I_{n,j}} \omega_{n,k} + \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz \right) + \varepsilon_{n,j} \right) \\ &= \frac{\psi_n}{(M_n)^x} \left(\sum_{0 \leq k < n} \omega_{n,k} + \int_0^1 \omega(z) dz \right) + \sum_{0 \leq j < M_n} \varepsilon_{n,j} = \frac{\psi_n}{(M_n)^x} \Theta(1) + \mathcal{O}(n^{-\rho}). \end{aligned}$$

Finally, taking into account that $\psi_n / (M_n)^x = \mathcal{O}(n^{-qx})$ when $0 < x \leq 1$ and $\psi_n / (M_n)^x = \mathcal{O}(n^{-q})$ when $x > 1$, we can bound the expression above by $\mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$ small enough. \square

PROPOSITION 7.5. *Let a function be defined by a continuous recursive definition. Then, for every $x > 0$, $\sum_{0 \leq k < n} \omega_{n,k} (k/n)^x = \varphi(x) + \mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$.*

PROOF. It is enough to observe that

$$\left| \sum_{0 \leq k < n} \omega_{n,k} \left(\frac{k}{n}\right)^x - \varphi(x) \right| = \left| \sum_{0 \leq j < M_n} \left(\sum_{k \in I_{n,j}} \omega_{n,k} \left(\frac{k}{n}\right)^x - \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) z^x dz \right) \right|,$$

which is not larger than $\sum_{0 \leq j < M_n} \varepsilon_{n,j}(x)$. But, according to Proposition 7.4, this sum is $\mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$. \square

LEMMA 7.6. *A continuous recursive definition is canonical if and only if $\varphi(0) = 1$.*

PROOF. For the “only if” proof, a reasoning similar to the one in Proposition 7.5 yields $|W_n - \varphi(0)| = \mathcal{O}(n^{-\rho})$. Hence, $\varphi(0)$ has to equal 1 for the recurrence to be canonical.

For the “if” proof, when $\varphi(0) = W = 1$, we have $|m_n| = \mathcal{O}(n^{-\rho})$. Moreover, $W_n Z_n = \sum_{0 \leq k < n} \omega_{n,k} k/n = \varphi(1) + \mathcal{O}(n^{-\sigma})$ for some $\sigma > 0$ small enough. Now, taking into account that $W_n = 1 + \mathcal{O}(n^{-\rho})$, we obtain $Z_n = \varphi(1) + \mathcal{O}(n^{-\sigma} + n^{-\rho})$, and conclude that $Z = \varphi(1) < 1$. \square

We require just two more technical propositions before proving the main results for continuous recurrences.

PROPOSITION 7.7. *Let $\omega_{n,k}$ be the weights of a continuous recursive definition, f_n be any function such that $f_n = \Omega(1)$ and $f_n = o(n)$, $x \geq 0$, and $\rho > 0$ and $0 < q \leq 1$ be the constants in the definition of continuous recursive definition. Define $h_n = \max\{f_n, n^{1-q}\}$. Then $\sum_{0 \leq k < f_n} \omega_{n,k} (k/n)^x = \mathcal{O}(\max\{n^{-\rho}, (h_n/n)^{-\mu}\})$, where $\mu < 0$ is such that $\varphi(\mu)$ converges.*

PROOF. Assume $x = 0$. Let $g_n = \Theta(h_n \cdot n^{q-1})$ be a function with integer values such that $g_n \cdot n/M_n \geq f_n$ for large n (notice that this implies $g_n \geq 1$). Then $\sum_{0 \leq k < f_n} \omega_{n,k} \leq \sum_{0 \leq k < g_n \cdot n/M_n} \omega_{n,k} = \sum_{0 \leq j < g_n} \sum_{k \in I_{n,j}} \omega_{n,k}$ for large n . On the other hand,

$$\left| \sum_{0 \leq j < g_n} \left(\sum_{k \in I_{n,j}} \omega_{n,k} - \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz \right) \right| \leq \sum_{0 \leq j < g_n} \varepsilon_{n,j} = \mathcal{O}(n^{-\rho}).$$

Furthermore, $\sum_{0 \leq j < g_n} \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz = \int_0^{g_n/M_n} \omega(z) z^\mu z^{-\mu} dz$, which can be bounded by

$$\left(\frac{g_n}{M_n}\right)^{-\mu} \int_0^{g_n/M_n} \omega(z) z^\mu dz \leq \mathcal{O}((h_n/n)^{-\mu}) \varphi(\mu) = \mathcal{O}((h_n/n)^{-\mu}).$$

Altogether, the proposition is proved for the case $x = 0$.

The proof for $x > 0$ is now trivial, since $\sum_{0 \leq k < f_n} \omega_{n,k} (k/n)^x \leq \sum_{0 \leq k < f_n} \omega_{n,k}$. \square

PROPOSITION 7.8. *Let $\omega_{n,k}$ be the weights of a continuous recursive definition, $x \geq 0$, and $N \geq 1$ be any constant. Then $\sum_{0 \leq k < N} \omega_{n,k} (k/n)^x = \mathcal{O}(n^{-u})$ for some $u > 0$.*

PROOF. It is enough to use Proposition 7.7 with $f_n = N$. Then $h_n = \Theta(n^{1-q})$ and the sum above is bounded by $\mathcal{O}(\max\{n^{-\rho}, n^{\mu q}\})$. \square

THEOREM 7.9. *Let C_n be defined by a canonical continuous recursive definition, and let $t_n = \mathcal{O}(\log^c n)$, where $c < -1$. Then $C_n = \mathcal{O}(1)$. Moreover, $C_n = \Theta(1)$ if $C_n \geq 0$ for every $n \geq 0$ and $t_n > 0$ for n large enough.*

PROOF. By Proposition 7.8, we have $\sum_{0 \leq k < N} w_{n,k} = \mathcal{O}(n^{-u})$ for some $u > 0$. Hence, we only need to use Theorems 4.3 and 4.4 to complete this proof. \square

THEOREM 7.10. *Let a function be defined by a canonical continuous recursive definition, and let $t_n = n^a \xi_n$, where $a > 0$ and $\xi_n = \lambda_n$ or $\xi_n = 1/\lambda_n$ for some subpolynomial function λ_n . Then $\mathcal{H}^{(1)} = 1 - \varphi(a)$.*

PROOF. Let us assume $\xi_n = 1/\lambda_n$. Then

$$\mathcal{H}_n^{(1)} = 1 - \sum_{N_\gamma \leq k < n} \omega_{n,k} \cdot \frac{t_k}{t_n} = 1 - \sum_{N_\gamma \leq k < n} \omega_{n,k} \left(\frac{k}{n}\right)^a \frac{\lambda_n}{\lambda_k}.$$

By hypothesis, for every $\varepsilon > 0$ there exists some N_ε such that $k^\varepsilon/\lambda_k \leq (k + 1)^\varepsilon/\lambda_{k+1}$ and $\lambda_k \leq \lambda_{k+1}$ for all $k \geq N_\varepsilon$. Choose any $0 < \varepsilon < a$, and let k and n be such that $N_\varepsilon \leq k < n$. Then we have $k^\varepsilon/\lambda_k \leq n^\varepsilon/\lambda_n$ and $\lambda_k \leq \lambda_n$. Therefore,

$$\sum_{N_\varepsilon \leq k < n} \omega_{n,k} \left(\frac{k}{n}\right)^a \leq \sum_{N_\varepsilon \leq k < n} \omega_{n,k} \left(\frac{k}{n}\right)^a \frac{\lambda_n}{\lambda_k} \leq \sum_{N_\varepsilon \leq k < n} \omega_{n,k} \left(\frac{k}{n}\right)^{a-\varepsilon}.$$

Now we can set $N_\gamma = N_\varepsilon$ to get

$$1 - \sum_{N_\gamma \leq k < n} \omega_{n,k} \left(\frac{k}{n}\right)^a \geq \mathcal{H}_n^{(1)} \geq 1 - \sum_{N_\gamma \leq k < n} \omega_{n,k} \left(\frac{k}{n}\right)^{a-\varepsilon}.$$

From Propositions 7.5 and 7.8, we deduce $1 - \varphi(a) \geq \mathcal{H}^{(1)} \geq 1 - \varphi(a - \varepsilon)$, which is true no matter how small ε is. Since $\varphi(x)$ is a continuous function, we conclude $\mathcal{H}^{(1)} = 1 - \varphi(a)$.

A similar argument proves the case $\xi_n = \lambda_n$. \square

THEOREM 7.11. *Let a function be defined by a canonical continuous recursive definition, and let t_n be a superpolynomial function. Then $\mathcal{H}^{(1)} = 1$.*

PROOF. The proof is very similar to that of Theorem 7.10. \square

THEOREM 7.12. *Let a function be defined by a canonical continuous recursive definition, and let $t_n = \ln^c n \cdot \xi_n$, where $c > -1$ and $\xi_n = \mu_n$ or $\xi_n = 1/\mu_n$ for some sublogarithmical function μ_n . Then $\mathcal{H}^{(\ln)} = -(c + 1) \int_0^1 \omega(z) \ln z \, dz$.*

PROOF. First of all, we must make sure that $\mathcal{H}^{(\ln)}$ exists and is strictly positive. It is easy to prove that $-\int_0^1 \omega(z) \ln z \, dz$, if exists, is strictly greater than 0. Proving that it does not diverge is not more difficult. Let $\mu < 0$ be such that $\varphi(\mu)$ converges, and let $0 < u < 1$ be such that $-\ln z \leq z^\mu$ in the interval $(0, u]$. Then,

$$-\int_0^1 \omega(z) \ln z \, dz \leq \int_0^u \omega(z) z^\mu \, dz - \int_u^1 \omega(z) \ln z \, dz \leq \varphi(\mu) - \varphi(0) \ln u < +\infty.$$

Now, assume $\xi_n = 1$, that is, $t_n = \ln^c n$. Then we have

$$\mathcal{H}_n^{(\ln)} = (1 - W_n) \ln n + W_n \ln n - \sum_{N_\gamma \leq k < n} \omega_{n,k} \cdot \frac{\ln^{c+1} k}{\ln^c n}.$$

Let $g_n = \Theta(n^d)$ be a function with integer values, where $d = q/2$. So, we have $0 < d < q$ and $0 < d + 1 - q < 1$. For large n ,

$$\begin{aligned} \mathcal{H}_n^{(\ln)} &= -m_n \ln n + \sum_{g_n n / M_n \leq k < n} \omega_{n,k} \cdot \frac{\ln^{c+1} n - \ln^{c+1} k}{\ln^c n} \\ &\quad + \sum_{0 \leq k < N_\gamma} \omega_{n,k} \ln n + \sum_{N_\gamma \leq k < g_n n / M_n} \omega_{n,k} \left(\ln n - \frac{\ln^{c+1} k}{\ln^c n} \right). \end{aligned}$$

The term $-m_n \ln n$ above vanishes because $m_n = \mathcal{O}(n^{-\rho})$. Moreover, the second line above is positive, and can be bounded by $(\sum_{0 \leq k < g_n n / M_n} \omega_{n,k}) \ln n$. Setting $f_n = \Theta(n^{1-q/2})$ in Proposition 7.7 proves that this sum also vanishes for large n . The rest of the proof is devoted to find the limit of the term that is left.

For every $g_n \leq j < M_n$, define

$$\Delta_{n,j} = \left| \sum_{k \in I_{n,j}} \omega_{n,k} \cdot \frac{\ln^{c+1} n - \ln^{c+1} k}{\ln^c n} - \left(-(c+1) \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \ln z \, dz \right) \right|.$$

Assume that the first term above is larger than the second one. Then

$$\begin{aligned} \frac{\Delta_{n,j}}{c+1} &\leq \sum_{k \in I_{n,j}} \omega_{n,k} \cdot \frac{\ln^{c+1} n - \ln^{c+1}(z_{n,j} \cdot n)}{(c+1) \ln^c n} + \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \ln z_{n,j+1} \, dz \\ &= \left(-\ln z_{n,j} + \mathcal{O}\left(\frac{1}{\log n}\right) \right) \sum_{k \in I_{n,j}} \omega_{n,k} + \ln z_{n,j+1} \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \, dz. \end{aligned}$$

But

$$\begin{aligned} &-\ln z_{n,j} \sum_{k \in I_{n,j}} \omega_{n,k} + \ln z_{n,j+1} \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \, dz \\ &= \ln z_{n,j} \cdot \left(\int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \, dz - \sum_{k \in I_{n,j}} \omega_{n,k} \right) + \ln \left(\frac{z_{n,j+1}}{z_{n,j}} \right) \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \, dz \\ &\leq -\ln \left(\frac{g_n}{M_n} \right) \cdot \varepsilon_{n,j} + \ln \left(\frac{g_n + 1}{g_n} \right) \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \, dz. \end{aligned}$$

Taking into account the facts that $-\ln(g_n/M_n) = \Theta(\log n^{q-d}) = \mathcal{O}(\log n)$ and $\ln(1 + 1/g_n) = \Theta(1/g_n) = \mathcal{O}(n^{-d})$, we have

$$\frac{\Delta_{n,j}}{c + 1} = \mathcal{O}(\log n)\varepsilon_{n,j} + \mathcal{O}(n^{-d}) \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz + \mathcal{O}\left(\frac{1}{\log n}\right) \sum_{k \in I_{n,j}} \omega_{n,k}.$$

If the second term in the definition of $\Delta_{n,j}$ above is larger than the first one, by means of a similar reasoning, we get

$$\frac{\Delta_{n,j}}{c + 1} = \mathcal{O}(\log n)\varepsilon_{n,j} + \mathcal{O}(n^{-d}) \sum_{k \in I_{n,j}} \omega_{n,k} + \mathcal{O}\left(\frac{1}{\log n}\right) \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) dz.$$

Thus, we can conclude

$$\sum_{g_n \leq j < M_n} \Delta_{n,j} = \mathcal{O}(\log n) \sum_{g_n \leq j < M_n} \varepsilon_{n,j} + \mathcal{O}\left(\frac{1}{\log n}\right) W_n + \mathcal{O}\left(\frac{1}{\log n}\right) \varphi(0),$$

which vanishes for large n , because the sum of $\varepsilon_{n,j}$'s is $\mathcal{O}(n^{-\rho})$ for some $\rho > 0$. Now we are ready to finish the proof, since

$$\left| \sum_{g_n n/M_n \leq k < n} \omega_{n,k} \cdot \frac{\ln^{c+1} n - \ln^{c+1} k}{\ln^c n} - \left(-(c + 1) \int_{g_n/M_n}^1 \omega(z) \ln z dz \right) \right|$$

is bounded by above by $\sum_{g_n \leq j < M_n} \Delta_{n,j} = o(1)$. Therefore,

$$\mathcal{H}^{(\ln)} = \lim_{n \rightarrow +\infty} -(c + 1) \int_{g_n/M_n}^1 \omega(z) \ln z dz = -(c + 1) \int_0^1 \omega(z) \ln z dz.$$

This ends the proof of the case $\xi_n = 1$.

The cases $\xi_n = \mu_n$ and $\xi_n = 1/\mu_n$ for sublogarithmical $\mu_n = \omega(1)$ can be proved by similar arguments (see the proof of Theorem 6.10). \square

THEOREM 7.13. *Let a function be defined by a canonical continuous recursive definition, and let $t_n = \ln^{-1} n$. Then $\mathcal{H}^{(\ln \cdot \ln \ln)} = -\int_0^1 \omega(z) \ln z dz$.*

PROOF. This proof follows the same pattern as the one of Theorem 7.12. Therefore, we skip some steps, for the sake of brevity.

Let $\nu_n = \exp(\sqrt{\ln n})$, and $g_n = \Theta(n^q/\nu_n)$ be a function with integer values. By the definitions of entropy and continuous recursive definition,

$$\begin{aligned} \mathcal{H}_n^{(\ln \cdot \ln \ln)} &= -m_n \ln n \cdot \ln \ln n + \ln n \cdot \sum_{g_n n/M_n \leq k < n} \omega_{n,k} (\ln \ln n - \ln \ln k) \\ &\quad + \ln n \cdot \ln \ln n \sum_{0 \leq k < N_\gamma} \omega_{n,k} + \ln n \cdot \sum_{N_\gamma \leq k < g_n n/M_n} \omega_{n,k} (\ln \ln n - \ln \ln k), \end{aligned}$$

as long as n is large enough. The term $-m_n \ln n \cdot \ln \ln n$ clearly vanishes for large n . Moreover, the second line above is positive, and bounded by $(\sum_{0 \leq k < g_n n/M_n} \omega_{n,k}) \ln n \ln \ln n$. By Proposition 7.7, this term is also $o(1)$, since

$f_n = h_n = n/v_n$ and $(h_n/n)^{-\mu} = v_n^\mu$. Below, we prove that the limit of the remaining term above is as stated by the theorem.

For every $g_n \leq j < M_n$, define

$$\Delta_{n,j} = \left| \ln n \cdot \sum_{k \in I_{n,j}} \omega_{n,k} (\ln \ln n - \ln \ln k) - \left(- \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \ln z \, dz \right) \right|.$$

Assume that the first term above is larger than the second one (the opposite produces a similar bound). Then

$$\Delta_{n,j} \leq \ln n \cdot \sum_{k \in I_{n,j}} \omega_{n,k} (\ln \ln n - \ln \ln(z_{n,j} \cdot n)) + \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \ln z_{n,j+1} \, dz.$$

Now it is not difficult to get

$$\begin{aligned} \Delta_{n,j} &\leq -\ln n \cdot \ln \left(1 - \frac{\ln(1/z_{n,j})}{\ln n} \right) \sum_{k \in I_{n,j}} \omega_{n,k} + \ln z_{n,j} \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \, dz \\ &\quad + \ln \left(1 + \frac{1}{g_n} \right) \int_{z_{n,j}}^{z_{n,j+1}} \omega(z) \, dz. \end{aligned}$$

The contribution to the entropy of the second line is asymptotically irrelevant, because $\ln(1 + 1/g_n) = \Theta(1/g_n) = o(1)$. For the next step, we show that the factor $\ln n \cdot \ln(1 - \ln(1/z_{n,j})/\ln n)$ equals $\ln z_{n,j} + o(1)$, which is easy if we take into account the fact that $1/z_{n,j} = \mathcal{O}(M_n/g_n) = \mathcal{O}(v_n)$. Hence, $\ln(1/z_{n,j}) = \mathcal{O}(\ln v_n) = \mathcal{O}(\sqrt[3]{\ln n})$, which implies $\ln(1/z_{n,j})/\ln n = \mathcal{O}(\ln^{-2/3} n) = o(1)$. Finally, we observe that $\ln(1 - x) = -x + \mathcal{O}(x^2)$ when $x = o(1)$ to conclude

$$\ln n \cdot \ln \left(1 - \frac{\ln(1/z_{n,j})}{\ln n} \right) = \ln n \cdot \left(\frac{\ln z_{n,j}}{\ln n} + \mathcal{O}(\ln^{-4/3} n) \right) = \ln z_{n,j} + o(1).$$

Now it is enough to follow the last steps in the proof of Theorem 7.12 to finish this proof. \square

THEOREM 7.14. *All continuous recursive definitions are proper DAC recursive definitions whose α is the unique solution of $\varphi(\alpha) = 1$.*

PROOF. From Lemma 7.3 and $\varphi(0) \geq 1$, the equation above has a unique solution. Define $C_n = F_n/n^\alpha$. Then, for every $n \geq N$,

$$C_n = \frac{t_n}{n^\alpha} + \frac{1}{n^\alpha} \sum_{0 \leq k < N} \omega_{n,k} F_k + \sum_{N \leq k < n} \omega_{n,k} \cdot \frac{k^\alpha}{n^\alpha} \cdot \frac{F_k}{k^\alpha} = t'_n + \sum_{0 \leq k < n} \omega'_{n,k} C_k,$$

where $t'_n = t_n/n^\alpha + (1/n^\alpha) \sum_{0 \leq k < N} \omega_{n,k} F_k$, and

$$\omega'_{n,k} = \begin{cases} 0, & \text{if } 0 \leq k < N \\ \omega_{n,k}(k/n)^\alpha, & \text{if } N \leq k < n. \end{cases}$$

From Propositions 7.4 and 7.8, it is a simple matter to prove that $\omega(z)z^\alpha$ is the shape function of \mathcal{C} (just set $x = \alpha$). Moreover, since $\varphi(\alpha) = 1$, we have $\int_0^1 \omega(z)z^\alpha dz = 1$. This proves that the recurrence for C_n is a canonical continuous recursive definition.

As a final remark, notice that the term $(1/n^\alpha) \sum_{0 \leq k < n} \omega_{n,k} F_k$ in t'_n is $\mathcal{O}(n^{-\alpha-u})$ for some $u > 0$ (use Proposition 7.8), and hence we are allowed to forget about the contribution of this term to the main term of C_n . \square

8. Computing the Variance

We have devoted previous sections to analyze the (expected) cost of algorithms. However, if the executions are random (in the different possible inputs of the same size; or in the different possible executions with the same input, if the algorithm is randomized), then the variance of the cost is also a valuable measure. In this section, we will analyze the variance of two types of algorithms.

8.1. ONE-BRANCH ALGORITHMS. The first class includes the algorithms that perform one recursive call with probability $1 - S_n$, and otherwise stop the recursion chain with probability S_n , where S_n tends to zero as long as n grows. FIND is a prototypical example of this class. It divides the original array into two parts, and afterwards recursively continues in the subarray to the left or in the subarray to the right of the current pivot, or either stops with probability $1/n$. In this setting, $\omega_{n,k}$ is the probability that the algorithm performs the recursive call over a subproblem with size k when the original problem has size n .

Let \mathcal{T}_n be the random variable that describes the nonrecursive cost, including both the cost to split the problem and the cost to combine the recursive solutions. We assume that these two costs do not depend on the particular input nor the recursive solutions, but eventually depend on which was the “pivot” (whatever it means depending on the algorithm). We also assume that the following symmetry holds:

$$\begin{aligned} T_{n,j}^{(k)} &= \Pr\{\mathcal{T}_n = j \mid \text{the } (k + 1)\text{-th element is the pivot}\} \\ &= \Pr\{\mathcal{T}_n = j \mid \text{the } (n - k)\text{-th element is the pivot}\}. \end{aligned}$$

Under this hypothesis, the size of the recursive call implicitly gives us which was the pivot (or its symmetric).

Let the total cost of the algorithm be described by the random variable \mathcal{F}_n , and let $P_{n,j} = \Pr\{\mathcal{F}_n = j\}$. Of course, we have $\sum_{j \geq 0} P_{n,j} = 1$. The $P_{n,j}$'s satisfy the following recurrence:

$$P_{n,j} \approx \sum_{0 \leq k < n} \omega_{n,k} \sum_{i \geq 0} P_{k,i} T_{n,j-i}^{(k)}.$$

The reason for the “ \approx ” above is that we dismiss the contribution to the cost of the executions that stop the chain of recursive calls, since, by hypothesis, S_n tends to zero, and in most practical situations this cost is asymptotically irrelevant.

Let

$$E(n, k) = E[\mathcal{T}_n \mid \text{the } (k + 1)\text{-th element is the pivot}] = \sum_{j \geq 0} T_{n,j}^{(k)} \cdot j, \quad (14)$$

and

$$Q(n, k) = E[(\mathcal{T}_n)^2 | \text{the } (k + 1)\text{-th element is the pivot}] = \sum_{j \geq 0} T_{n,j}^{(k)} \cdot j^2. \quad (15)$$

Define $F_n = E[\mathcal{F}_n] = \sum_{j \geq 0} P_{n,j} \cdot j$ and $G_n = E[(\mathcal{F}_n)^2] = \sum_{j \geq 0} P_{n,j} \cdot j^2$. On the one hand, by means of simple rewriting we obtain

$$F_n \simeq t_n + \sum_{0 \leq k < n} \omega_{n,k} F_k, \quad (16)$$

as expected. Recall that, despite the approximation above, the main term of the solution of the recurrence is indeed the main term of F_n . On the other hand, from the equality $j^2 = (j - i)^2 + 2(j - i)i + i^2$, it is straightforward to get

$$\begin{aligned} G_n &\simeq \sum_{0 \leq k < n} \omega_{n,k} \sum_{i \geq 0} P_{k,i} \sum_{j \geq 0} T_{n,j-i}^{(k)} \cdot j^2 \\ &= \sum_{0 \leq k < n} \omega_{n,k} (Q(n, k) + 2E(n, k)F_k) + \sum_{0 \leq k < n} \omega_{n,k} G_k. \end{aligned} \quad (17)$$

To solve this recurrence we need to estimate the sums $\sum_k \omega_{n,k} Q(n, k)$ and $\sum_k \omega_{n,k} E(n, k)F_k$.

Assume that the set of $\omega_{n,k}$'s corresponds to a canonical continuous recursive definition, and that \mathcal{T}_n is not random, that is, $\mathcal{T}_n = t_n$. Then $Q(n, k) = (t_n)^2$, and from $E(n, k) = t_n$ and (16) we have $\sum_{0 \leq k < n} \omega_{n,k} 2E(n, k)F_k \simeq 2t_n F_n - 2(t_n)^2$. Thus, (17) reduces to

$$G_n \simeq 2t_n F_n - (t_n)^2 + \sum_{0 \leq k < n} \omega_{n,k} G_k. \quad (18)$$

Notice that this recurrence is a continuous recursive definition with the same shape function as (16).

Suppose first $t_n \sim n^a \xi_n$, where $a > 0$ and ξ_n includes subpolynomial terms. According to the Continuous MT, $F_n \sim t_n / (1 - \varphi(a))$. Therefore, the toll function in (18) is $\sim 2n^{2a} (\xi_n)^2 / (1 - \varphi(a)) - n^{2a} (\xi_n)^2 = n^{2a} (\xi_n)^2 (1 + \varphi(a)) / (1 - \varphi(a))$, and

$$G_n \sim \frac{1 + \varphi(a)}{(1 - \varphi(2a))(1 - \varphi(a))} \cdot (t_n)^2.$$

Hence, the variance of \mathcal{F}_n , $V[\mathcal{F}_n] = E[(\mathcal{F}_n)^2] - E[\mathcal{F}_n]^2$, can be computed as

$$V[\mathcal{F}_n] \sim \frac{\varphi(2a) - \varphi(a)^2}{(1 - \varphi(2a))(1 - \varphi(a))^2} \cdot (t_n)^2.$$

Suppose now $t_n \sim \ln^c n \cdot \xi_n$, where $c > -1$ and ξ_n includes sublogarithmical terms. Then $F_n \sim t_n \ln n / (c + 1)I$, where $I = -\int_0^1 \omega(z) \ln z \, dz$. Replacing this result in (18) yields $G_n \sim (t_n)^2 \ln^2 n / (c + 1)^2 I^2$, and we can only conclude $V[\mathcal{F}_n] = o(F_n)^2$, but no more precise statement can be made in general.

8.2. TWO-BRANCH ALGORITHMS. These algorithms perform two recursive calls to subproblems with size k and $n - 1 - k$, respectively, with probability $1 - S_n$. Otherwise, the algorithm makes no recursive call. Again, S_n tends to zero as long as n grows. Quicksort is a prototypical example of this class. It divides the array into two parts, and afterwards it recursively sorts the left and the right subarrays.

Let $\psi_{n,k}$ be the probability that the original problem of size n is broken into two subproblems of sizes k and $n - 1 - k$, respectively. Like for one-branch algorithms, assume that the algorithm's cost to break and recombine does not depend on the particular input nor the recursive solutions, but does depend on k , the "pivot." Again, let $P_{n,j} = \Pr\{\mathcal{F}_n = j\}$. Then

$$P_{n,j} \simeq \sum_{0 \leq k < n} \psi_{n,k} \sum_{i,y \geq 0} P_{k,i} P_{n-1-k,y} T_{n,k,j-i-y}.$$

Let $E(n, k)$ be defined as in (14), and let $F_n = E[\mathcal{F}_n]$. It is easy to see that F_n follows the recurrence

$$F_n \simeq t_n + \sum_{0 \leq k < n} \psi_{n,k} (F_k + F_{n-1-k}).$$

Let $Q(n, k)$ be defined as in (15), and let $G_n = E[(\mathcal{F}_n)^2]$. Then

$$G_n \simeq \sum_{0 \leq k < n} \psi_{n,k} \sum_{i,y \geq 0} P_{k,i} P_{n-1-k,y} \sum_{j \geq 0} T_{n,k,j-i-y} \cdot j^2.$$

Using $j^2 = (j - i - y)^2 + 2(j - i - y)i + 2(j - i - y)y + 2iy + i^2 + y^2$ yields

$$\begin{aligned} G_n &\simeq \sum_{0 \leq k < n} \psi_{n,k} (Q(n, k) + 2E(n, k)(F_k + F_{n-1-k}) + 2F_k F_{n-1-k}) \\ &\quad + \sum_{0 \leq k < n} \psi_{n,k} (G_k + G_{n-1-k}). \end{aligned}$$

We can solve the recurrences for F_n and G_n under some particular hypotheses. Assume that $E(n, k) = t_n$, $Q(n, k) = (t_n)^2$, and that there exists some function $\psi(z)$ such that

$$\sum_{0 \leq j < M_n} \left| \sum_{k \in I_{n,j}} \psi_{n,k} - \int_{z_{n,j}}^{z_{n,j+1}} \psi(z) dz \right| = \mathcal{O}(n^{-\rho}),$$

where $\psi(z)$, M_n , $I_{n,j}$, $z_{n,j}$ and ρ are like in Definition 3.2. Then F_n follows the continuous recurrence

$$F_n \simeq t_n + \sum_{0 \leq k < n} \omega_{n,k} F_k,$$

where $\omega_{n,k} = \psi_{n,k} + \psi_{n,n-1-k}$, and thus $\omega(z) = \psi(z) + \psi(1 - z)$. Moreover, $\alpha = 1$, because

$$\int_0^1 \omega(z)z^1 dz = \int_0^1 (\psi(z)z + \psi(z)(1 - z)) dz = \int_0^1 \psi(z) dz = 1.$$

Also, under the assumptions above, we have

$$G_n \approx 2t_n F_n - (t_n)^2 + 2 \sum_{0 \leq k < n} \psi_{n,k} F_k F_{n-1-k} + \sum_{0 \leq k < n} \omega_{n,k} G_k.$$

Suppose first $t_n \sim n^a \xi_n$, where $a > 1$ and ξ_n includes subpolynomial terms. Then $F_n \sim t_n / (1 - \varphi(a))$, and

$$G_n \sim \left(\frac{1 + \varphi(a)}{(1 - \varphi(2a))(1 - \varphi(a))} + \frac{2 \int_0^1 \psi(z)z^a(1 - z)^a dz}{(1 - \varphi(2a))(1 - \varphi(a))^2} \right) (t_n)^2,$$

where the integral above results from a reasoning similar to those used to derive the Continuous MT. We can conclude

$$V[\mathcal{F}_n] \sim \frac{\varphi(2a) - \varphi(a)^2 + 2 \int_0^1 \psi(z)z^a(1 - z)^a dz}{(1 - \varphi(2a))(1 - \varphi(a))^2} \cdot (t_n)^2.$$

Suppose now $t_n \sim n \ln^c n \cdot \xi_n$, where $c > -1$ and ξ_n includes sublogarithmical terms. Then $F_n \sim t_n \ln n / \mathcal{H}$, where $\mathcal{H} = -(c + 1) \int_0^1 \omega(z)z \ln z dz$. Like for one-branch algorithms, we can only conclude $V[\mathcal{F}_n] = o(F_n)^2$, since

$$G_n \sim \frac{2 \int_0^1 \psi(z)z(1 - z) dz}{(1 - \varphi(2))\mathcal{H}^2} \cdot (t_n)^2 \ln^2 n = \frac{(t_n)^2 \ln^2 n}{\mathcal{H}^2}.$$

For this last step, it is enough to notice that $2 \int_0^1 \psi(z)z(1 - z) dz = 1 - \varphi(2)$.

9. Open Problems

The results of this work could be extended in other directions. For instance, consider the system of equations

$$\begin{aligned} F_n^{(1)} &= t_n^{(1)} + \sum_{0 \leq i < n} \omega_{n,i}^{(1,1)} F_i^{(1)} + \dots + \sum_{0 \leq i < n} \omega_{n,i}^{(1,k)} F_i^{(k)}, \\ F_n^{(2)} &= t_n^{(2)} + \sum_{0 \leq i < n} \omega_{n,i}^{(2,1)} F_i^{(1)} + \dots + \sum_{0 \leq i < n} \omega_{n,i}^{(2,k)} F_i^{(k)}, \\ &\dots \\ F_n^{(k)} &= t_n^{(k)} + \sum_{0 \leq i < n} \omega_{n,i}^{(k,1)} F_i^{(1)} + \dots + \sum_{0 \leq i < n} \omega_{n,i}^{(k,k)} F_i^{(k)}, \end{aligned}$$

where the weights above fulfill the discrete and/or the continuous pattern. The problem is: Can we systematically solve such a system, for a wide set of toll functions and weight distributions?

On the other hand, it would be interesting to dispose of theorems to systematically solve recurrences on more than one variable, like

$$F_{n,m} = t_{n,m} + \sum_{0 \leq i < n} \sum_{0 \leq j < n} w_{n,m,i,j} F_{i,j},$$

where the weights above follow a discrete or continuous pattern.

Finally, and mostly of theoretical interest, we could deal with toll functions $t_n = n^\alpha \delta_n$ for some δ_n subpolynomial and superlogarithmical (like $\delta_n = \exp(\sqrt{\ln n})$), or $\delta_n = \omega(\log^{-1} n)$ and $\delta_n = o(\log^{-1+\varepsilon} n)$ for every $\varepsilon > 0$, or $\delta_n = o(\log^{-1} n)$ and $\delta_n = \omega(\log^{-1-\varepsilon} n)$ for every $\varepsilon > 0$. The main interest of the later case would be finding the maximum growing order for δ_n such that $F_n = \mathcal{O}(n^\alpha)$.

Appendixes

A. Examples of the Use of the Discrete Master Theorem

We start solving (1). First, we identify the main term of toll function, and the sets of values $\{w_d\}_{1 \leq d \leq D}$ and $\{z_d\}_{1 \leq d \leq D}$. This yields $t_n \sim n^1 \ln^0 n$, $w_1 = w_2 = 1$ and $z_1 = z_2 = 1/2$. Note that the bounds for $r_{d,n}$ and $s_{d,n}$ trivially hold here, because Definition 2.1 always covers floors and ceilings. We now define $\Phi(x) = 2(1/2)^x$, and evaluate $\mathcal{H} = 1 - \Phi(1) = 0$. Since $\mathcal{H} = 0$, we compute $\mathcal{H}' = -(0 + 1)(2(1/2)^1 \ln(1/2)) = \ln 2$. Therefore, $M_n \sim n \ln n / \ln 2 = n \log_2 n$.

Let us analyze (2). The main term in its toll function is $n^0 \ln^0 n$. Moreover, $w_1 = w_2 = 1/2$ and $z_1 = z_2 = 1/2$. So, we define $\Phi(x) = (1/2)^x$. Since $\mathcal{H} = 1 - \Phi(0) = 0$, we compute $\mathcal{H}' = -(0 + 1)((1/2)^0 \ln(1/2)) = \ln 2$, and conclude $B_n \sim \ln n / \ln 2 = \log_2 n$.

We now consider (3). Suppose $t_n = 6n^2 / \ln^5 n$. The weights are $w_1 = 2$, $w_2 = 4$ and $w_3 = 1/2$; the fractions are $z_1 = 1/3$, $z_2 = 1/2$ and $z_3 = 4/5$. It is a simple matter to check that this recurrence follows Definition 2.1. So, we define $\Phi(x) = 2(1/3)^x + 4(1/2)^x + 1/2 \cdot (4/5)^x$, and $\mathcal{H} = 1 - \Phi(2) = -122/225$. Since $\mathcal{H} < 0$, we have $F_n = \Theta(n^\alpha)$, where α is the unique solution of the equation $\Phi(\alpha) = 1$, which numerically is $\alpha \approx 2.68723$.

Solving

$$f(n) = \Theta(n) + 2f\left(\frac{n}{2}\right) + 2f\left(\frac{n}{3}\right) + 2f\left(\frac{n}{6}\right)$$

is very similar. It yields $f(n) = \Theta(n^\alpha)$, where $\alpha \approx 1.72121$ is the unique solution of $2(1/2)^\alpha + 2(1/3)^\alpha + 2(1/6)^\alpha = 1$. This is an upper bound to the system of equations presented in Kuo and Chang [1994, pp. 79–80], where a very good approximation, $\alpha \approx 1.722$, is given.⁴

Finally, set $t_n = n^2$ for (11). Solving it is very easy: $w_1 = 1$ and $z_1 = 1/4$; $\Phi(x) = (1/4)^x$ and $\mathcal{H} = 1 - \Phi(2) = 15/16$; since $\mathcal{H} > 0$, we deduce $F_n \sim n^2 / (15/16) = 16n^2 / 15$.

In this example, we can get more information by means of a simple trick. Let $G_n = F_n - 16n^2 / 15$. Then,

⁴ Thanks are due to Fatos Xhafa for suggesting this example.

$$G_n = n^2 + F_{\lfloor n/4 \rfloor} - \frac{16n^2}{15} = -\frac{n^2}{15} + \frac{16}{15} \left\lfloor \frac{n}{4} \right\rfloor^2 + G_{\lfloor n/4 \rfloor}.$$

We can now find bounds for G_n . Taking into account that $n/4 - 3/4 \leq \lfloor n/4 \rfloor \leq n/4$, we have $n^2/16 - 3n/8 + 9/16 \leq \lfloor n/4 \rfloor^2 \leq n^2/16$, and hence

$$-\frac{2n}{5} + \frac{9}{15} + G_{\lfloor n/4 \rfloor} \leq G_n \leq G_{\lfloor n/4 \rfloor}.$$

We can use the Discrete MT again to solve $G'_n = -2n/5 + 9/15 + G'_{\lfloor n/4 \rfloor}$. This is even simpler, since the distribution of weights remains unchanged, and so does $\Phi(x)$. The main term in the toll function is negative ($-2n/5$); recall that this is not a problem. Computing \mathcal{H} yields $\mathcal{H} = 1 - \Phi(1) = 3/4$, and therefore $G'_n \sim -2/5 \cdot n/(3/4) = -8n/15$. We now solve $G''_n = G''_{\lfloor n/4 \rfloor}$, and get $G''_n = \mathcal{O}(n^\alpha) = \mathcal{O}(1)$; not $\Theta(1)$, since every term could be zero. Finally, we conclude $F_n = 16n^2/15 + G_n$, where $-8n/15 + o(n) \leq G_n \leq \mathcal{O}(1)$, thus achieving tighter bounds for F_n . We will see that for stochastic recurrences this technique provides more accurate results.

B. Examples of the Use of the Continuous Master Theorem

We first analyze (4). It is trivial to show that $\omega(z) = 2$. Therefore, $\varphi(x) = \int_0^1 \omega(z)z^x dz = 2/(x + 1)$ if $x > -1$ (and $\varphi(x) = +\infty$ otherwise). Since $\mathcal{H} = 1 - \varphi(1) = 0$, we define $\mathcal{H}' = -\int_0^1 \omega(z)z^1 \ln z dz = 1/2$, and finally $Q_n \sim n \ln n/(1/2) = (2 \ln 2)n \log_2 n \approx 1.38n \log_2 n$.

To solve (5) we start identifying the shape function for the weights, using Lemma 7.2. First, we replace the term $n + 1$ in the denominator of $\omega_{n,k}$ by n . This yields $\sigma_{n,k} = 4(n - k)/n^2$. Second, we compute $\omega(z) = n \cdot \sigma_{n,zn} = n \cdot 4(n - zn)/n^2 = 4(1 - z)$. Third and last, we check that $\int_0^1 \omega(z) dz = 2 \geq 1$ indeed. Now we define $\varphi(x) = 4 \int_0^1 (1 - z)z^x dz = 4(1/(x + 1) - 1/(x + 2))$ if $x > -1$ (and $\varphi(x) = +\infty$ otherwise). Since $\mathcal{H} = 1 - \varphi(0) = -1 < 0$, we conclude $F_n = \Theta(n^\alpha)$, where α is the unique solution of $\varphi(\alpha) = 1$. This yields $\alpha = (\sqrt{17} - 3)/2 \approx 0.56155$ (the only solution to $\alpha^2 + 3\alpha - 2 = 0$ that is larger than -1).

We now solve (9), for which we already know that $\omega(z) = 2z$. In this case $\varphi(x) = 2/(x + 2)$ if $x > -2$ (and $\varphi(x) = +\infty$ otherwise). Since $\mathcal{H} = 1 - \varphi(1) = 1/3 > 0$, it follows that $S_n \sim n/(1/3) = 3n$.

In this example, we can get more information, using the technique that was presented in Appendix A. Define $G_n = S_n - 3n$. Then,

$$G_n = n + 1 + \frac{2}{n^2} \sum_{0 \leq k < n} kS_k - 3n = -2n + 1 + \frac{6}{n^2} \sum_{0 \leq k < n} k^2 + \frac{2}{n^2} \sum_{0 \leq k < n} kG_k.$$

But $\sum_{0 \leq k < n} k^2 = n^3/3 - n^2/2 + n/6$. Thus,

$$G_n = -2 + \frac{1}{n} + \frac{2}{n^2} \sum_{0 \leq k < n} kG_k.$$

Notice that, in contrast to what happens with discrete recurrences, the recurrence for G_n above does not contain noise terms. Now we use the Continuous MT again. The first step is already done, since the distribution of weights remains the same, and $\varphi(x)$ too. Since $\mathcal{H} = 1 - \varphi(0) = 0$, we compute $\mathcal{H}' = -\int_0^1 \omega(z)z^0 \ln z dz = 1/2$, and get $G_n \sim -2 \ln n/(1/2) = -4 \ln n$.

We can go one step further, defining $I_n = G_n + 4H_n$, which is slightly simpler than defining $I_n = G_n + 4 \ln n$. Notice that we are extracting the main terms of S_n one by one. This step yields

$$I_n = \frac{4H_n}{n} - \frac{1}{n} + \frac{2}{n^2} \sum_{0 \leq k < n} kI_k,$$

where we have used the fact that $\sum_{0 \leq k < n} kH_k = n^2H_n/2 - n^2/4 - nH_n/2 + n/4$. Now we get $\mathcal{H} = 1 - \varphi(-1) = -1 < 0$, and hence $I_n = \mathcal{O}(n^\alpha) = \mathcal{O}(1)$; we cannot deduce $I_n = \Theta(1)$, since the toll function in the definition of I_n includes positive and negative terms together. At this point we cannot go any further, because we have reached the core of the recurrence. As a conclusion, we have $S_n = 3n - 4 \ln n + \mathcal{O}(1)$.

As a final remark, hybrid recurrences like

$$F_n = n^2 + F_{\lfloor n/2 \rfloor} + \frac{1}{n} \sum_{0 \leq k < n} F_k$$

can be analyzed by combining both MTs. So, to solve this recurrence is enough to find out that $w_1 = 1$, $z_1 = 1/2$ and $\omega(z) = 1$; compute $\mathcal{H} = 1 - (1/2)^2 - \int_0^1 z^2 dz = 5/12 > 0$; and conclude $F_n \sim n^2/(5/12) = 12n^2/5$.

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