QUANTUM RANDOM WALKS ON $\mathbb{Z}^2$

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Contents

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1 Introduction

The theory of quantum mechanics explains the world that we live in more accurately than classical physics. Sometimes quantum mechanics and classical physics agree and other times the behavior differences can be very strange. Researchers have been intrigued by the counterintuitive behavior of quantum mechanics, and the interesting interference properties displayed. Aside from their intrinsic interest, these differences lead to situations in which non-classical behaviors may be used to solve problems in engineering.

Recently, theoreticians have developed several possible architectures for creating computer components that behave according to the laws of quantum mechanics [NC00]. While such circuits have yet to be constructed on a large enough scale, they have the potential to compute in polynomial time problems that require exponential time under the best known classical algorithms. As an example, Peter Shor [Sho97] gives a polynomial time algorithm using a quantum computer for calculating prime factorization. Such an algorithm could have major impact on security protocols such as RSA. The possibility of finding polynomial time solutions for some known exponentially hard problems gives major motivation to study quantum behaviors.

One such quantum system to study is the quantum random walk. Many algorithms used today stem from the understanding of the classical random walk. The random walk has been used to develop algorithms for checking $S-T$ connectivity, $2-SAT$, Markov Chain Monte Carlo, Counting, and Uniform Sampling. The usefulness of random walks in classical computing naturally leads the quantum computing world to turn to the quantum random walk. As discussed in [ABN$^+$01], "quantum random walks have the potential to offer new tools for quantum algorithms". In comparison to the classic symmetric random walk, the Hadamard quantum random walk (the symmetric version of the quantum random walk) propagates quadratically faster. The classic symmetric random walk after t steps has variance t, so the expected distance traveled is of order $\sqrt{t}$. On the other hand, the Hadamard quantum random walk has a variance that scales with t^2, so its expected distance traveled is of order t [Kem05].

The bulk of the study of quantum random walks so far has been limited to walks on the integer lattice. There has been limited work done on higher dimensional quantum random walks; such work can be found in [BMSS02]. The goal of this paper is to study quantum random walks on the two dimensional integer lattice. While we do not perform a full asymptotic analysis, we will derive a method using generating functions for finding the region for which the walk amplitudes will have non-exponential decay.

In section 2, we describe quantum random walks on $\mathbb{Z}$ and $\mathbb{Z}^2$. In the case of a quantum random walk on $\mathbb{Z}$, we give some known results. We then go through basic generating function background and derive pertinent generating functions for quantum random walks on both $\mathbb{Z}$ and $\mathbb{Z}^2$ in section 3. From there we proceed to section 4, where we state and prove a theorem on asymptotics of generating functions, and we develop a method to plot the region of non-exponential decay. In section 5, we apply this method to a particular quantum random walk and a one parameter family of quantum random walks determined by a one parameter subfamily of unitary matrices. We conclude with section 6, where we state a conjecture based on the results seen in section 5.

2 Quantum Random Walks on $\mathbb{Z}$ and $\mathbb{Z}^2$

2.1 Quantum Random Walk on $\mathbb{Z}$

Before we begin to discuss a quantum random walk (QRW), we first need to understand the classical random walk on $\mathbb{Z}$. By classic random walk (CRW) we mean the nearest neighbor translation invariant random walk on $\mathbb{Z}$ (viewed as an infinite horizontal lattice labeled in order by the integers). The CRW can be described as a particle starting at 0, which at every time step has probability $p \in [0, 1]$ of moving right (plus 1) and probability $1 - p$ of moving left (minus 1). Another view of the CRW is as on operator on $\ell_1(\mathbb{Z})$, which consists of elements $a = (..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...)$ with $a_i \in \mathbb{R}$ such that $|a|_1 = \sum_i |a_i| < \infty$. In particular, we view the CRW as an operator U_p on the subset of $\ell_1(\mathbb{Z})$ which can be described as probability distributions. For this view, a probability distribution $a \in \ell_1(\mathbb{Z})$, is a list of absolutely summable nonnegative real numbers indexed by $\mathbb{Z}$ that sum to 1. Thus, U_p operates on probability distribution a by $(U_p(a))_i = pa_{i-1} + (1 - p)a_{i+1}$. From this operation it is clear that the entries stay positive and $|U_p(a)|_1 = 1$ so U_p is a map from probability distributions to probability distributions.

The CRW induced by U_p gives the nearest neighbor random walk with probability p of moving right. To see this we start with our particle in the 0th position, given by $a \in \ell_1(\mathbb{Z})$ where $a_i = 0$ if $i \neq 0$, and $a_0 = 1$ which tells us with probability 1 our particle is at the 0th position. Now we have $U_p(a) = b \in \ell(\mathbb{Z})$ with $b_i = 0$ if

$i \neq 1, -1$, and $b_{-1} = 1 - p$ and $b_1 = p$, which tells us that with probability $(1 - p)$ the particle is in position -1 (left one unit of 0), and with probability p it is in the 1 position (right one unit of 0). In general, $U_p(\mu)$ is the distribution at time $n + 1$ if the distribution at time n is μ. This is the CRW we first described.

We described above the CRW as an operator on $\ell_1(\mathbb{Z})$ that preserves the probability measure. The quantum analogue of the CRW is an operator on the Hilbert space $\ell_2(\mathbb{Z})$ over $\mathbb{C}$ that preserves the probability measure, and it thus represents a particle moving under the laws of quantum mechanics on $\mathbb{Z}$. For $a \in \ell_2(\mathbb{Z})$ to be a probability measure, we cannot just look at the a_i entry to find the probability at i, but instead we must look at $|a_i|^2$. This is to say that, unlike probability measures described by $\ell_1(\mathbb{Z})$ which simply listed the probabilities, in the $\ell_2(\mathbb{Z})$ case the probability measure is now described by a list of complex amplitudes, where the square of the absolute value of the amplitude gives the probability. Now consider the QRW that corresponds to the symmetric CRW (where $p = 1/2$). This QRW should move both left and right with equal amplitude, but this is impossible because the ℓ_2 norm of such a process will not remain 1. Furthermore, the only p values which have a direct relation from CRW and QRW are for $p = 0$ or $p = 1$, which correspond to the trivial single direction motion [ABN$^+$01].

One way to deal with this problem is to allow the particle an extra degree of freedom that assists in its motion [ABN$^+$01]. That is to say we give the particle a spin or a chirality of R (right) or L (left), and this chirality dictates movement. Let

$\Sigma = \{L, R\}$ and now we can describe the space that the particle exists on as $\mathbb{Z} \times \Sigma$. Thus, we wish to describe a QRW as an operator on $\ell_2(\mathbb{Z} \times \Sigma) \cong \ell_2(\mathbb{Z}) \otimes \ell_2(\Sigma)$. It will be easier to think of the state space of this walk as the tensor product of Hilbert spaces. Much of the following notation and terminology is borrowed from [ABN$^+$01]. An actual quantum state would be given by (n, d) where $n \in \mathbb{Z}$ and $d \in \Sigma$, and these states correspond to basis vectors on $\ell_2(\mathbb{Z}) \otimes \ell_2(\Sigma)$. To see this, (n, d) corresponds to the element $\bar{n} \in \ell_2(\mathbb{Z})$ with a 1 in the nth position and 0s elsewhere, and the element $\bar{d} \in \Sigma$ which is a 1 in the dth position and a 0 in the other, so we get $\bar{n} \otimes \bar{d} \in \ell_2(\mathbb{Z}) \otimes \ell_2(\Sigma)$. The elements $\bar{n} \otimes \bar{d}$ form an orthonormal basis.

Now we use this notation to describe what is called the Hadamard walk as it undergoes a Hadamard transformation. We first define the Hadamard transform H_1 [ABN$^+$01], which acts on $\ell_2(\Sigma)$ as follows:

$$H_1(\overline{R}) = \frac{1}{\sqrt{2}}(\overline{R} + \overline{L}), \qquad H_1(\overline{L}) = \frac{1}{\sqrt{2}}(\overline{R} - \overline{L})$$

This transform is first defined on the basis elements of $\ell_2(\Sigma)$ and is then extended to all elements linearly. Furthermore we see this operator is unitary since $\left|\left|\frac{1}{\sqrt{2}}(\overline{R} \pm \overline{L})\right|\right|_2 = 1$. We can describe the transform H_1 as a matrix

$$\mathbf{H} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \tag{2.1}$$

which acts by left multiplication on the vectors

$$\vec{R} = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}; \qquad \vec{L} = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}; \tag{2.2}$$

This is clear as $\mathbf{H}\vec{R} = \frac{1}{\sqrt{2}}\vec{R} + \frac{1}{\sqrt{2}}\vec{L}$ and $\mathbf{H}\vec{L} = \frac{1}{\sqrt{2}}\vec{R} - \frac{1}{\sqrt{2}}\vec{L}$.

Now to apply the Hadamard transform to $\ell_2(\mathbb{Z}) \otimes \ell_2(\Sigma)$, we simply tensor H_1 with the identity. So now we have

$$I \otimes H_1(\overline{n} \otimes \overline{R}) = \frac{1}{\sqrt{2}}(\overline{n} \otimes (\overline{R} + \overline{L}))$$

$$I \otimes H_1(\overline{n} \otimes \overline{L}) = \frac{1}{\sqrt{2}}(\overline{n} \otimes (\overline{R} - \overline{L}))$$

Again, the operator $I \otimes H_1$ can be linearly extended to all of $\ell_2(\mathbb{Z}) \otimes \ell_2(\Sigma)$ and it is unitary since I and H_1 are unitary.

So far we have defined an operator that changes the chirality of our particle. Now we simply use the chirality to move the particle using a translation operator T:

$$T(\overline{n} \otimes \overline{R}) = \overline{(n+1)} \otimes \overline{R}, \quad T(\overline{n} \otimes \overline{L}) = \overline{(n-1)} \otimes \overline{L}$$

Here we see again that T is unitary. Now we simply define $W = T \cdot (I \otimes H_1)$, and the unitary operator W defines our Hadamard walk.

Now we examine W. Suppose our particle starts in the $(0, L)$ state, which corresponds to $(\overline{0} \otimes \overline{L})$. We apply W:

$$W(\overline{0} \otimes \overline{L}) = T(I \otimes H_1)(\overline{0} \otimes \overline{L}) = T\left(\frac{1}{\sqrt{2}}(\overline{0} \otimes (\overline{R} - \overline{L}))\right) = \frac{1}{\sqrt{2}}\left(\overline{1} \otimes \overline{R} - (\overline{-1} \otimes \overline{L})\right)$$

An illustration of W can be see in figure 1. The behavior of a QRW is as follows: given a particular position and chirality, at each time step, the chirality may change according to some probability (given by H_1 in our case), then the particle moves one to the right if in the R chirality or one to left if it is in the L chirality. We can think of the quantum random walk as occurring on two parallel $\mathbb{Z}$ lattices, the top lattice corresponding to chirality R and the bottom lattice corresponding to L.

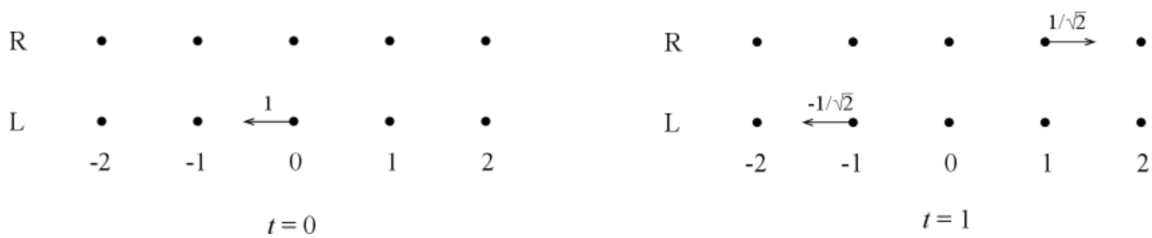


Figure 1: QRW initial position of $\overline{0} \otimes \overline{L}$ on the left. The right is after one time step.

Now to find the probability that the particle is in position n, we simply sum the squares of the numbers in the nth position with both chirality R and L. So in the above, after applying W to $(\overline{0} \otimes \overline{L})$ we see the probability the particle is in the 1 position is $|\frac{1}{\sqrt{2}}|^2 + 0^2 = \frac{1}{2}$, and the probability the particle is in the -1 position is $0^2 + |\frac{-1}{\sqrt{2}}|^2 = \frac{1}{2}$. Thus we see the Hadamard walk behaves just like the symmetric CRW after one time step. But if we take three or more steps before measuring, we would see that the walks are not the same because of interference that will occur.

To show this, we apply W^3 to $(\overline{0} \otimes \overline{L})$. We see

$$\begin{aligned} W^3(\overline{0} \otimes \overline{L}) &= \frac{1}{\sqrt{2}} W^2 \left(\overline{1} \otimes \overline{R}\right) - \frac{1}{\sqrt{2}} W^2 \left(\overline{-1} \otimes \overline{L}\right)) = \cdots \\ &= \frac{1}{2\sqrt{2}} \left((\overline{3} \otimes \overline{R}) + (\overline{1} \otimes \overline{L}) - 2(\overline{-1} \otimes \overline{L}) + (\overline{-1} \otimes \overline{R}) - (\overline{-3} \otimes \overline{L})\right) \end{aligned}$$

Thus, after three steps of the Hadamard QRW there is a $5/8$ probability of being in the -1 position which is not the case for the symmetric CRW after three steps.

With this basic understanding of the Hadamard QRW, we now survey three major cases of interest, which can be found in [ABN$^+$01]. They are the "two-way infinite timed Hadamard walk", "semi-infinite Hadamard walk", and the "finite Hadamard walk".

For the "Two-way infinite timed Hadamard walk", we simply start the walk in the $\overline{0} \otimes \overline{R}$ state and for any time $t \in \mathbb{Z}^+$, apply W^t to the walk, then observe the probabilities for each location. We define the probability of being at position n on $\mathbb{Z}$ to be $P(n,t) = p_L(n,t) + p_R(n,t)$, where $p_L(n,t)$ and $p_R(n,t)$ are the probability for being at position n with chirality L and R, respectively, after t steps.

Theorem 2.1. *[ABN$^+$01, Theorem 1] Let $n = \alpha t \to \infty$ with α fixed. In case $-1 < \alpha < -1/\sqrt{2}$ or $1/\sqrt{2} < \alpha < 1$, there exists $c > 1$ for which $p_R(n,t) = O(c^{-n})$ and $p_L(n,t) = O(c^{-n})$.*

The above theorem tells us that for time t (large), the probability of finding the particle outside the interval $[\frac{-t}{\sqrt{2}}, \frac{t}{\sqrt{2}}]$ is exponentially small.

Furthermore, let π_t be the distribution having density $\frac{c}{\sqrt{1-2(n/t)^2}}$ on $\mathbb{Z}\cap[\frac{-t}{\sqrt{2}}, \frac{t}{\sqrt{2}}]$. Then we have $p_L(n,t) \sim \cos^2(A_t + w_t n) \cdot \pi_t(n)$ and a similar result for $p_R(n,t)$. This result is taken from Theorem 2 of [ABN$^+$01]. Thus if we take a measurement of the walk for a time t, the behavior of the probability of the walk, in the interval $[\frac{-t}{\sqrt{2}}, \frac{t}{\sqrt{2}}]$, is oscillating and growing larger as it approaches the end points (see Figures 2 and 3 in [ABN$^+$01]). The current work [BP07b] gives similar results of Theorem 1 and 2 found in [ABN$^+$01], but using methods which are more similar to the methods that will be used later in this thesis.

For the "semi-infinite Hadamard walk", our start state is $\overline{1} \otimes \overline{R}$ and we have an absorbing boundary at 0. Let p_∞ be the probability that the particle is absorbed, and then we have

Theorem 2.2. *[ABN$^+$01, Theorem 8]* $p_\infty = 2/\pi$.

This theorem for the Hadamard QRW is again in sharp contrast to the symmetric CRW which has a probability of 1 of eventually exiting to the left. Furthermore, in the Hadamard QRW, if we change our initial starting point (or move the boundary) our absorption probability is still non-zero [Kem05].

Lastly we look at the "finite Hadamard walk", which is just like the semi-infinite walk except that we add another absorbing boundary to the right of our start state $\overline{1} \otimes \overline{R}$. For $n > 1$, let p_n be the probability that the walk eventually exits left (absorbed at 0) given that an absorption boundary exists at n.

Theorem 2.3. *[ABN$^+$01, Theorem 10]* $\lim_{n\to\infty} p_n = 1/\sqrt{2}$

This result is again in sharp contrast to the symmetric CRW for which the probability of exiting left is $1 - 1/n$. Now notice that $1/\sqrt{2} > 2/\pi$, which implies for large n that the probability of exiting left surprisingly goes up when we place an absorbing boundary at n [ABN$^+$01].

As the above results show, the QRW behaves much different than the CRW. These behavior differences give the QRW, when applied to algorithms on a quantum computer, the potential for dramatic consequences.

2.2 Quantum Random Walk on $\mathbb{Z}^2$

Now that we have an understanding of a QRW on $\mathbb{Z}$, we can generalize to higher dimensions. The results in this paper concern only $\mathbb{Z}^2$, but the following construction is easily generalized to any d. In this case we now use the chiralities U (up), R (right), D (down), and L (left), which gives $\Sigma = \{R, L, U, D\}$. We shall define such a walk similarly to the one dimensional case. The possible states for a particle are now $\mathbb{Z}^2 \times \Sigma$, and hence we will define a QRW as an operator on the space $\ell_2(\mathbb{Z}^2) \otimes \ell_2(\Sigma)$. We will use the same notation as we did above but now an actual quantum state is given by $((s,t), d)$ where $(s,t) \in \mathbb{Z}^2$ and $d \in \Sigma$, and these states correspond to basis vectors on $\ell_2(\mathbb{Z}^2) \otimes \ell_2(\Sigma)$ in the usual way.

Now just as in the QRW on $\mathbb{Z}$, we define a Hadamard transform H_2 on the basis elements of $\ell_2(\Sigma)$.

$$\begin{aligned}
H_2(\overline{R}) &= \frac{1}{2}(\overline{R} - \overline{L} - \overline{U} - \overline{D}) \\
H_2(\overline{L}) &= \frac{1}{2}(-\overline{R} + \overline{L} - \overline{U} - \overline{D}) \\
H_2(\overline{U}) &= \frac{1}{2}(-\overline{R} - \overline{L} + \overline{U} - \overline{D}) \\
H_2(\overline{D}) &= \frac{1}{2}(-\overline{R} - \overline{L} - \overline{U} + \overline{D})
\end{aligned}$$

Here again, as we saw in section 2.1, this transform can be described by a matrix

$$\mathbf{H}' = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{2.3}$$

where this matrix acts by left multiplication on the following vectors

$$R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad L = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad U = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad D = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \tag{2.4}$$

Now, as before, we tensor H_2 with the identity I on $\ell_2(\mathbb{Z}^2)$ to get a transform on $\ell_2(\mathbb{Z}^2) \otimes \ell_2(\Sigma)$. Lastly we define a translation operator T:

$$T(\overline{(s,t)} \otimes \overline{R}) = \overline{(s+1,t)} \otimes \overline{R}$$

$$T(\overline{(s,t)} \otimes \overline{L}) = \overline{(s-1,t)} \otimes \overline{L}$$

$$T(\overline{(s,t)} \otimes \overline{U}) = \overline{(s,t+1)} \otimes \overline{U}$$

$$T(\overline{(s,t)} \otimes \overline{D}) = \overline{(s,t-1)} \otimes \overline{D}$$

We now define $W = T \cdot (I \otimes H_2)$, which acts on $\ell_2(\mathbb{Z}^2) \otimes \ell_2(\Sigma)$. Just as with the QRW on $\mathbb{Z}$, we see W first updates the chirality and then causes the particle's amplitude to shift depending on its chirality.

Recall how the Hadamard transform H_2 was encoded into the matrix H'. We see that this matrix is unitary, and in fact, can be replaced by any unitary matrix to give us a transform $\tilde{H}$. We can then define our walk with $\widetilde{W} = T \cdot (I \otimes \tilde{H})$. Therefore a general QRW can be defined by its associated unitary matrix for chirality updates. Also, one should now be able to describe a QRW on any $\mathbb{Z}^d$ in a similar manner using $2d$ chiralities. To analyze these QRWs, we require combinatorial techniques for computing amplitudes for a given position, chirality, and time step.

3 Generating Functions for QRWs on $\mathbb{Z}$ and $\mathbb{Z}^2$

In this section, we will show a combinatorial approach to finding the amplitudes of a particle on a QRW using generating functions.

3.1 Formal Power Series

As Herbert Wilf aptly puts it in [Wil94], "a generating function is a clothesline on which we hang up a sequence of numbers for display". More generally, we will utilize multivariate generating functions to describe an array of numbers. In this section and all the following we will be focusing on finding information about arrays of numbers, $\{a_{r_1,\ldots,r_d} : r_1, \ldots, r_d \in \mathbb{Z}^+\}$. In particular, given such an array, we define the associated generating function

$$F(z_1, \ldots, z_d) = \sum_{r_1,\ldots,r_d \in \mathbb{Z}^+} a_{r_1,\ldots,r_d} z_1^{r_1} z_2^{r_2} \cdots z_d^{r_d}.$$

This generating function is a formal power series, so before we proceed with generating functions we first discuss formal power series.

The set of formal power series over $\mathbb{C}$ in one and several variables form rings, $\mathbb{C}[[x]]$ and $\mathbb{C}[[z_1, \ldots, z_d]]$, respectively. Elements of $\mathbb{C}[[x]]$ are parameterized by collections of complex numbers, $\{a_r\}_{r=0}^{\infty}$, with the associated power series $f(z) = \sum_{r=0}^{\infty} a_r z^r$. Similarly for elements of $\mathbb{C}[[z_1, \ldots, z_d]]$, we parameterize by $\{a_{r_1,\ldots,r_d} : r_1, \ldots, r_d \in \mathbb{Z}^+\}$ with associated power series

$$F(z_1, \ldots, z_d) = \sum_{r_1,\ldots,r_d \in \mathbb{Z}^+} a_{r_1,\ldots,r_d} z_1^{r_1} z_2^{r_2} \cdots z_d^{r_d}.$$

We will use the notation $\mathbf{r} = r_1, \dots, r_d$, $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} z_2^{r_2} \cdots z_d^{r_d}$, and $[\mathbf{z}^{\mathbf{r}}]F(z_1, \dots, z_d) = a_{\mathbf{r}}$. Addition of two formal power series F and G with parameterizations $\{a_{\mathbf{r}}\}$ and $\{b_{\mathbf{r}}\}$, respectively, is defined by $c_{\mathbf{r}} = a_{\mathbf{r}} + b_{\mathbf{r}}$. Multiplication is defined by the convolution $c_{\mathbf{r}} = \sum_{\mathbf{s}} a_{\mathbf{s}} b_{\mathbf{r}-\mathbf{s}}$, the sum having finitely many terms and therefore avoiding problems with non-summability.

Furthermore, every element of $\mathbb{C}[[z]]$ with parametrization a_r such that $a_0 \neq 0$ has a reciprocal. That is, if $F \in \mathbb{C}[[z]]$, $F(0) \neq 0$ (equivalently $a_0 \neq 0$), then there exists G such that $FG = 1$ [Wil94]. For our purposes we will be concerned with the following:

Proposition 3.1. *Given a polynomial expression $G(\mathbf{z})$ in $z_1, \dots, z_d$ such that $G(\mathbf{0}) = 0$, the reciprocal of $\sum_{t \geq 0} G(\mathbf{z})^t$ is given by $1 - G(\mathbf{z})$ in the ring $\mathbb{C}[[z_1, .., z_d]]$.*

Proof: First, since $G(\mathbf{0}) = 0$, the constant term of the infinite sum is 1 and it comes from $G(z)^0$. We have $(1 - G(\mathbf{z})) \sum_{t \geq 0} G(\mathbf{z})^t = 1$. Thus all we need to show now is that the coefficient of $\mathbf{z}^{\mathbf{r}}$ in $\sum_{t \geq 0} G(\mathbf{z})^t$ is finite. Since $G(\mathbf{0}) = 0$, then $\mathbf{z}^{\mathbf{r}}$ is unchanged by adding powers of $G(\mathbf{z})$ which are greater than $|\mathbf{r}|$. Thus we have $[\mathbf{z}^{\mathbf{r}}] \sum_{t \geq 0} G(\mathbf{z})^t = [\mathbf{z}^{\mathbf{r}}] \sum_{0 \leq t \leq |\mathbf{r}|} G(\mathbf{z})^t$, which is finite as it is the coefficient of a finite sum. $\square$

We use Proposition 3.1 to see that $\sum_{t \geq 0} G(\mathbf{z})^t = \frac{1}{1 - G(\mathbf{z})}$. As an example consider the power series $\sum_{r=0} 2^r x^r$ (clearly the parametrization is $\{2^r\}$), which has reciprocal $1 - 2x$.

Now we can consider matrices with elements from the formal power series ring

$\mathbb{C}[[z_1, \ldots, z_d]]$. We have the following:

Proposition 3.2. *Given a square matrix* $\mathbf{A}(\mathbf{z})$ *with polynomial entries in* $z_1, \ldots, z_d$ *such that no entry has a constant term, then the* $\sum_{t\geq 0} \mathbf{A}^t = (I - \mathbf{A})^{-1}$.

Proof: First since $\mathbf{A}(\mathbf{0}) = 0$, the constant term in the infinite sum is the identity matrix from $\mathbf{A}^0$. Now we have $(I - \mathbf{A}) \sum_{t\geq 0} \mathbf{A}^t = I$. So all we need to show is that the coefficient of $\mathbf{z}^{\mathbf{r}}$ is finite in each entry of $\sum_{t\geq 0} \mathbf{A}^t$. Since the constant term of each entry in $\mathbf{A}(\mathbf{z})$ is zero then for all $t > |\mathbf{r}|$, we see that $\mathbf{A}(\mathbf{z})^t$ has a zero coefficent on the $\mathbf{z}^{\mathbf{r}}$ term. Therefore the coefficient of $\mathbf{z}^{\mathbf{r}}$ is determined in each entry of $\sum_{t\geq 0} \mathbf{A}^t$ by the finite sum $\sum_{0\leq t\leq |\mathbf{r}|} \mathbf{A}^t$. $\square$

In this paper, we will be working with Laurent series $F(\mathbf{x}, z)$ which allow negative exponents on the x_i but not z, and such that $F(\mathbf{x}, \mathbf{x}z)$ is a formal power series (no negative exponents). In our case, the above results all hold for $F(\mathbf{x}, \mathbf{x}z)$, so they will hold for $F(\mathbf{x}, z)$ if we first apply them them to $F(\mathbf{x}, \mathbf{x}z)$ and then make the substitution $z \leftarrow \mathbf{x}^{-\mathbf{1}} z$.

Before deriving the generating functions for QRW on $\mathbb{Z}$ and $\mathbb{Z}^2$, we develop the generating function for the symmetric CRW as an example. Assume the particle starts at 0, and let the probability of the particle being at position r after t steps be given by $P(r, t)$. Fix t and set $P_t(r) = P(r, t)$. We now find the generating function $f_t(x) = \sum_r P_t(r) x^r$ for the sequence $\{P_t(r)\}_{r\in\mathbb{Z}}$. During one step of CRW we have with probability $1/2$ the particles moves left and with probability $1/2$ the particle moves right. Therefore $P_t(r) = (\frac{1}{2}P_{t-1}(r+1) + \frac{1}{2}P_{t-1}(r-1))$, and hence the following:

$$\begin{aligned} f_t(x) &= \sum_r P_t(r)x^r = \sum_r \left(\frac{1}{2}P_{t-1}(r+1) + \frac{1}{2}P_{t-1}(r-1)\right) x^r \\ &= \left(\frac{1}{2}x^{-1} + \frac{1}{2}x\right) \sum_r P_{t-1}(r)x^r = \left(\frac{1}{2}x^{-1} + \frac{1}{2}x\right) f_{t-1}(x) \end{aligned}$$

In the above, multiplying by x corresponds to a step right and by x^{-1} to a step left. Since the generating function for the sequence $\{P_0(r)\}$ is just x^0 (i.e., 1), we see that the the generating function for $\{P_t(r)\}$ is given by $(\frac{1}{2}x^{-1} + \frac{1}{2}x)^t$.

We have developed the single variable generating function for $\{P(r,t)\}_{(r,t\in\mathbb{Z},t\geq 0)}$ for a fixed t. Using Proposition 3.1, we extend this to the multivariate generating function:

$$F(x,z) = \sum_{r,t\in\mathbb{Z},t\geq 0} P(r,t)x^r z^t = \sum_{t\geq 0} \left(\frac{1}{2}x^{-1} + \frac{1}{2}x\right)^t z^t = \frac{1}{1-(\frac{1}{2}x^{-1} + \frac{1}{2}x)z}$$

This completes the development of the generating function for the symmetric CRW.

3.2 Developing the generating function for a QRW on $\mathbb{Z}$

Before we discuss the generating functions for QRWs we need to define the family of numbers that is of interest to us. Recall that for a general QRW to describe a particle's state, we have a location $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{Z}^d$ and a chirality $c \in \Sigma$. Let $\psi_{(c,c')}(\mathbf{r}, t)$ be the amplitude of a particle starting at $\mathbf{v} \in \mathbb{Z}^d$ with chirality c and ending at location $\mathbf{v} + \mathbf{r}$ with chirality c' after exactly t steps. Since the QRW is translation invariant, then $\psi_{(c,c')}(\mathbf{r}, t)$ does not depend on $\mathbf{v}$, so we assume the walk starts at the origin. In this paper we will only be dealing with $d = 1$ and $d = 2$.

As an example consider the Hadamard QRW on $\mathbb{Z}$. If our particle starts in the 0 location with chirality L and we let $t = 3$, then from section 2.1 we see that $\psi_{(L,L)}(-1, 3) = \frac{-1}{\sqrt{2}}$ and $\psi_{(L,R)}(-1, 3) = \frac{1}{2\sqrt{2}}$.

The family of numbers we will be concerned with is $\psi_{c,c'}(\mathbf{r}, t)$ with $c, c' \in \Sigma$ fixed. So we define a generating function for each of the possible pairs (c, c') of chiralities:

$$F_{(c,c')}(\mathbf{x}, z) = \sum_{\mathbf{r},t} \psi_{(c,c')}(\mathbf{r}, t)\mathbf{x}^{\mathbf{r}} z^t$$

where $\mathbf{x}^{\mathbf{r}} = x_1^{r_1} \cdots x_d^{r_d}$. Now we wish to find $F_{(c,c')}(\mathbf{x}, z)$ for $\mathbf{r} \in \mathbb{Z}$ with $\Sigma = \{L, R\}$ and $\mathbf{r} \in \mathbb{Z}^2$ with $\Sigma = \{U, R, D, L\}$.

Let $\mathbf{r} = r \in \mathbb{Z}$ with $\Sigma = \{L, R\}$. To develop the generating function for the QRW we examine what happens in one time step. First the chirality is updated and then, based on the updated chirality, the particle's position shifts. Consider the amplitude $\psi_{c,R}(r, t)$ for some $c \in \Sigma$. Since it is in the chirality R, that means the last thing the particle did was move right. This implies that $\psi_{c,R}(r, t)$ is the sum of two updated amplitudes that were in position $r - 1$ before the right shift. These are amplitudes that, after the chirality is updated, are in the chirality of R. One contributor is $\frac{1}{\sqrt{2}}\psi_{c,R}(r - 1, t - 1)$ since if the particle is in the R state the chance (in terms of amplitude) of staying in the R state is $\frac{1}{\sqrt{2}}$ as we see from the Hadamard transform H_1 discussed in section 2.1. The other contributing term is $\frac{1}{\sqrt{2}}\psi_{c,L}(r - 1, t - 1)$ since if the particle is in the L state the chance of it switching to the R state is $\frac{1}{\sqrt{2}}$. Thus we see

$$\psi_{c,R}(r,t) = \frac{1}{\sqrt{2}}\psi_{c,R}(r-1,t-1) + \frac{1}{\sqrt{2}}\psi_{c,L}(r-1,t-1) \tag{3.1}$$

By the same logic we have

$$\psi_{c,L}(r,t) = \frac{1}{\sqrt{2}}\psi_{c,R}(r+1,t-1) - \frac{1}{\sqrt{2}}\psi_{c,L}(r+1,t-1) \tag{3.2}$$

Now recall how H_1 could be given as a matrix acting on vectors. If we let the top entry of a 2-vector correspond to the ending chirality of R and the bottom entry correspond to ending in the L chirality, we see

$$\mathbf{H}\begin{bmatrix} \psi_{c,R}(r,t) \\ \psi_{c,L}(r,t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \psi_{c,R}(r,t) \\ \psi_{c,L}(r,t) \end{bmatrix} =$$
$$\begin{bmatrix} \frac{1}{\sqrt{2}}\psi_{c,R}(r,t) + \frac{1}{\sqrt{2}}\psi_{c,L}(r,t) \\ \frac{1}{\sqrt{2}}\psi_{c,R}(r,t) - \frac{1}{\sqrt{2}}\psi_{c,L}(r,t) \end{bmatrix} = \begin{bmatrix} \psi_{c,R}(r+1,t+1) \\ \psi_{c,L}(r-1,t+1) \end{bmatrix} \tag{3.3}$$

So applying this unitary matrix $\mathbf{H}$ gives us equations (3.1) and (3.2)

Let $f^t_{(c,c')}(x)$ be the generating function associated with the numbers $\{\psi_{c,c'}(r,t)\}_r$ for a fixed t. Now we see using (3.1) gives

$$\begin{aligned} f^t_{(R,R)}(x) &= \sum_r \psi_{(R,R)}(r,t)x^r \\ &= \sum_r \left(\frac{1}{\sqrt{2}}\psi_{R,R}(r-1,t-1) + \frac{1}{\sqrt{2}}\psi_{R,L}(r-1,t-1)\right)x^r \\ &= \frac{x}{\sqrt{2}}\left(\sum_r \psi_{R,R}(r-1,t-1)x^{r-1}\right) + \frac{x}{\sqrt{2}}\left(\sum_r \psi_{R,L}(r-1,t-1)x^{r-1}\right) \\ &= \frac{x}{\sqrt{2}}f^{t-1}_{(R,R)}(x) + \frac{x}{\sqrt{2}}f^{t-1}_{(R,L)}(x). \end{aligned}$$

So we have

$$f^t_{(R,R)}(x) = \frac{x}{\sqrt{2}} f^{t-1}_{(R,R)}(x) + \frac{x}{\sqrt{2}} f^{t-1}_{(R,L)}(x). \tag{3.4}$$

Similarly we have

$$f^t_{(L,R)}(x) = \frac{x}{\sqrt{2}} f^{t-1}_{(L,R)}(x) + \frac{x}{\sqrt{2}} f^{t-1}_{(L,L)}(x). \tag{3.5}$$

Using (3.2) we can also show

$$f^t_{(R,L)}(x) = \frac{x^{-1}}{\sqrt{2}} f^{t-1}_{(R,R)}(x) - \frac{x^{-1}}{\sqrt{2}} f^{t-1}_{(R,L)}(x). \tag{3.6}$$

$$f^t_{(L,L)}(x) = \frac{x^{-1}}{\sqrt{2}} f^{t-1}_{(L,R)}(x) - \frac{x^{-1}}{\sqrt{2}} f^{t-1}_{(L,L)}(x). \tag{3.7}$$

Now let

$$\mathbf{F}^t(x) = \begin{bmatrix} f^t_{(R,R)}(x) & f^t_{(L,R)}(x) \\ f^t_{(R,L)}(x) & f^t_{(L,L)}(x) \end{bmatrix}, \tag{3.8}$$

and we see

$$\mathbf{HF}^{t-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} f^{t-1}_{(R,R)}(x) + \frac{1}{\sqrt{2}} f^{t-1}_{(R,L)}(x) & \frac{1}{\sqrt{2}} f^{t-1}_{(L,R)}(x) + \frac{1}{\sqrt{2}} f^{t-1}_{(L,L)}(x) \\ \frac{1}{\sqrt{2}} f^{t-1}_{(R,R)}(x) - \frac{1}{\sqrt{2}} f^{t-1}_{(R,L)}(x) & \frac{1}{\sqrt{2}} f^{t-1}_{(L,R)}(x) - \frac{1}{\sqrt{2}} f^{t-1}_{(L,L)}(x) \end{bmatrix}. \tag{3.9}$$

The righthand side of (3.8) matrix gives us equations (3.4), (3.5) and (3.6), (3.7) but not multiplied by x or x^{-1}, respectively. To add the missing x and x^{-1} we simply multiply by a translation matrix

$$\mathbf{T} = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}. \tag{3.10}$$

Lastly we note

$$\mathbf{F}^0(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.11}$$

Using all the above we see that

$$\mathbf{F}^t(x) = \left(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \right)^t = \begin{bmatrix} \frac{x}{\sqrt{2}} & \frac{x}{\sqrt{2}} \\ \frac{x^{-1}}{\sqrt{2}} & \frac{-x^{-1}}{\sqrt{2}} \end{bmatrix}^t. \tag{3.12}$$

This is clear as left multiplication by H corresponds to applying the transform $I \otimes H_1$ and then left multiplication by matrix T corresponds to the application of transform T, where multiplying $f^t_{(c,c')}$ by x or x^{-1} corresponds to a right or left step, respectively.

We can now make this a multivariate generating function by keeping track of t using the invariant z and proposition 3.2.

$$\begin{aligned} \mathbf{F}(x,z) &= \sum_{t\geq 0} \begin{bmatrix} \frac{x}{\sqrt{2}} & \frac{x}{\sqrt{2}} \\ \frac{x^{-1}}{\sqrt{2}} & \frac{-x^{-1}}{\sqrt{2}} \end{bmatrix}^t z^t = \sum_{t\geq 0} \begin{bmatrix} \frac{xz}{\sqrt{2}} & \frac{xz}{\sqrt{2}} \\ \frac{x^{-1}z}{\sqrt{2}} & \frac{-x^{-1}z}{\sqrt{2}} \end{bmatrix}^t \\ &= \left(\mathbf{I} - \begin{bmatrix} \frac{xz}{\sqrt{2}} & \frac{xz}{\sqrt{2}} \\ \frac{x^{-1}z}{\sqrt{2}} & \frac{-x^{-1}z}{\sqrt{2}} \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \frac{-2x+z\sqrt{2}}{-2x+z\sqrt{2}+zx^2\sqrt{2}} & \frac{-zx^2\sqrt{2}}{-2x+z\sqrt{2}+zx^2\sqrt{2}} \\ \frac{-z\sqrt{2}}{-2x+z\sqrt{2}+zx^2\sqrt{2}} & \frac{x(-2+xz\sqrt{2})}{-2x+z\sqrt{2}+zx^2\sqrt{2}} \end{bmatrix}. \end{aligned} \tag{3.13}$$

The last matrix above gives the four generating functions $F_{(c,c')}$ for the QRW on $\mathbb{Z}$.

3.3 Developing the generating function for a QRW on $\mathbb{Z}^2$

Now we consider $\mathbf{r} = (s,t) \in \mathbb{Z}^2$ with $\Sigma = \{R, L, U, D\}$. We again want to consider $\psi_{(c,c')}((r,s),t)$ where $c, c' \in \Sigma$. Lets begin by looking at a particular ending chirality, say R. We want to determine the contribution to $\psi_{(c,R)}((r,s),t)$ from time step $t-1$. Again the last thing to occur was a shift right from $(r-1,s)$. To find what amplitudes shifted right we need to find what amplitudes had a chirality of R after their updates. Examining H_2 we conclude

$$\begin{aligned} \psi_{(c,R)}((r,s),t) &= \frac{1}{2}\psi_{(c,R)}((r-1,s),t-1) - \frac{1}{2}\psi_{(c,L)}((r-1,s),t-1) \\ &\quad -\frac{1}{2}\psi_{(c,U)}((r-1,s),t-1) - \frac{1}{2}\psi_{(c,D)}((r-1,s),t-1) \end{aligned} \tag{3.14}$$

If we do the same analysis for ending chiralities L, U, D, then we have the following matrix multiplication property, which we also so in equation (3.3) for the QRW on $\mathbb{Z}$, but now using H' from section 2.2. In this case, the chiralities are associated as R, L, U, D going down the vector.

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \psi_{c,R}((r,s),t) \\ \psi_{c,L}((r,s),t) \\ \psi_{c,U}((r,s),t) \\ \psi_{c,D}((r,s),t) \end{bmatrix} = \begin{bmatrix} \psi_{c,R}((r+1,s),t+1) \\ \psi_{c,L}((r-1,s),t+1) \\ \psi_{c,U}((r,s+1),t+1) \\ \psi_{c,D}((r,s-1),t+1) \end{bmatrix} \tag{3.15}$$

For a fixed t we let

$$f^t_{(c,c')}(x,y) = \sum_{r,s} \psi_{(c,c')}((r,s),t)x^r y^s$$

Just as we developed equations (3.4), (3.5), (3.6), and (3.7), we derive similar equations for the Hadamard walk on $\mathbb{Z}^2$. Since there are sixteen such equations, we give the following four; each corresponds to starting in the R chirality.

$$f^t_{(R,R)}(x,y) = \frac{x}{2}\left(f^{t-1}_{(R,R)}(x,y) - f^{t-1}_{(R,L)}(x,y) - f^{t-1}_{(R,U)}(x,y) - f^{t-1}_{(R,D)}(x,y)\right) \tag{3.16}$$

$$f^t_{(R,L)}(x,y) = \frac{x^{-1}}{2}\left(-f^{t-1}_{(R,R)}(x,y) + f^{t-1}_{(R,L)}(x,y) - f^{t-1}_{(R,U)}(x,y) - f^{t-1}_{(R,D)}(x,y)\right) \tag{3.17}$$

$$f^t_{(R,U)}(x,y) = \frac{y}{2}\left(-f^{t-1}_{(R,R)}(x,y) - f^{t-1}_{(R,L)}(x,y) + f^{t-1}_{(R,U)}(x,y) - f^{t-1}_{(R,D)}(x,y)\right) \tag{3.18}$$

$$f^t_{(R,D)}(x,y) = \frac{y^{-1}}{2}\left(-f^{t-1}_{(R,R)}(x,y) - f^{t-1}_{(R,L)}(x,y) - f^{t-1}_{(R,U)}(x,y) + f^{t-1}_{(R,D)}(x,y)\right) \tag{3.19}$$

In the above multiplication by x, x^{-1}, y, or y^{-1} corresponds to the appropriate shift on $\mathbb{Z}^2$ due to the chirality. These shifts are given by the following translation matrix

$$\mathbf{T}' = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x^{-1} & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & y^{-1} \end{bmatrix} \tag{3.20}$$

Now just as we saw in (3.8), we define

$$\mathbf{F}^t(x,y) = \begin{bmatrix} f^t_{(R,R)}(x,y) & f^t_{(L,R)}(x,y) & f^t_{(U,R)}(x,y) & f^t_{(D,R)}(x,y) \\ f^t_{(R,L)}(x,y) & f^t_{(L,L)}(x,y) & f^t_{(U,L)}(x,y) & f^t_{(D,L)}(x,y) \\ f^t_{(R,U)}(x,y) & f^t_{(L,U)}(x,y) & f^t_{(U,U)}(x,y) & f^t_{(D,U)}(x,y) \\ f^t_{(R,D)}(x,y) & f^t_{(L,D)}(x,y) & f^t_{(U,D)}(x,y) & f^t_{(D,D)}(x,y) \end{bmatrix} \tag{3.21}$$

And using the above developed relations between $f^t_{(c,c')}(x,y)$ and $f^{t-1}_{(c,c')}(x,y)$ we have

$$\mathbf{F}^t(x,y) = \mathbf{T}'\mathbf{H}'\mathbf{F}^{t-1}(x,y)$$

Now we see

$$\mathbf{F}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we can conclude

$$\mathbf{F}^t(x,y) = (\mathbf{T}'\mathbf{H}')^t$$

Thus the multivariate generating functions $F_{(c,c')}(x,y,z)$ are given in the following matrix:

$$\mathbf{F}(x,y,z) = \sum_{t\geq 0} \mathbf{F}^t(x,y)z^t = (\mathbf{I} - z\mathbf{T}'\mathbf{H}')^{-1}$$

The actual calculation of $(\mathbf{I} - z\mathbf{T}'\mathbf{H}')^{-1}$ can easily be done with software such as Maple. As a result, we obtain sixteen generating functions. For example, the generating function for starting chirality R and ending chirality R is

$$F_{(R,R)}(x,y,z) = \frac{-(zx + xy^2z - 2yx - z^3y + zy)}{x^2z^3y - yzx^2 - 2xz^4y + z^3x + xy^2z^3 - zx - xy^2z + 2yx + z^3y - zy} \tag{3.22}$$

By Crammer's Rule we see that a matrix M of polynomials has an inverse matrix of rational functions, all with denominator $\det(M)$. Thus, all the $F_{(c,c')}(x,y,z)$ have the same denominator, just as we saw for QRW on $\mathbb{Z}$.

All the above can be generalized to any QRW described by transform $\tilde{H}$ as discussed at the end of section 2.2, by simply replacing H' by the unitary matrix representation of $\tilde{H}$. Thus we have the following theorem.

Theorem 3.3. *For a general QRW on $\mathbb{Z}^2$ with chirality transform given by the unitary 4×4 matrix U, the matrix of generating functions for the general QRW, $F(x, y, z)$, is given by $(I - zT'U)^{-1}$.*

3.4 Basic Facts about the Hadamard QRW on $\mathbb{Z}^2$

In generating functions, the denominator of a rational generating function can give us much information about the parameter array associated with the generating function. In this section we take a closer look at the denominator of the Hadamard QRW on $\mathbb{Z}^2$ and note some properties of the generating functions of the Hadamard QRW that will be useful later in the paper. From (3.22), the generating functions for the Hadamard QRW have the following denominator:

$$x^2z^3y - yzx^2 - 2xz^4y + z^3x + xy^2z^3 - zx - xy^2z + 2yx + z^3y - zy$$
$$= (z-1)(z+1)(-2z^2yx + yzx^2 + xy^2z + zx + zy - 2yx)$$

Here we focus on the term $-2z^2yx + yzx^2 + xy^2z + zx + zy - 2yx$ which we denote by H. We show H has two properties which imply other facts about the poles of the generating functions for the Hadamard QRW, $F_{(c,c')}(x, y, z)$, which will be useful in the following sections. But first we need the following definition:

Definition 3.4. T_r^d will be the d dimensional complex torus centered at the origin with radius r. That is, if $\mathbf{x} \in T_r^d$, then for $1 \leq i \leq d$ we have $|x_i| = r$. We will use T_r in place of T_r^1. So T_1 is simply the complex unit circle.

Lemma 3.5. *If $x, y \in T_1$ and $H(x, y, z) = 0$, then we have $z \in T_1$.*

Proof. We first observe if $z = 0$ then $H(x, y, 0) = -2yx \neq 0$ and since we have $H(x, y, z) = 0$ then we conclude $z \neq 0$. So, we divide the equation $H(x, y, z) = 0$ through by xyz and rearrange to get

$$2\left(z + \frac{1}{z}\right) = x + \frac{1}{x} + y + \frac{1}{y}$$

The right hand side is real as both x and y are units in $\mathbb{C}$, and so the left hand side must also be real, but this is only possible if z is a unit in $\mathbb{C}$. Furthermore we have $\Re(z) = \frac{\Re(x)+\Re(y)}{2}$, so with $x = e^{ia}$, $y = e^{ib}$ and $H(x, y, z) = 0$ imply

$$z = \frac{cos(a) + cos(b)}{2} \pm i\sqrt{1 - \left(\frac{\cos(a) + \cos(b)}{2}\right)^2}$$

$\square$

Lemma 3.6. *$\frac{\partial H}{\partial z} = 0$ if and only if $\nabla H = 0$.*

Proof. Clearly if $\nabla H = 0$ then so does $\frac{\partial H}{\partial z} = 0$. We have $\frac{\partial H}{\partial z} = -4zyx + yx^2 + xy^2 + x + y$. Setting $\frac{\partial H}{\partial z} = 0$ and then dividing through by xy we get

$$x + \frac{1}{x} + y + \frac{1}{y} = 4z \tag{3.23}$$

If $x, y \in T_1$ then $x + \frac{1}{x} + y + \frac{1}{y}$ is real, so the only solutions to (3.23) are $x = y = z = 1$ and $x = y = z = -1$. We see the same holds for $\frac{\partial H}{\partial x} = 0$ and $\frac{\partial H}{\partial y} = 0$. So ∇H and $\frac{\partial H}{\partial z}$ only vanishes at $(1, 1, 1)$ and $(-1, -1, -1)$. $\square$

Corollary 3.7. *The singularities of $F_{(c,c')}(x,y,z)$ on T_1^3 are simple as long as $z \neq \pm 1$.*

Proof. The previous two lemmas give the only non-simple pole of $F_{(c,c')}(x,y,z)$ due to H are at $(1,1,1)$ and $(-1,-1,-1)$. $\square$

Lemma 3.8. *$F_{(c,c')}(x,y,xyz)$ is a power series (no negative powers).*

Proof. We simply note that every time an x^{-1} or y^{-1} is present, so is z. $\square$

Let $\tilde{F}_{(c,c')}(x,y,z) = F_{(c,c')}(x,y,xyz)$.

Lemma 3.9. *$\tilde{F}_{(c,c')}(x,y,z)$ converges absolutely for $|x|,|y| \leq 1$ and $|z| < 1$.*

Proof. Recall $F_{(c,c')}(x,y,z) = \sum_{t \geq 0, r, s} \psi_{(c,c')}((r,s),t)x^r y^s z^t$ where $0 \leq |\psi_{(c,c')}((r,s),t)| \leq 1$ as it is an amplitude, and furthermore we have $\psi_{(c,c')}((r,s),t) = 0$ if $|r| > t$ or $|s| > t$ as the walk cannot take more steps to the left, right, up, or down then the number of time steps that have passed. From this, we have

$$\begin{aligned}\tilde{F}_{(c,c')}(x,y,z) &= \sum_{t \geq 0, r, s} \psi_{(c,c')}((r,s),t)x^{r+t}y^{s+t}z^t \\ &= \sum_{t \geq 0} z^t \sum_{|r| \leq t, |s| \leq t} \psi_{(c,c')}((r,s),t)x^{r+t}y^{s+t}\end{aligned} \tag{3.24}$$

which is absolutely summable for $|x|,|y| \leq 1$ and $|z| < 1$, since

$$\sum_{|r| \leq t, |s| \leq t} |\psi_{(c,c')}((r,s),t)x^{r+t}y^{s+t}| \leq \sum_{|r| \leq t, |s| \leq t} 1 = 4t^2 + 4t + 1$$

Therefore with $|z| < 1$ we have

$$\sum_{t \geq 0, r, s} |\psi_{(c,c')}((r,s),t) x^{r+t} y^{s+t} z^t| \;\; \leq \;\; \sum_{t \geq 0} |z^t|(t^2 + 4t + 1) < \infty$$

We conclude that $\tilde{F}_{(c,c')}(x,y,z)$ has no poles in the region where $|x|, |y| \leq 1$ and $|z| < 1$. A rational function is holomorphic away from its poles, and a function holomorphic in the disk centered at the origin with radius s has absolutely convergent Taylor Series there. $\square$

4 Asymptotics from Generating Functions

Now that we have developed generating functions for QRWs on $\mathbb{Z}^2$, we want to extract information about the region in which the walk is most likely to be found. We want to know the region for which the $\psi_{(c,c')}(\mathbf{r},t)$ do not exponentially decay with respect to time t. Here we state and prove a general result for asymptotics for the coefficients of a particular kind of power series. We then broaden the scope of the result to a particular kind of Laurent series. The section ends with a general method for plotting the directions of non-exponential decay for these particular power and Laurent series. We will use the usual asymptotic notation. For real valued functions f and g, we use $f = O(g)$ to mean $\limsup_{x\to\infty} |f(x)|/|g(x)| < \infty$. We use the notation H_x to mean $\frac{\partial H}{\partial x}$.

4.1 Main Result for Asymptotic Behavior of $a_{\mathbf{r},t}$

We let $\mathbf{z}$ denote the $(d+1)$-dimensional vector $(\mathbf{x}, y)$. Let $F(\mathbf{z}) = \sum_{\mathbf{r},t\geq 0} a_{\mathbf{r},t}\mathbf{x}^{\mathbf{r}}y^t = \frac{G(\mathbf{z})}{H(\mathbf{z})}$ be a $(d+1)$-variate rational generating function for $a_{\mathbf{r},t}$, where $\mathbf{r} = (r_1,\ldots,r_d)$ and $r_1,\ldots,r_d,t \in \mathbb{Z}^+$. Call the set of points $\mathbf{z} \in \mathbb{C}^{d+1}$ such that $H(\mathbf{z}) = 0$ the *singular variety* of $F(\mathbf{z})$, and denote it by $\mathcal{V} \subseteq \mathbb{C}^{d+1}$. We call the region with $|z_j| \leq 1$ for all j the unit *polydisk* in $\mathbb{C}^{d+1}$, labeling it D. For a small neighborhood on T_1^d of $\mathbf{x}$, we pick a branch of the function Arg, so that $\mathrm{Arg}(x_j) = \theta_j$ where $x_j = e^{i\theta_j}$, is well defined. Denote $\mathrm{Arg}(\mathbf{x}) = \boldsymbol{\theta} = (\theta_1,\ldots,\theta_d)$ and $e^{i\boldsymbol{\theta}} = \mathbf{x}$. Define $\hat{\mathbf{r}} = (\mathbf{r},t)/t = (\hat{r}_1,\ldots,\hat{r}_d,1)$, and we view $\hat{\mathbf{r}}$ as an element of $\mathbb{R}^{d+1}$ and as an element of $\mathbb{RP}^d$. Thus we can view a cone K containing $\hat{\mathbf{r}}$ as a set in $\mathbb{R}^{d+1}$ or a set in $\mathbb{RP}^d$.

We now make the following hypotheses:

Hypothesis 4.1. *Both $G(\mathbf{z})$ and $H(\mathbf{z})$ are analytic on D, with $H(\mathbf{x}, y) \neq 0$ for $|x_i| \leq 1$ and $|y| < 1$.*

Hypothesis 4.2. *If $\mathbf{x} \in T_1^d$ and $H(\mathbf{x}, y) = 0$ then $|y| = 1$, and these singularities by previous hypothesis are minimal or nearest to the origin.*

Hypothesis 4.3. *$H_y(\mathbf{z}) = 0$ if and only if $\nabla H(\mathbf{z}) = 0$.*

Remark. Lemmas 3.5, 3.6, and 3.8 show these hypotheses to be true for H from Section 3.4. Moreover, these hypotheses will still hold for $(y - y_0)H$ where $y_0 \in T_1$.

Define $S_{\hat{\mathbf{r}}}$ to be the set of points $\mathbf{z} \in T_1^{d+1} \cap \mathcal{V}$ such that $(x_1 H_{x_1}, \ldots, x_d H_{x_d}, y H_y)$ is a complex multiple of $\hat{\mathbf{r}}$ and H_y does not vanish. Define $S'_{\hat{\mathbf{r}}}$ to be the set of points $\mathbf{z} \in T_1^{d+1} \cap \mathcal{V}$ where H_y vanishes and hence ∇H vanishes. For $\mathbf{z} \in S_{\hat{\mathbf{r}}}$, we know that $H_y(\mathbf{z})$ does not vanish, and hence the gradient of H does not vanish, so by the implicit function theorem, there is a neighborhood of $\mathbf{z}$ where y can be smoothly parameterized by $\mathbf{x}$. For $\mathbf{z} \in S_{\hat{\mathbf{r}}}$, we use $Y_{\mathbf{z}}(\mathbf{x})$ to denote the particular parametrization of y at the point $\mathbf{z}$. For each $\mathbf{z} \in S_{\hat{\mathbf{r}}} \cup S'_{\hat{\mathbf{r}}}$ we define neighborhoods of $\mathbf{x} \in T_1^d$ with diameter less than $1/2$ which are mutually disjoint, so that for each point $\mathbf{z}$ in $S_{\hat{\mathbf{r}}} \cup S'_{\hat{\mathbf{r}}}$ the $\mathbf{x}$-coordinates are contained in a unique neighborhood. We further restrict the neighborhoods of $\mathbf{x}$ such that for each $\mathbf{z} \in S_{\hat{\mathbf{r}}}$ the parameterization $Y_{\mathbf{z}}$ is smooth, and note that there may be multiple parameterizations associated to each $\mathbf{x}$.

For each $\mathbf{z} \in S_{\hat{\mathbf{r}}}$, define $\phi_{\mathbf{z}}(\boldsymbol{\theta}) = \sum_{j=1}^{d} \hat{r}_j \theta_j - \operatorname{Arg}(Y_{\mathbf{z}}(e^{i\boldsymbol{\theta}}))$ as long as $e^{i\boldsymbol{\theta}}$ stays in the neighborhood of $\mathbf{x}$ where $Y_{\mathbf{z}}$ is smooth. We define the *Hessian* of $\phi_{\mathbf{z}}$ at $\boldsymbol{\theta}$ to be the matrix

$$\left[\frac{\partial^2 \phi_{\mathbf{z}}}{\partial \theta_i \partial \theta_j}\right](\boldsymbol{\theta})$$

with corresponding eigenvalues of $\mu_{\mathbf{z},m}$ for $1 \leq m \leq d$.

Hypothesis 4.4. *For $\mathbf{z} \in S_{\hat{\mathbf{r}}}$ with Arg$(\mathbf{x}) = \boldsymbol{\theta}$, the Hessian of $\phi_{\mathbf{z}}$ is invertible at $\boldsymbol{\theta}$.*

Let

$$\psi_{\mathbf{z}}(\boldsymbol{\theta}) = Y_{\mathbf{z}}^{-1}(\mathbf{x}) \frac{G(\mathbf{x}, Y_{\mathbf{z}}(\mathbf{x}))}{\partial H(\mathbf{x}, Y_{\mathbf{z}}(\mathbf{x}))/\partial y}\bigg|_{x=e^{i\boldsymbol{\theta}}}. \tag{4.1}$$

Let $\mathtt{deg}(H) = \mathtt{deg}(H; \mathbf{z})$ be the least degree of the non-zero terms in the Taylor expansion of H at z. For $F = \frac{G}{H}$, define $\mathtt{deg}(F) = \mathtt{deg}(G) - \mathtt{deg}(H)$. Define

$$\mathcal{G} = \{\hat{\mathbf{r}} \in \mathbb{RP}^d \ : \ \text{the set } S_{\hat{\mathbf{r}}} \text{ is non-empty}\}. \tag{4.2}$$

Let $\log(D)$ be the *closed logarithmic domain* of F, that is to say the closure of the set $(w_1, w_2, \ldots, w_{d+1}) \in \mathbb{R}^{d+1}$ such that $F(e^{w_1}, e^{w_2}, \ldots, e^{w_{d+1}})$ converges. Define

$$\mathcal{G}' = \{v \ : \ v \text{ is the normal of a support hyperplane on } \log(D) \text{ at } 0\}. \tag{4.3}$$

Theorem 4.5. *Under hypotheses 4.1, 4.2, 4.3 and 4.4, with $t > 0$,*

a) *When* $\hat{\mathbf{r}}' \in \mathcal{G}$ *(i.e.,* $S_{\hat{\mathbf{r}}'}$ *is nonempty), then there is an open cone* K *containing* $\hat{\mathbf{r}}'$, *such that if we assume for all* $\hat{\mathbf{r}} \in K$ *and all* $\mathbf{z} \in S_{\hat{\mathbf{r}}}$ *that* $\mathtt{deg}(F) \geq \frac{d}{2} - 2$, *then*

$$a_{\mathbf{r},t} = \sum_{\mathbf{z} \in S_{\hat{\mathbf{r}}}} t^{-d/2} C_{\mathbf{z}}(\hat{\mathbf{r}}) e^{it\phi_{\mathbf{z}}(Arg(\mathbf{x}))} + O\left(t^{-\frac{d}{2}-1}\right) \tag{4.4}$$

where $C_{\mathbf{z}}(\hat{\mathbf{r}}) = \psi_{\mathbf{z}}(Arg(\mathbf{x})) \cdot (2\pi)^{d/2} \prod_{m=1}^{d}(-i\mu_{\mathbf{z},m})^{-1/2}$.

b) *If* $\hat{\mathbf{r}} \notin \mathcal{G}'$ *then there is an open cone* K *containing* $\hat{\mathbf{r}}$, *such for all* $(\mathbf{r}, t) \in K$ *we have* $a_{\mathbf{r},t} = O(e^{-ct})$.

c) *When* F *is the generating function for the Hadamard QRW on* $\mathbb{Z}^2$, *then the convex hull of* $\mathcal{G}$ *is equal to* $\mathcal{G}'$.

Conjecture 4.6. *The convex hull of* $\mathcal{G}$ *is equal to* $\mathcal{G}'$ *for every* F *satisfying the hypotheses of Theorem 4.5.*

Corollary 4.7. *For* $\hat{\mathbf{r}} \neq (\mathbf{0}, 1)$ *and* $F = \frac{G(\mathbf{x},y)/(1-y^2)}{H(\mathbf{x},y)}$, *Theorem 4.5 holds as long as* $\frac{G}{H}$ *satisfies the hypotheses. In particular, the Theorem 4.5 holds for the Hadamard Quantum Random Walk generating functions, once Hypothesis 4.4 is verified.*

4.2 Proof of part (a) of Theorem 4.5

Note that Theorem 3.5 and Corollary 3.7 found in [PW02] are the conclusion of Theorem 4.5(a), but under the stricter assumption that $|\mathcal{V} \cap \partial D| < \infty$.

4.2.1 Multivariate Cauchy Integral

Let $\{Y_1(\mathbf{x}), \ldots, Y_k(\mathbf{x})\}$ be the set of y parameterizations such that $H(\mathbf{x}, Y_j(\mathbf{x})) = 0$ when $\mathbf{x} \in T_1^d$, and hence by hypothesis $|Y_j(\mathbf{x})| = 1$. We shall refer to $Y_j(\mathbf{x})$ as just Y_j. Define

$$\psi_j(\mathbf{x}) = Y_j^{-1} \frac{G(\mathbf{x}, Y_j)}{\partial H(\mathbf{x}, Y_j)/\partial y}. \tag{4.5}$$

Lemma 4.8. *We have*

$$a_{(\mathbf{r},t)} = O(1+\epsilon)^{-t} + \frac{1}{(2\pi i)^d} \sum_j \left[\int_{T_1^d} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t} \psi_j(\mathbf{x}) d\mathbf{x} \right].$$

Here $\sum_j \left[\int_{T_1^d} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t} \psi_j(\mathbf{x}) d\mathbf{x} \right]$ *denotes* $\sum_{N \in \mathcal{C}} \int_N \left(\sum_j \left[\int_{T_1^d} \chi_N \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t} \psi_j(\mathbf{x}) d\mathbf{x} \right] \right)$ *where* $\mathcal{C}$ *and* χ_N *form a partition of unity of* T_1^d *such that the* Y_j *parameterizations hold.*

Proof. We begin with the multivariate Cauchy integral. By hypothesis, $F(\mathbf{z})$ has no singularities when $\mathbf{z} \in T_1^d \times T_{1-\epsilon}$. To find $a_{\mathbf{r},t}$, we integrate around $(\mathbf{0}, 0)$ through the region with no singularities using the positive orientation.

$$\begin{aligned}
a_{\mathbf{r},t} &= \frac{1}{(2\pi i)^{d+1}} \int_{T_1^d \times T_{1-\epsilon}} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} y^{-t-1} \frac{G(\mathbf{x}, y)}{H(\mathbf{x}, y)} dy d\mathbf{x} \\
&= \frac{1}{(2\pi i)^{d+1}} \int_{T_1^d \times T_{1+\epsilon}} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} y^{-t-1} \frac{G(\mathbf{x}, y)}{H(\mathbf{x}, y)} dy d\mathbf{x} \\
&\quad + \frac{1}{(2\pi i)^{d+1}} \int_{T_1^d} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} \left[\int_{T_{1-\epsilon}} y^{-t-1} \frac{G(\mathbf{x}, y)}{H(\mathbf{x}, y)} dy - \int_{T_{1+\epsilon}} y^{-t-1} \frac{G(\mathbf{x}, y)}{H(\mathbf{x}, y)} dy \right] d\mathbf{x}
\end{aligned} \tag{4.6}$$

For the first integral in (4.6), since hypothesis 4.2 ensures that $H(\mathbf{x}, y) \neq 0$ on $T_1^d \times T_{1+\epsilon}$, we have

$$\left| \int_{T_1^d \times T_{1+\epsilon}} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} y^{-t-1} \frac{G(\mathbf{x}, y)}{H(\mathbf{x}, y)} dy d\mathbf{x} \right| \leq (1+\epsilon)^{-t} \sup_{T_1^d \times T_{1+\epsilon}} \left| \frac{G(\mathbf{x}, y)}{H(\mathbf{x}, y)} \right| \cdot v(T_1^d \times T_{1+\epsilon})$$

where v is the volume in the $(d+1)$ dimensional sense. Thus the first integral is exponentially decaying with respect to t; that is, it is $O(e^{-ct})$ for some positive constant c.

The difference of integrals,

$$\int_{T_{1-\epsilon}} y^{-t-1}\frac{G(\mathbf{x},y)}{H(\mathbf{x},y)}dy - \int_{T_{1+\epsilon}} y^{-t-1}\frac{G(\mathbf{x},y)}{H(\mathbf{x},y)}dy, \tag{4.7}$$

found in the brackets of (4.6) is univariate as $\mathbf{x}$ is fixed by the outer integral. Hence (4.7) is the integral on the boundary of an annulus containing the unit torus. This integral can be reduced to a sum of residues via the univariate Cauchy integral formula. Except on the the set of points where $\nabla H = 0$, using the parameterizations Y_j, the residue for a given $\mathbf{x}$ at Y_j of $y^{-t-1}\frac{G(\mathbf{x},y)}{H(\mathbf{x},y)}$ is given by

$$y^{-t-1}\frac{G(\mathbf{x},y)}{\partial H(\mathbf{x},y)/\partial y}\bigg|_{y=Y_j}.$$

Thus applying the Cauchy integral formula to (4.7),

$$\int_{T_{1-\epsilon}} y^{-t-1}\frac{G(\mathbf{x},y)}{H(\mathbf{x},y)}dy - \int_{T_{1+\epsilon}} y^{-t-1}\frac{G(\mathbf{x},y)}{H(\mathbf{x},y)}dy = 2\pi i\sum_j Y_j^{-t}\psi_j(\mathbf{x}). \tag{4.8}$$

Therefore using (4.8) in (4.6) with a partition of unity of T_1^d such that the Y_j parameterizations hold in each neighborhood, away from the measure zero set where $\nabla H = 0$, gives

$$\begin{aligned} a_{\mathbf{r},t} &= O(1+\epsilon)^{-t} + \frac{1}{(2\pi i)^{d+1}}\int_{T_1^d}\mathbf{x}^{-\mathbf{r}-\mathbf{1}}\left[2\pi i\sum_j Y_j^{-t}\psi_j(\mathbf{x})\right]d\mathbf{x} \\ &= O(1+\epsilon)^{-t} + \frac{1}{(2\pi i)^{d}}\sum_j\left[\int_{T_1^d}\mathbf{x}^{-\mathbf{r}-\mathbf{1}}Y_j^{-t}\psi_j(\mathbf{x})d\mathbf{x}\right] \end{aligned} \tag{4.9}$$

□

4.2.2 Oscillatory Integrals

The integrals in the final sum in (4.9) are actually oscillating integrals. Recall that for $\mathbf{x} = (x_1, \ldots, x_d)$ we have $\mathrm{Arg}(\mathbf{x}) = (\theta_1, \ldots, \theta_d) = \boldsymbol{\theta}$. So for $\mathbf{x} \in T_1^d$ and $Y_j(\mathbf{x}) \in T_1$,

$$\mathbf{x}^{-\mathbf{r}} Y_j^{-t} = e^{i(-\mathbf{r}\cdot \mathrm{Arg}(\mathbf{x}) - t\mathrm{Arg}(Y_j))} = e^{-it(\frac{\mathbf{r}}{t}\cdot\boldsymbol{\theta} + \mathrm{Arg}(Y_j))}.$$

We therefore define

$$\phi_{j,\mathbf{r}/t}(\theta_1, \ldots, \theta_d) = -\frac{\mathbf{r}}{t}\cdot\boldsymbol{\theta} - \mathrm{Arg}(Y_j). \tag{4.10}$$

This yields

$$\mathbf{x}^{-\mathbf{r}} Y_j^{-t} = e^{it\phi_{j,\mathbf{r}/t}(\boldsymbol{\theta})}.$$

Lastly, define $\widetilde{\psi}_j(\boldsymbol{\theta}) = \psi_j(e^{i\boldsymbol{\theta}})$. Therefore

$$\mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t}\psi_j(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}} e^{it\phi_{j,\mathbf{r}/t}(\boldsymbol{\theta})}\widetilde{\psi}_j(\boldsymbol{\theta}). \tag{4.11}$$

We will make this more precise shortly, but if we are on a small enough region of T_1^d so that a chosen branch of Arg is well defined, then using the change of variables of $\mathbf{x}$ to $\boldsymbol{\theta}$ we see that sum (4.9) becomes

$$\frac{1}{(2\pi)^d}\sum_j \int e^{it\phi_{j,\mathbf{r}/t}(\boldsymbol{\theta})}\widetilde{\psi}_j(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

This form of an integral can be found in [Ste93] as an oscillatory integral. So we turn to some results on such integrals.

For a real-valued function ϕ defined in a neighborhood of a point $x_0 \in \mathbb{R}^n$, we call x_0 a *critical point* if

$$(\nabla\phi)(x_0) = \left(\frac{\partial\phi}{\partial x_1}, \cdots, \frac{\partial\phi}{\partial x_n}\right)\Big|_{x=x_0} = 0.$$

Proposition 4.9. *[Ste93, Chap.8.2 prop. 4] Suppose ψ is smooth (on $\mathbb{R}^n$), has compact support, and that ϕ is a smooth real-valued function (on $\mathbb{R}^n$) that has no critical points in the support of ψ. Then*

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx = O(\lambda^{-N})$$

as $\lambda \to \infty$, for every $N \geq 0$.

Suppose ϕ has a critical point at x_0. If the symmetric $n \times n$ matrix (the Hessian)

$$\left[\frac{\partial^2\phi}{\partial x_i \partial x_j}\right](x_0) \tag{4.12}$$

is invertible, then call the critical point x_0 *nondegenerate*. Otherwise, call the critical point x_0 *degenerate*.

Proposition 4.10. *[Ste93, Chap.8.2 prop. 6] Suppose $\phi(x_0) = 0$, and ϕ has a non-degenerate critical point at x_0. If ψ is supported in a sufficiently small neighborhood of x_0, then*

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx \sim \lambda^{-n/2}\sum_{j=0}^{\infty} a_j\lambda^{-j}, \quad \text{as } \lambda \to \infty,$$

with

$$a_0 = \psi(x_0) \cdot (2\pi)^{n/2}\prod_{j=1}^{n}(-i\mu_j)^{-1/2},$$

where $\mu_1, \ldots, \mu_n$ *are the eigenvalues of the matrix* (4.12)*. These asymptotics hold in the sense that, for all nonnegative integers* N *and* r*,*

$$\left(\frac{d}{d\lambda}\right)^r \left[I(\lambda) - \lambda^{-n/2} \sum_{j=0}^{\infty} a_j \lambda^{-j}\right] = O(\lambda^{-r-(N+1)}), \quad \textit{as } \lambda \to \infty.$$

Remark. The $O(\lambda^{-\mathbf{r}-(N+1)})$ term is uniform up to the $(N+1)$-partial derivatives of ϕ, ψ, and μ_j^{-1}.

The above propositions will be used with $\phi = \phi_{j,\mathbf{r}/t}$. To apply these propositions we must first find the critical points of $\phi_{j,\mathbf{r}/t}$.

4.2.3 Critical Points

Definition 4.11 (direction map, [PW02, Definition 2.2]). Suppose $(\mathbf{x}, y)$ has all coordinates nonzero. Define $\mathbf{dir}(\mathbf{x}, y)$ to be the equivalence class of (complex) scalar multiples of the vector $(x_1 H_1, \ldots, x_d H_d, y H_y)$. Thus, $\mathbf{dir}$: $(\mathbb{C}^*)^{d+1} \to \mathbb{CP}^d$.

For $\mathbf{z} \in \mathcal{V}$, if the gradient of H at $\mathbf{z}$ does not vanish, then $\mathbf{z}$ is a simple pole of F. The following Lemma 4.12, tells us that for $\mathbf{z} \in \mathcal{V} \cap T_1^{d+1}$ (these are *locally minimal* points according to [PW02]) which are simple poles, $\mathbf{dir}(\mathbf{z})$ contains an all real nonnegative vector, namely $(x_1 H_1/(y H_y), \ldots, x_d H_d/(y H_y), 1)$.

Lemma 4.12. *[PW02, Lemma 2.1] Let* $(\mathbf{x}, y)$ *be a simple pole of* F *and suppose that* $y H_y$ *does not vanish there. If* $(\mathbf{x}, y)$ *is locally minimal then for all* $j \leq d$*, the* *quantity $x_j H_j/(yH_y)$ is real and nonnegative.*

Along the lines of [PW02, Lemma 4.3], we develop the following proposition.

Proposition 4.13. *If $(\mathbf{r}, t) \in \mathbf{dir}(\mathbf{x}, Y_j(\mathbf{x}))$ with $(\mathbf{x}, Y_j) \in (T_1^d \times T_1) \cap \mathcal{V}$ a simple pole, then $\nabla \phi_{j,\mathbf{r}/t}(\boldsymbol{\theta}) = 0$ and hence $(\mathbf{x}, Y_j)$, under the correspondence $\mathbf{x} = e^{i\boldsymbol{\theta}}$, is a critical point of $\phi_{j,\mathbf{r}/t}$ in the direction $(\mathbf{r}, t)$.*

Proof. Let $f(\mathbf{x}) = \frac{\mathbf{r}}{t} \cdot \log(\mathbf{x}) + \log(Y_j)$. For $1 \leq k \leq d$ we have

$$t f_k(\mathbf{x}) = \frac{r_k}{x_k} + \frac{t\frac{\partial Y_j}{\partial x_k}}{Y_j}. \tag{4.13}$$

By the definition of $(\mathbf{r}, t) \in \mathbf{dir}(\mathbf{x}, Y_j)$ and $(\mathbf{x}, Y_j)$ being a simple pole, we have $(\mathbf{r}, t) = c(x_1 H_1, \ldots, x_d H_d, Y_j H_y(Y_j))$, and thus we have for all k that $r_k/(x_k H_k) = t/(Y_j H_y) = c$, where c is some nonzero constant. Hence

$$c^{-1} t f_k = H_k + H_y \frac{\partial Y_j}{\partial x_k}. \tag{4.14}$$

Now we also know that $H(\mathbf{x}, Y_j(x)) = 0$ by definition of Y_j. We take the derivative of $H(\mathbf{x}, Y_j(x))$ with respect to x_k to get

$$H_k + H_y \frac{\partial Y_j}{\partial x_k} = 0$$

Thus we conclude that $\nabla f(\mathbf{x}) = \mathbf{0}$. Now by (4.10) we have $-i f(\mathbf{x}) = \phi_{j,\mathbf{r}/t}(\boldsymbol{\theta})$ under the map $\theta_i \to e^{i\theta_i} = x_i$, therefore $\frac{\partial \phi_{j,\mathbf{r}/t}}{\partial \theta_k}(\theta) = x_k f_k(\mathbf{x}) = 0$. Thus $\nabla \phi_{j,\mathbf{r}/t}(\boldsymbol{\theta}) = \mathbf{0}$. $\square$

We get the following corollary for the critical points of $\phi_{j,\mathbf{r}/t}$.

Corollary 4.14. *If $H_y \neq 0$, then the critical points of $\phi_{j,\mathbf{r}/t}$ are exactly the points $(\mathbf{x}, Y_j)$ under the usual correspondence between $\mathbf{x}$ and $\boldsymbol{\theta}$, where the direction $(\mathbf{r}, t)$ is parallel to*

$$\log_{\nabla} H(\mathbf{x}, Y_j) = \left(x_1 \cdot \frac{\partial H}{\partial x_1}(\mathbf{x}, Y_j), \ldots, x_d \cdot \frac{\partial H}{\partial x_d}(\mathbf{x}, Y_j),\ y \cdot \frac{\partial H}{\partial y}(\mathbf{x}, Y_j) \right).$$

We can find all the critical points, and now we isolate them with a partition of unity.

4.2.4 Finishing the Proof of Theorem 4.5(a)

The ψ_j will not be smooth at $(\mathbf{x}, Y_j)$ if and only if the gradient of H vanishes at $(\mathbf{x}, Y_j)$, and only at such points may the parameterization Y_j be non-smooth. So for a fixed $\mathbf{r}/t$, which is the same as fixing $\hat{\mathbf{r}}$, we define $S'_{\hat{\mathbf{r}}}$ to be the set of points $\mathbf{z}$ where the gradient vanishes, which by Hypothesis 4.3 is the same as H_y vanishing. We define $S_{\hat{\mathbf{r}}}$ to be the set of $\mathbf{z}$ such that $\mathbf{z} \notin S'_{\hat{\mathbf{r}}}$ and such that if $\boldsymbol{\theta} = \operatorname{Arg}(\mathbf{x})$, then $\phi_{j,\mathbf{r}/t}(\boldsymbol{\theta}) = 0$ for at least one j. Note by Corollary 4.14 that $S_{\hat{\mathbf{r}}}$ is the same set defined in section 4.1. Now we relabel $\phi_{j,\mathbf{r}/t}$ to $\phi_{\mathbf{z}}$ for $\mathbf{z} \in S_{\hat{\mathbf{r}}}$, as $\mathbf{z}$ can only be in one $S_{\hat{\mathbf{r}}}$ and defines the correct parameterization for y. Similarly we relabel $\tilde{\psi}_j$ to just $\psi_{\mathbf{z}}$. Recall Hypothesis 4.4 that the Hessian of $\phi_{\mathbf{z}}$ is invertible at $\boldsymbol{\theta}$ when $\mathbf{z} \in S_{\hat{\mathbf{r}}}$. This hypothesis gives that the set $S_{\hat{\mathbf{r}}}$ contains only nondegenerate critical points.

For all $\mathbf{x}$ such that for some y we have $(\mathbf{x}, y) \in S_{\hat{\mathbf{r}}} \cup S'_{\hat{\mathbf{r}}}$, we now define functions $\chi_{\mathbf{x}}$ and a function χ_0 on T_1^d to form a partition of unity. The image of any χ will be in $[0,1]$. Let the support of $\chi_{\mathbf{x}}$, denoted by $\mathbf{R}_{\mathbf{x}}$, be a neighborhood of $\mathbf{x}$ on T_1^d such that $\chi_{\mathbf{x}}$ is 1 near $\mathbf{x}$ and 0 outside of $\mathbf{R}_{\mathbf{x}}$. Furthermore, the diameter of $\mathbf{R}_{\mathbf{x}}$ is less than $1/2$ and $\mathbf{R}_{\mathbf{x}} \cap \mathbf{R}_{\mathbf{x}'} = \emptyset$ if $\mathbf{x} \neq \mathbf{x}'$. Define

$$\chi_0 = 1 - \sum_{\mathbf{x} \in S_{\hat{\mathbf{r}}} \cup S'_{\hat{\mathbf{r}}}} \chi_{\mathbf{x}}.$$

We have, by the definition of $\chi_{\mathbf{x}}$, that $\mathbf{R}_{\mathbf{x}}$ is small enough that we can choose a branch of Arg and log to make them all well defined. Let $\mathbf{R}_0$ be the region of T_1^d on which χ_0 is not zero.

We use a change of variables from $\mathbf{x}$ to $\boldsymbol{\theta}$ so that for $\bar{\mathbf{x}} \in \mathcal{CP}_{\mathbf{r}/t}$ we have

$$\frac{1}{(2\pi i)^d} \int_{T_1^d} \chi_{\mathbf{x}}(\mathbf{x}) \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t} \psi_j(\mathbf{x}) d\mathbf{x} = \frac{1}{(2\pi)^d} \int_{\mathbf{R}_{\mathbf{x}}} e^{it\phi_{\mathbf{z}}(\boldsymbol{\theta})} \chi_{\mathbf{x}}(e^{i\boldsymbol{\theta}}) \psi_{\mathbf{z}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

In the change of variables from $\mathbf{x}$ to $\boldsymbol{\theta}$ we have the cancelation of $\mathbf{x}^{-\mathbf{1}} i^{-d}$, and we freely think of $\mathbf{R}_{\mathbf{x}}$ as a region in $\boldsymbol{\theta}$-coordinates. Using the above we can now rewrite the sum in (4.9) as follows:

$$\begin{aligned}
&\frac{1}{(2\pi i)^d} \sum_j \left[\int_{T_1^d} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t} \psi_j(\mathbf{x}) d\mathbf{x} \right] \\
&\qquad = \quad \frac{1}{(2\pi)^d} \left[\sum_{\mathbf{z} \in S_{\hat{\mathbf{r}}}} \int_{\mathbf{R}_{\mathbf{x}}} e^{it\phi_{\mathbf{z}}(\boldsymbol{\theta})} \chi_{\mathbf{x}}(e^{i\boldsymbol{\theta}}) \psi_{\mathbf{z}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \right] &(4.15)\\
&\qquad + \quad \frac{1}{(2\pi i)^d} \sum_j \left[\int_{\mathbf{R}_0} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t} \chi_0(\mathbf{x}) \psi_j(\mathbf{x}) d\mathbf{x} \right] &(4.16)\\
&\qquad + \quad \frac{1}{(2\pi i)^d} \sum_{\mathbf{z} \in S'_{\hat{\mathbf{r}}}} \left[\int_{\mathbf{R}_{\mathbf{x}}} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t} \chi_{\mathbf{x}}(\mathbf{x}) \psi_{\mathbf{z}}(\mathbf{x}) d\mathbf{x} \right] &(4.17)
\end{aligned}$$

We now handle the three sums (4.15), (4.16), and (4.17) separately.

For (4.16), if we now partition $\mathbf{R}_0$ into small enough regions $\mathbf{R}_0^k$ so that Arg and log are well defined and the parameterizations Y_j are smooth, we can write this sum using the $\boldsymbol{\theta}$-substitution as

$$\frac{1}{(2\pi i)^d}\sum_j \left[\int_{\mathbf{R_0}} \mathbf{x}^{-\mathbf{r}-\mathbf{1}} Y_j^{-t}\chi_0(\mathbf{x})\psi_j(\mathbf{x})d\mathbf{x}\right]$$
$$= \frac{1}{(2\pi)^d}\sum_j \left[\sum_k \int_{\mathbf{R}_0^k} e^{it\phi_{j,\mathbf{r}/t}(\boldsymbol{\theta})}\chi_0(\boldsymbol{\theta})\tilde{\psi}_j(\boldsymbol{\theta})d\boldsymbol{\theta}\right]. \tag{4.18}$$

Now by the definition of χ_0 each one of the integrals on the right of (4.18) contains no critical points. Hence, by Proposition 4.9 with $\phi = \phi_{j,\mathbf{r}/t}$ and $\lambda = t$ the contribution from the $\mathbf{R}_0$ region is $O(t^{-N})$ for all $N \geq 0$. So as t grows, the contribution from χ_0 is negligible.

By the Homogenous Estimate from [BP07a], for $\mathbf{z} \in S'_{\hat{\mathbf{r}}}$ the contribution will be $O(t^{-3-\mathtt{deg}(F)})$. Thus, (4.17) will contribute at most $O(t^{-3-\mathtt{deg}(F)})$, and if $\mathtt{deg}(F) \geq \frac{d}{2} - 2$ this is at most $O(t^{-\frac{d}{2}-1})$.

Therefore, $a_{\mathbf{r},t}$ by Lemma 4.8 is equal to the sum (4.15) plus $O(t^{-\frac{d}{2}-1})$. Now for (4.15), we want to apply Proposition 4.10 with $\phi = \phi_{\mathbf{z}}$ and $\lambda = t$, as this sum is over the nondegenerate points where the $\psi_{\mathbf{z}}$'s are smooth. For these points, to apply Proposition 4.10 to

$$\int_{\mathbf{R_x}} e^{it\phi_{\mathbf{z}}(\boldsymbol{\theta})}\chi_{\mathbf{x}}(\boldsymbol{\theta})\psi_{\mathbf{z}}(\boldsymbol{\theta})d\boldsymbol{\theta}$$

we need $\phi_{\mathbf{z}}(\boldsymbol{\theta}) = 0$ at the critical point $\boldsymbol{\theta}$ corresponding to $\mathbf{x}$. We define $\phi_{\mathbf{z}}^{\boldsymbol{\theta}}(\boldsymbol{\theta}') = \phi_{\mathbf{z}}(\boldsymbol{\theta}') - \phi_{\mathbf{z}}(\boldsymbol{\theta})$, and therefore $e^{it\phi_{\mathbf{z}}(\boldsymbol{\theta}')} = e^{it\phi_{\mathbf{z}}^{\boldsymbol{\theta}}(\boldsymbol{\theta}')}e^{it\phi_{\mathbf{z}}(\boldsymbol{\theta})}$. Thus we pull the constant $e^{it\phi_{\mathbf{z}}(\boldsymbol{\theta})}$ out of the integral.

Next, from Proposition 4.10, letting $\phi = \phi_{\mathbf{z}}^{\boldsymbol{\theta}}$, $\psi = \chi_{\mathbf{x}}\psi_{\mathbf{z}}$, $\lambda = t$, $a_n = a_{\mathbf{z},n}(\hat{\mathbf{r}})$, and

$\boldsymbol{\theta}$ correspond to each $\mathbf{z} \in S_{\hat{\mathbf{r}}}$, the sum 4.15 becomes

$$\sum_{\mathbf{z} \in S_{\hat{\mathbf{r}}}} \left(t^{-d/2} \left(\sum_{n=0}^{\infty} a_{\mathbf{z},n}(\hat{\mathbf{r}}) t^{-n} \right) e^{it\phi_{\mathbf{z}}(\boldsymbol{\theta})} \right) + O\left(t^{-\frac{d}{2}-1} \right).$$

Taking only the leading term of the sum $t^{-d/2} \sum_{n=0}^{\infty} a_{\mathbf{z},n}(\hat{\mathbf{r}}) t^{-n}$ as the remaining terms are accounted for by $O(t^{-\frac{d}{2}-1})$ and recalling that $\boldsymbol{\theta} = \mathrm{Arg}(\mathbf{x})$, we conclude with $C_{\mathbf{z}}(\hat{\mathbf{r}}) = a_{\mathbf{z},0}(\hat{\mathbf{r}})$,

$$a_{\mathbf{r},t} = \sum_{\mathbf{z} \in S_{\hat{\mathbf{r}}}} t^{-d/2} C_{\mathbf{z}}(\hat{\mathbf{r}}) e^{it\phi_{\mathbf{z}}(\mathrm{Arg}(\mathbf{x}))} + O\left(t^{-\frac{d}{2}-1} \right).$$

And Theorem 4.5(a) is proved, where the uniformity in a neighborhood of $\hat{\mathbf{r}}'$ is inherent in Proposition 4.10 (see remark following the proposition).

4.3 Proof of part (b) of Theorem 4.5

We now wish to prove part b) of theorem 4.5. Recall, the *logarithmic domain of convergence* of F is the closure of the set $(w_1, w_2, \ldots, w_{d+1}) \in \mathbb{R}^{d+1}$ such that $F(e^{w_1}, e^{w_2}, \ldots, e^{w_{d+1}})$ converges. We denote this set as $\log(D)$. It is known that the logarithmic domain of convergence of F is convex [Hör90, Corollary 2.5.8].

Definition 4.15. A point $\mathbf{z} \in T_1^{d+1} \cap \mathcal{V}$ is a *cone point* for polynomial $H(\mathbf{z})$ if $H(\mathbf{z} + \mathbf{w}) = A(\mathbf{w}) + O(|\mathbf{w}|^3)$ as $\mathbf{w} \to 0$, where $A(\mathbf{w})$ is an irreducible homogeneous quadratic in $\mathbb{C}[w_1, \ldots, w_{d+1}]$.

Now let $\mathcal{G}'$ be the set of outward normals to support hyperplanes on the logarithmic domain of F at $\mathbf{0}$. These can be viewed as elements of $\mathbb{RP}^d$ by considering all scalar multiples of each vector.

Lemma 4.16. *If direction* $\hat{\mathbf{r}}$ *is not in* $\mathcal{G}'$*, then there exists a cone* K *about* $\hat{\mathbf{r}}$*, such that for all* $(\mathbf{r}, t) \in K$*,* $a_{\mathbf{r},t}$ *decays exponentially with respect to* t*.*

Proof. Suppose $\hat{\mathbf{r}}$ is not an outward normal to a support hyperplane to $\log(D)$ at $\mathbf{0}$. Then $\sup\{v \cdot \hat{\mathbf{r}} \ : \ v \in \log D\} > 0$. Let $v = (v_1, \ldots, v_{d+1}) \in \log D$ such that $\hat{\mathbf{r}} \cdot v = c > 0$. Now to find $a_{\mathbf{r},t}$ we consider the Cauchy multivariate integral over the torus T given by $|z_i| = e^{v_i}$. Since $v \in \log D$, T is completely contained in the domain of convergence of F. Thus

$$a_{\mathbf{r},t} = \int_T \mathbf{z}^{-t\hat{\mathbf{r}}} F(\mathbf{z}) d\mathbf{z} \leq |\mathbf{z}^{-t\hat{\mathbf{r}}}| \cdot \sup_{\mathbf{z} \in T} |F(\mathbf{z})| \cdot v(T).$$

Now we have $|\mathbf{z}^{-t\hat{\mathbf{r}}}| = e^{-t(\hat{\mathbf{r}} \cdot v)} = e^{-ct}$. Thus for some c' such that $0 < c' < c$, we have by continuity that in a conic neighborhood of $\hat{\mathbf{r}}$, any $(\mathbf{r}, t)$ in this neighborhood has $a_{(\mathbf{r},t)}$ being of order $O(e^{-c't})$. □

With the last lemma we now have part b) of theorem 4.5.

4.4 Proof of part (c) of Theorem 4.5

Definition 4.17. The map $\mathfrak{G} : \mathcal{V} \cap T_1^{d+1} \to \mathbb{RP}^d$, where $\mathfrak{G}(\mathbf{x}, y) = \mathbf{dir}(\mathbf{x}, y) \cap \mathbb{RP}^d$, is called the *Gauss map*, as it is related to the Gauss map for $\mathcal{V}$.

We see that the image of the Gauss map is the set $\mathcal{G}$, which is all the $\hat{\mathbf{r}}$ such that $S_{\hat{\mathbf{r}}}$ is non-empty.

Proof. (Part (c) of Theorem 4.5) Using the denominator of the generating function for Hadamard QRW on $\mathbb{Z}^2$, we can compute both $\mathcal{G}$ and $\mathcal{G}'$. In this case they are both the disk with radius $1/\sqrt{2}$ when viewed with last coordinate fixed to 1. To compute $\mathcal{G}$, we find all directions $(xH_x/(zH_z), yH_y/(zH_z), 1)$ where $(x, y, z) \in T_1^3 \cap \mathcal{V}$. We can do this using the parameterizations for z given at the end of the proof of Lemma 3.5. $\square$

Thus, the image of the Gauss map in the Hadamard case gives the direction $\hat{\mathbf{r}} \in \mathbb{RP}^d$ for which $a_{\mathbf{r},t}$ does not exponentially decay as $t \to \infty$.

4.5 Laurent Case and Proof of Corollary 4.7

Now we wish to relax the assumption that the r_j are non-negative, which was made at the beginning of this section. Let $F(x_1, \ldots, x_d, y) = \sum a_{\mathbf{r},t}\mathbf{x}^{\mathbf{r}}y^t$ be a Laurent series in which we allow the r_j to be negative. Suppose there exist integers $\alpha_j > 0$ for which any nonzero term $a_{\mathbf{r},t}\mathbf{x}^{\mathbf{r}}y^t$ satisfies $r_j + \alpha_j t \geq 0$ for all $1 \leq j \leq d$. In this case, write

$$\tilde{F}(\mathbf{x}, y) = F(\mathbf{x}, \mathbf{x}^\alpha y), \quad \tilde{G}(\mathbf{x}, y) = G(\mathbf{x}, \mathbf{x}^\alpha y), \quad \tilde{H}(\mathbf{x}, y) = H(\mathbf{x}, \mathbf{x}^\alpha y).$$

Corollary 4.18. *Suppose that $\tilde{F}$ satisfied all the hypotheses for Theorem 4.5. Then the results of Theorem 4.5 hold for F.*

Proof. Let $\mathbf{z} = (\mathbf{x}, y)$ be a nondegenerate critical point of F in the direction $(\mathbf{r}, t)$

and thus by Proposition 4.14, we have that $(\mathbf{r}, t)$ is parallel to $\log_{\nabla} H(\mathbf{z})$. Assume $(\mathbf{x}, y)$ is on the unit torus. The point $\tilde{\mathbf{z}} = (\mathbf{x}, \mathbf{x}^{-\alpha} y)$ is therefore on the pole variety of $\tilde{F}$ and on the unit torus. Furthermore (by the product rule for derivatives), $\log_{\nabla} \tilde{H}(\tilde{\mathbf{z}})$ will be parallel to $(\tilde{\mathbf{r}}, t) = (r_1 + \alpha_1 t, \ldots, r_d + \alpha_d t, t)$, so $\tilde{z}$ is a critical point for the direction $(\tilde{\mathbf{r}}, t)$. We have

$$a_{\mathbf{r},t} = [\mathbf{x}^{\mathbf{r}}, y^t] F(\mathbf{x}, y) = [\mathbf{x}^{\tilde{\mathbf{r}}}, y^t] \tilde{F}(\mathbf{x}, y) = \tilde{a}_{\tilde{\mathbf{r}},t}. \tag{4.19}$$

By assumption, asymptotics for the coefficients of $\tilde{F}$ in the direction $(\tilde{\mathbf{r}}, t)$ are given by Theorem 4.5. Thus under the usual map for $x \in T_1$ where $x \mapsto \theta$ such that $e^{i\theta} = x$,

$$a_{\mathbf{r},t} = \tilde{a}_{\tilde{\mathbf{r}},t} = \sum_{\tilde{\mathbf{z}}} C_{\tilde{\mathbf{z}}} e^{it\tilde{\phi}_{\tilde{\mathbf{z}}}(\mathrm{Arg}(\tilde{\mathbf{x}}))} + O(t^{-\frac{d}{2}-1}). \tag{4.20}$$

But the nondegenerate critical points $\tilde{z}$ for $\tilde{F}$ in direction $(\tilde{\mathbf{r}}, t)$ are just the nondegenerate critical points $(\mathbf{x}, y)$ on T^{d+1} of F in direction $(\mathbf{r}, t)$ with $\tilde{\mathbf{z}} = (\mathbf{x}, \mathbf{x}^{-\alpha} y)$ (as shown above). The sum in (4.20) under the substitution $(\mathbf{x}, \mathbf{x}^{\alpha} y)$ for $\tilde{\mathbf{z}}$ gives us the same result as Theorem 4.5 applied to F for the coefficient $a_{\mathbf{r},t}$. $\square$

In the special Laurent series case of F described above, Lemma 4.12 still holds but the quantity $x_j H_j / (y H_y)$ may be negative.

Proof of Corollary 4.7. The term $(1 - z^2)$ will add two more Y_j parameterizations to the result of Lemma 4.8. These parameterizations are the constant parameterizations of $Y = \pm 1$. Consider the parameterization $Y_a = 1$ corresponding to the factor $(1-z)$. Then we have from (4.10) that $\phi_{a,\mathbf{r}/t} = \frac{\mathbf{r}}{t} \cdot \boldsymbol{\theta}$. But now we notice that $\nabla\phi_{a,\mathbf{r}/t}$ only vanishes if $\mathbf{r} = \mathbf{0}$. Hence the factor will only contribute in the direction $\hat{\mathbf{r}} = (\mathbf{0}, 1)$. Similarly for the parameterization $Y_b = -1$ corresponding to the factor $(1+z)$. Thus for any $(\mathbf{r}, t)$ such that $\hat{\mathbf{r}} \neq (\mathbf{0}, 1)$, the results of $a_{\mathbf{r},t}$ will be obtained from Theorem 4.1 applied to just $\frac{G}{H}$. $\square$

Note that at critical points $\mathbf{z} = (\mathbf{x}, y)$ with $y^2 \neq 1$, the $1/(1-y^2)$ is present in $\psi_{\mathbf{z}}$ as a locally smooth factor.

5 Non-exponential Decay Plots of QRWs on $\mathbb{Z}^2$

5.1 General Plotting of Non-exponential Decay Region

Using the Gauss map, we develop a method to plot the relative directions $\hat{\mathbf{r}}$ of non-exponential decay for a function satisfying Theorem 4.5, Corollary 4.18 or Corollary 4.7. We first map $T^d \hookrightarrow \mathcal{V} \cap T^{d+1} \subset (\mathbb{C}^*)^{d+1}$ by $\mathbf{x} \mapsto (\mathbf{x}, Y_j)$ where $Y_j \in T^1$ such that $H(\mathbf{x}, Y_j) = 0$, and we denote by Y_j the multiple solutions of $H(\mathbf{x}, Y_j) = 0$ as we have all along. Then we apply $\mathfrak{G}$ which maps $\mathcal{V} \cap T^{d+1} \to \mathbb{RP}^d$. Lastly, take the resulting elements in $\mathbb{RP}^d$ and use the standard coordinate map $\mathbb{RP}^d \to \mathbb{R}^d$ which fixes the last coordinate of elements in $\mathbb{RP}^d$ to 1. The image of the final map gives the direction $\hat{\mathbf{r}}$.

5.2 Hadamard QRW on $\mathbb{Z}^2$

To apply the method described above to the generating functions $F_{(c,c')}(x, y, z)$ derived in Section 3.3, we will use Corollary 4.7, with $H(x, y, z) = -2z^2yx + yzx^2 + xy^2z + zx + zy - 2yx$ and thus ignoring the $(1 - z^2)$ term. We have from Lemmas 3.5, 3.6 and 3.8 that H meets Hypotheses 4.1, 4.2 and 4.3. For simplicity we just assume that the Hessians of the $\phi_{\mathbf{z}}$ are invertible. We pick $x, y \in T_1$ and find z such that $H(x, y, z) = 0$, then calculate $(\frac{xH_x}{zH_z}, \frac{yH_y}{zH_z})$ to plot the corresponding direction $(r/t, s/t)$. Using Matlab and $x = e^{ia}$, $y = e^{ib}$, we select a grid of a and b in the interval $[0, 2\pi]$, solve for z and plot the direction to get figure 2 below.

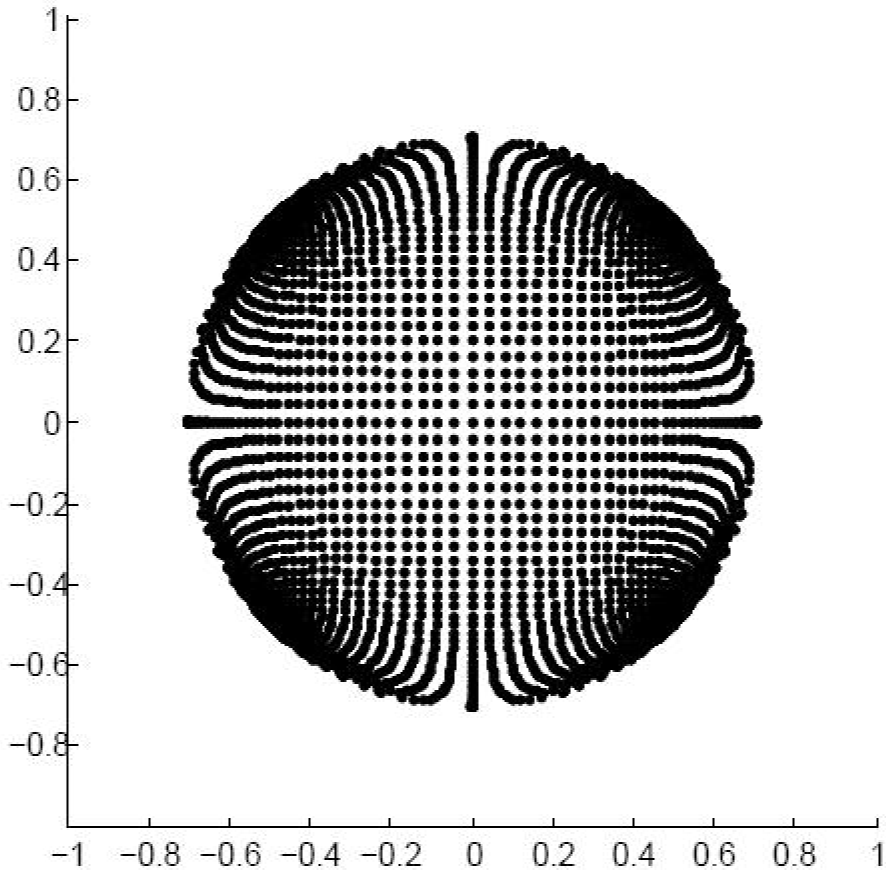


Figure 2: Plot of non-exponential decay directions $\mathbf{r}/t$.

The grid chosen for T^2 parameterized by $e^{i\theta}$ was evenly spaced on $[0, 2\pi]^2$, but as we see in the resulting figure, the map of the grid has denser dots in some locations. We compare figure 2 to figure 3 which is a simulation of the QRW run on Matlab for 200 steps. We plot the probability of the particle being at each point after the 200 steps, with point (200,200) being the starting location. The simulation was run using equal starting chiralities of 1/2. Note in figure 3 that every other grid point

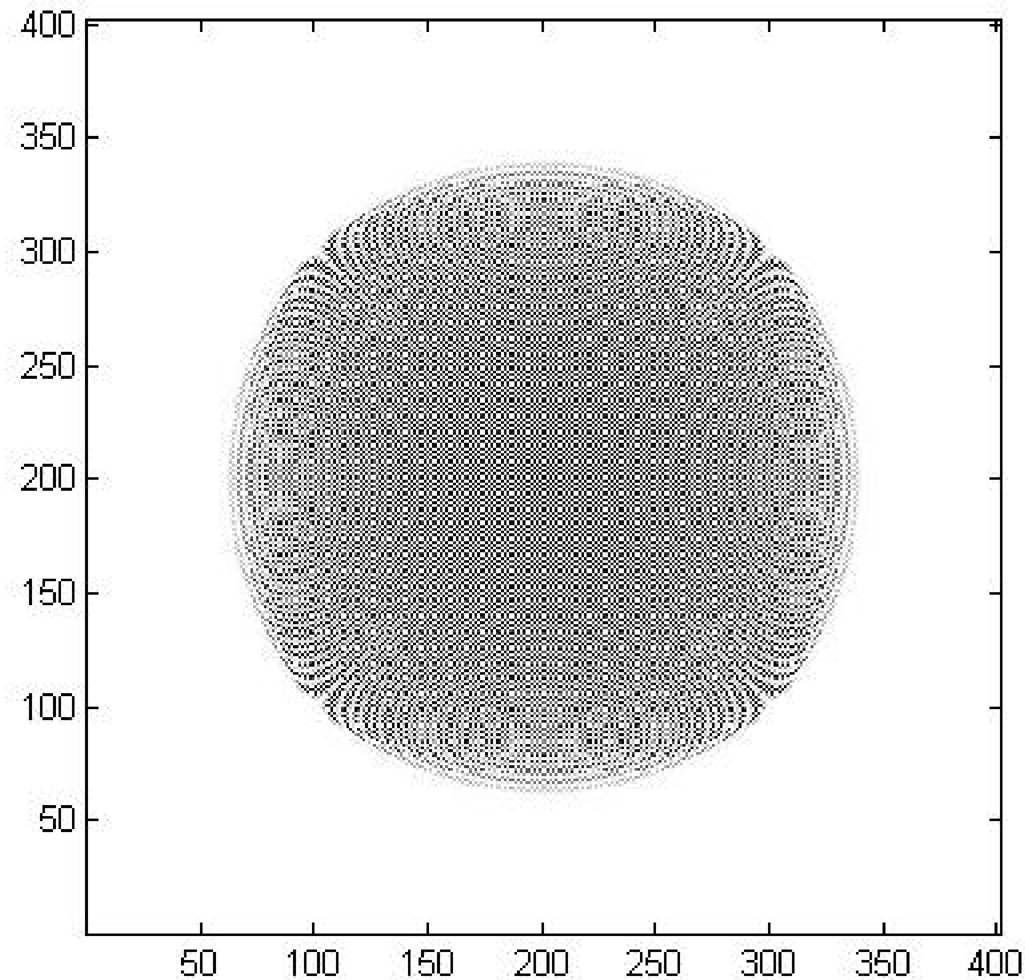


Figure 3: Hadamard QRW after 200 steps

is zero by definition of the walk.

As proven in section 4, the figures have the same general shape. Notice the northeast, northwest, southwest, and southeast edges of figure 3 contain the darkest regions which correspond to greater probability. In figure 2, the same edges have the greatest density of plotted points. This may not seem significant here, but in a moment we shall exam some more QRWs and their Gauss maps. There, we will see more clearly this relationship of higher density of dots in the Gauss map image corresponding to where the probability plot is darker.

As a side note, in the Hadamard QRW on $\mathbb{Z}^2$ case, the cone points themselves will only contribute $O(t^{-2})$ in any non-exponentially decaying direction as the homogenous degrees of $F_{(c,c')}$ at $(1,1,1)$ and $(-1,-1,-1)$ are just -1. This is easy to verify; for example, $F_{(R,R)}$ vanishes to degree 3 in the denominator at both points $(1,1,1)$ and $(-1,-1,-1)$, once for z^2-1 and twice in H. Also, the numerator vanishes to degree 2 at these points as one can easily check, by seeing that the gradient of the numerator vanishes at these points. So for main term asymptotics, we need only consider the sum in part (a) of Theorem 4.5. Also, at the origin the probability will be greater than predicted by the formula in Theorem 4.5 as we are using Corollary 4.7 and thus not including the contribution of the $1/(1-z^2)$ term that contributes to the direction $\hat{\mathbf{r}} = (0,0,1)$, which corresponds to the origin.

5.3 A Family of QRW on $\mathbb{Z}^2$

As discussed at the end of section 3.3, we can describe a QRW in general by any unitary matrix. Consider the following one parameter family of unitary matrices for

$0 \leq p \leq 1$:

$$B(p) = \begin{bmatrix} \frac{\sqrt{p}}{\sqrt{2}} & \frac{\sqrt{p}}{\sqrt{2}} & \frac{\sqrt{1-p}}{\sqrt{2}} & \frac{\sqrt{1-p}}{\sqrt{2}} \\ \frac{-\sqrt{p}}{\sqrt{2}} & \frac{\sqrt{p}}{\sqrt{2}} & \frac{-\sqrt{1-p}}{\sqrt{2}} & \frac{\sqrt{1-p}}{\sqrt{2}} \\ \frac{-\sqrt{1-p}}{\sqrt{2}} & \frac{\sqrt{1-p}}{\sqrt{2}} & \frac{\sqrt{p}}{\sqrt{2}} & \frac{-\sqrt{p}}{\sqrt{2}} \\ \frac{-\sqrt{1-p}}{\sqrt{2}} & \frac{-\sqrt{1-p}}{\sqrt{2}} & \frac{\sqrt{p}}{\sqrt{2}} & \frac{\sqrt{p}}{\sqrt{2}} \end{bmatrix} \tag{5.1}$$

We develop the generating functions for the family of QRWs corresponding to this family of unitary matrices using Theorem 3.3 from section 3.3. If we wish to apply the method from section 5.1, we need the denominator of these generating functions, which we get by taking the determinant of $(I - zT'B(p))$. The denominator of these generating functions is

$$\begin{aligned} H(x,y,z) \quad = \quad & \frac{1}{2xy}\Big(z^2x^2 - \sqrt{2p}z^3x - zx\sqrt{2p} + z^2 - \sqrt{2p}yz^3x^2 \\ & -\sqrt{2p}yzx^2 + 2yz^4x + 4pyz^2x + 2yx - z^3\sqrt{2p}y - zy\sqrt{2p} \\ & +y^2z^2x^2 - \sqrt{2p}y^2z^3x - \sqrt{2p}y^2zx + y^2z^2\Big). \end{aligned}$$

Now by definition of the QRW generating function, we know Lemma 3.8, shown in section 3.4 will still hold here. We therefore have the correct kind of Laurent series needed for Corollary 4.18. Assume the hypotheses of Theorem 4.5 hold. Then we can apply the Gauss map here just as we did for the Hadamard QRW above. We get the plot of non-exponential decay found in figures 4, 6 and 8 corresponding to $p = 1/3$, $p = 1/2$ and $p = 2/3$ respectively. Compare these plots with the simulations for 200 steps found in figures 5, 7 and 9.

As the simulations show, the general region of the particle's location is a 45 degree rotated square with slightly bowed out edges. And as expected in all figure pairs, the general shape of the region of the Gauss map is identical. But more surprising is that the pattern of the plotted dots on the interior of the Gauss map and the pattern seen in the simulation are the same. In particular, notice the four-pointed star in all the figures. As the p value increases the star becomes thinner in the simulation figures 4, 6, and 8, and this is reflected in the corresponding Gauss map images as well. The star in each simulation appears the same height, width, and thickness as it does in the corresponding Gauss map images. The four corners east, north, west, and south of all the simulations have darker regions corresponding to higher probability of the particle being there. We see the same phenomenon in the Gauss map images.

Overall, the Gauss map images seem not only to show the general shape of the region of non-exponential decay but also show, via higher density of plotted dots, where the particle is more likely to be found. That is, the Gauss map also gives information on the magnitude of the amplitude. This is unexpected as the work in section 4 regarding the Gauss map image, only showed that the general region would be plotted and said nothing as to the density of plotted points corresponding to greater amplitude. Thus the Gauss map may contain information about the $C_{\mathbf{z}}(\hat{\mathbf{r}})$ found in Theorem 4.5.

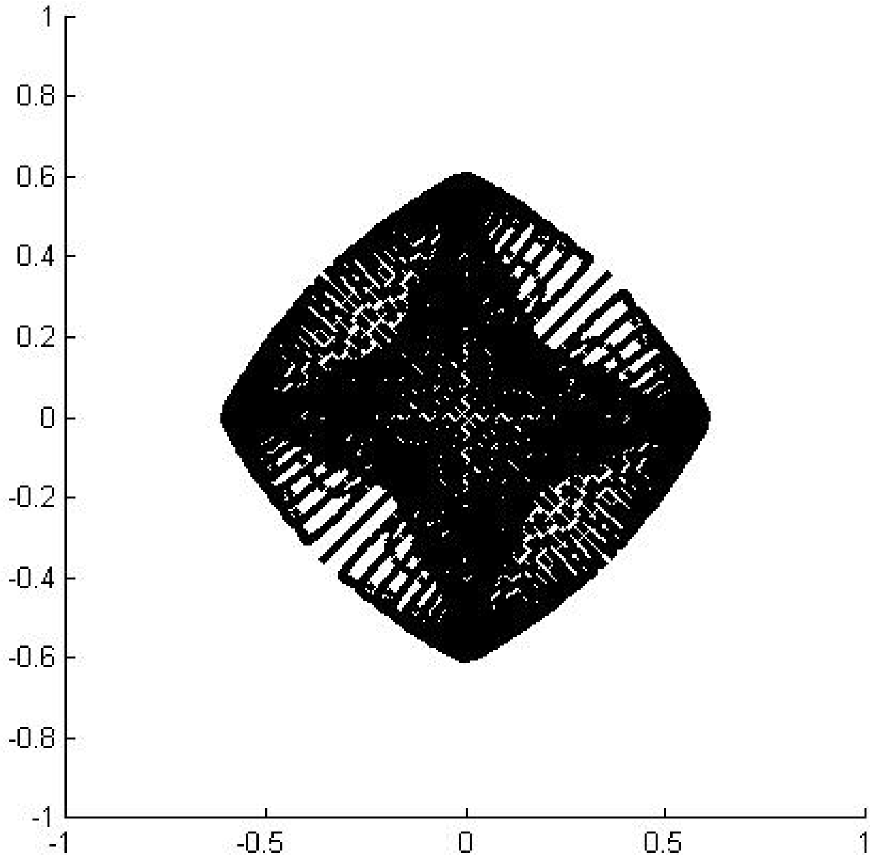
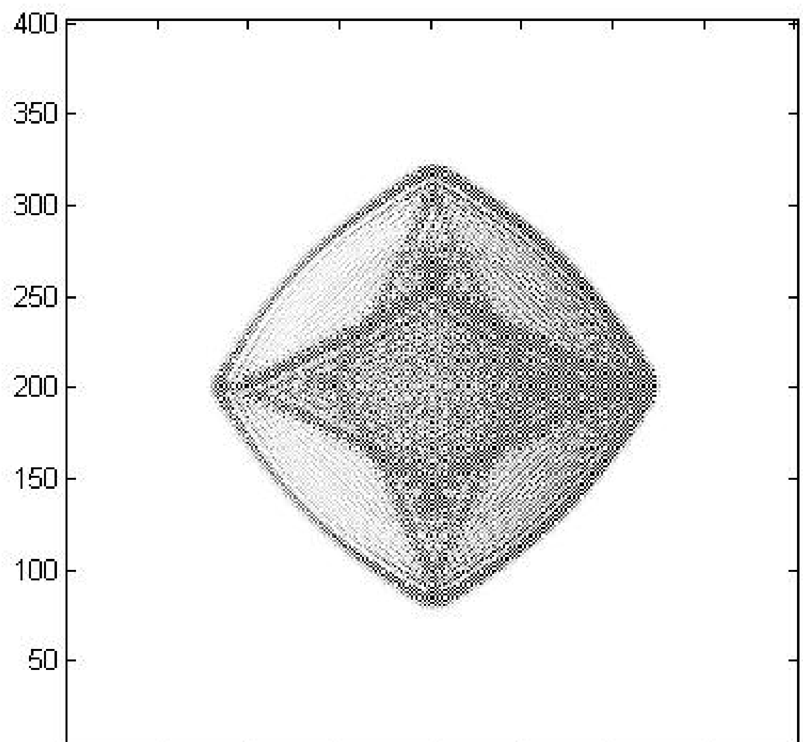


Figure 4: Non-exponential decay plot with $p = 1/3$.



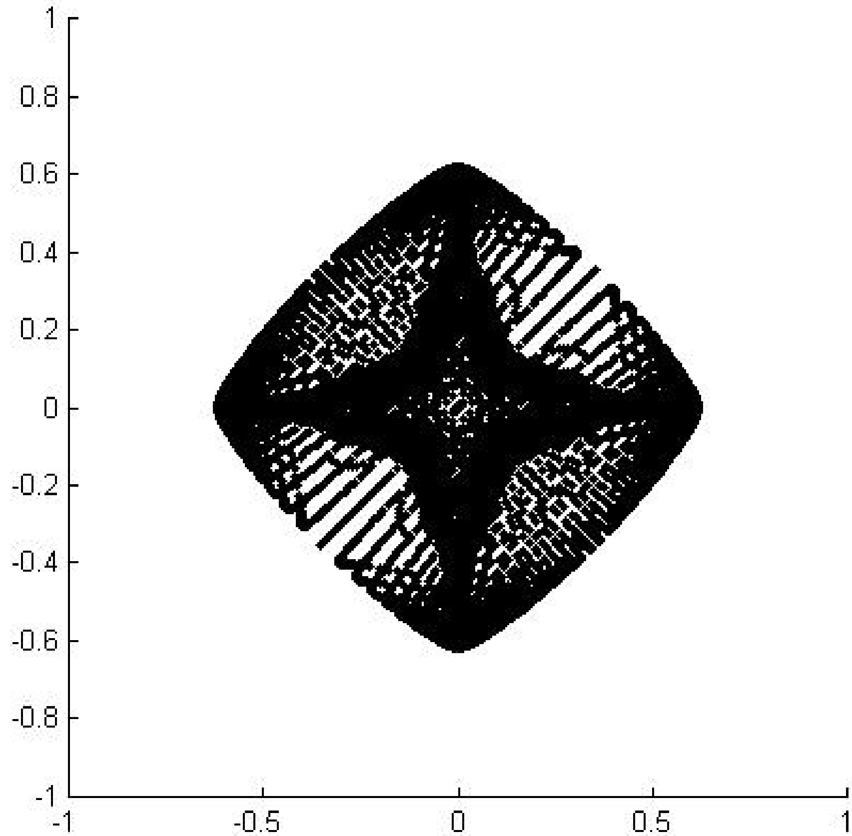
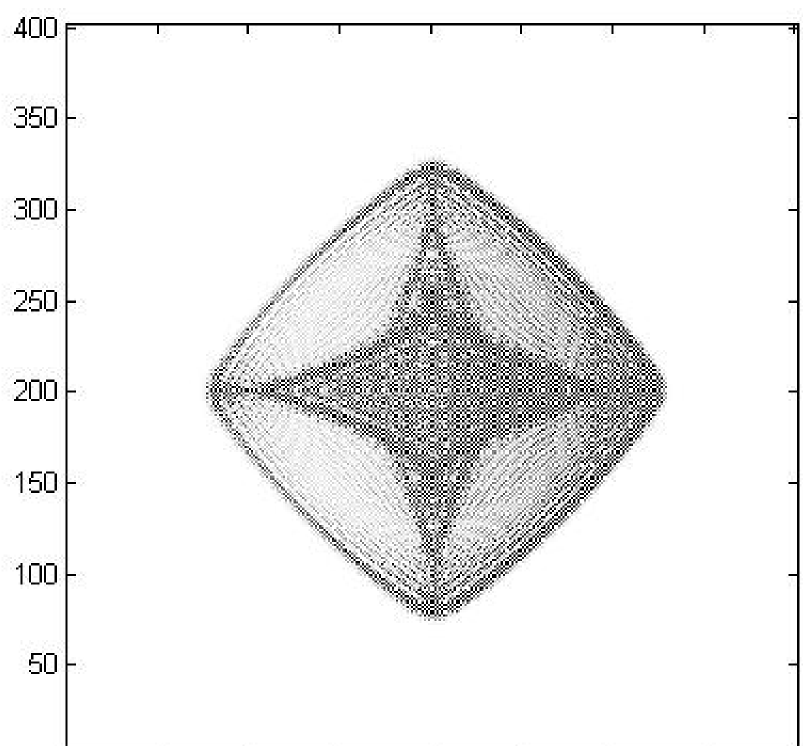


Figure 6: Non-exponential decay plot with $p = 1/2$.



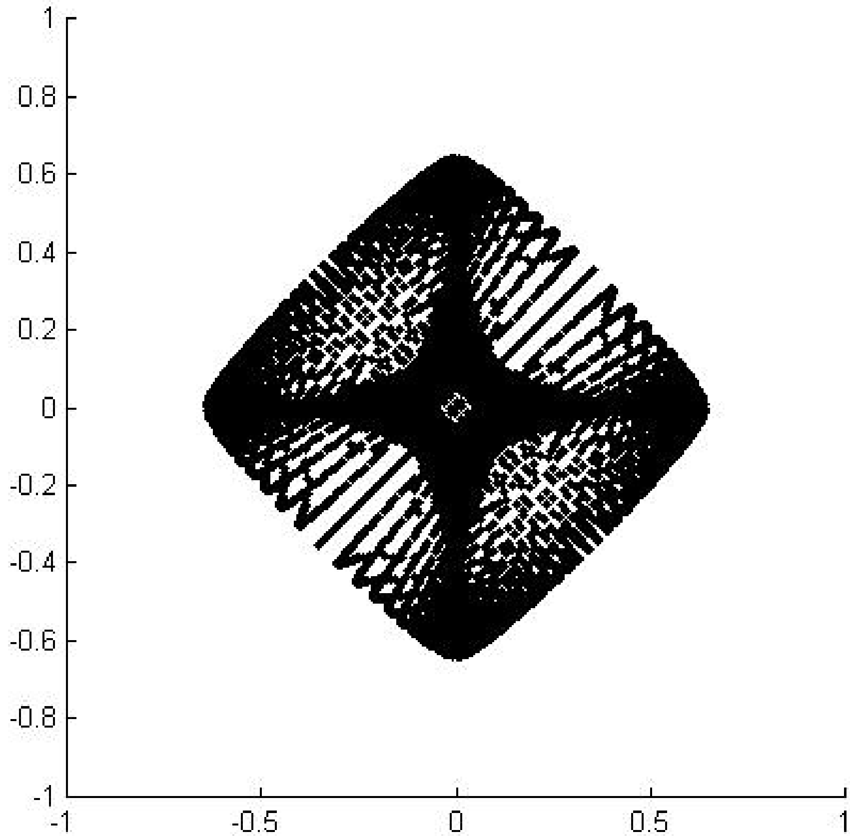
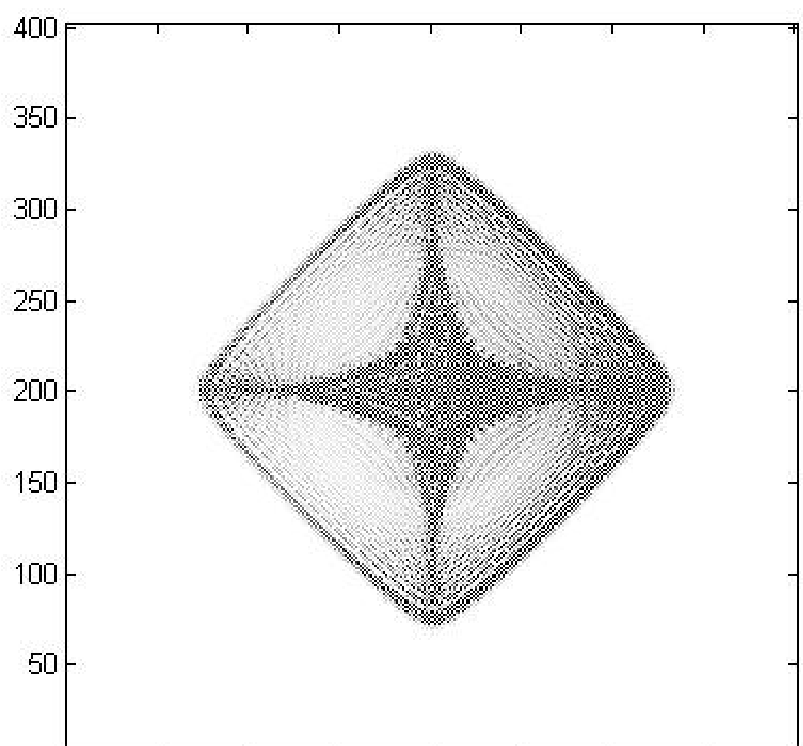


Figure 8: Non-exponential decay plot with $p = 2/3$.



6 Conclusion

The results seen in section 5 not only showed that the method for finding non-exponential decay worked, but also appeared to predict amplitudes. Let H be the common denominator for the generating functions of a QRW on $\mathbb{Z}^2$, with $\mathcal{V}$ being the singular variety of these generating functions. The Gauss map, $\mathfrak{G} : \mathcal{V} \cap T_1^3 \to \mathbb{RP}^2$, is already defined, but we now view it as $\mathfrak{G} : \mathcal{V} \cap T_1^3 \to \mathbb{R}^3$ with constant z-coordinate of 1. We define the function J to be the absolute value of the determinat of the matrix of partial derivatives (the Jacobian). For a given map f the density of dots found in the image of f applied to a uniform density grid of dots on the domain of f, is given by $1/J(f)$.

In particular, in section 5, we took a uniform grid of $[0, 2\pi]^2$ and mapped it to $T_1^3 \cap \mathcal{V}$, via the maps $S_j(x, y) = (x, y, Y_j)$, where j corresponds to a particular parameterization. We then used the Gauss map (ignoring the $z = 1$ coordinate) to get an image in $\mathbb{R}^2$. So the density of the dots in the image of our plots is given by

$$\frac{1}{\sum_j J(\mathfrak{G} \circ S_j)}.$$

We saw the density of the plotted dots was closely related to the darkness (probability) of the simulations. This would indicate that the $C_{\mathbf{z}}$ found in Theorem 4.1 is proportional to one over the Jacobian of our map. The S_j were used to cover the space $T_1^3 \cap \mathcal{V}$, and it is the belief of this author that these maps do not contribute much the plot densities seen. Thus, with the above notation and the results seen in section 5, we have the following conjecture:

Conjecture 6.1. *Suppose* $\mathbf{z} \in T_1^3 \cap \mathcal{V}$ *and* $\mathfrak{G}(\mathbf{z}) = \hat{\mathbf{r}}$*. Then for the* $C_{\mathbf{z}}(\hat{\mathbf{r}})$ *found in Theorem 4.5 in the* $d = 2$ *case, we have*

$$C_{\mathbf{z}}(\hat{\mathbf{r}}) = \frac{c}{J(\mathfrak{G})(\mathbf{z})}.$$

Overall, we have that the Gauss map gives us the region of non-exponential decay, but moreover, if the above conjecture is true then we would be able to get exact asymptotics from the Gauss map. As the need for understanding of QRWs grows, the Gauss map and method given in this thesis may prove very useful.

A Appendix: Matlab Code

Below is all the code used to plot directions of non-exponential decay and to run simulations.

A.1 Hadamard Case

To plot the non-exponential decay region we used the following function:

```
function returnVal = plotDirection(N,R)

step = 2*pi/(N+1) + (sqrt(2)/(2 + 2.5*N))*R;
newplot
hold on
a =[step:step:2*pi - (step/(5*N))];
b = [step:step:2*pi - (step/(5*N))];
L = length(b);

for k = 1:L
    for l = 1:L

       [x,y,z1,z2] = getTriples(a(k),b(l));
       point1 = gradF(x,y,z1);
       point2 = gradF(x,y,z2);

       plot(point1(1),point1(2), '.', 'MarkerEdgeColor', 'k');
       plot(point2(1),point2(2), '.', 'MarkerEdgeColor', 'k');
    end
end
```

The above function, takes `N` and `R` as plotting parameters. `N` controls how many points are plotted and `R`, if set to 1 causes the selected plotting grid in $[0, 2\pi] \times [0, 2\pi]$ is perturbed to be picked with irrational coordinates, while if `R` is 0 then the selected points are all rational. The two interior function calls are below.

```
function [x,y,z1,z2] = getTriples(a,b)

x = exp(i*a);
y = exp(i*b);

z1 = ((cos(a)+cos(b))/2) + i*(sqrt(1-((cos(a) + cos(b))/2)^2));
z2 = ((cos(a)+cos(b))/2) - i*(sqrt(1-((cos(a) + cos(b))/2)^2));
%end of getTriples

function point = gradF(x,y,z)

a = x*(-2*y*z^2 + z + z*y^2 + 2*x*y*z - 2*y);
b = y*(-2*x*z^2 + z + z*x^2 + 2*x*y*z - 2*x);
c = z*(-4*x*y*z +y +x +x*y^2 + y*x^2);

point(1) = real(a/c);
point(2) = real(b/c);
%end of gradF
```

A.2 Family of QRW Case

To plot the non-exponential decay region we used the following function:

```
function returnVal = plotDirectionFamilyB(N,r,R)

step = 2*pi/(N+1) + (sqrt(2)/(2 + 2.5*N))*R;
newplot
hold on
a = [step:step:2*pi - (step/(5*N))];
b = [step:step:2*pi - (step/(5*N))];
L = length(b);

for k = 1:L
    for l = 1:L

       [x,y,z1,z2,z3,z4] = getTriplesFamilyB(a(k),b(l),r);

       point1 = gradFfamilyB(x,y,z1,r);
       point2 = gradFfamilyB(x,y,z2,r);
       point3 = gradFfamilyB(x,y,z3,r);
       point4 = gradFfamilyB(x,y,z4,r);
```

```
        plot(point1(1),point1(2), '.', 'MarkerEdgeColor', 'k');
        plot(point2(1),point2(2), '.', 'MarkerEdgeColor', 'k');
        plot(point3(1),point3(2), '.', 'MarkerEdgeColor', 'k');
        plot(point4(1),point4(2), '.', 'MarkerEdgeColor', 'k');
    end
end
```

The N and R parameters are just as in the Hadamard Case. The r parameter take the value p from section 5.3. The internal functions are below.

```
function [x,y,z1,z2,z3,z4] = getTriplesFamilyB(a,b,p)

x = exp(i*a);
y = exp(i*b);

%the z below are the roots of H interms of x and y, solved my matlab
%the actual expressions are extremly long.
z1 = 1/8*(y*2^(1/2)*p^(1/2)+x*2^(1/2)*p^(1/2)+x^2*2^(1/2)*p^(1/2)*y+...
z2 = 1/8*(y*2^(1/2)*p^(1/2)+x*2^(1/2)*p^(1/2)+x^2*2^(1/2)*p^(1/2)*y+...
z3 = 1/8*(y*2^(1/2)*p^(1/2)+x*2^(1/2)*p^(1/2)+x^2*2^(1/2)*p^(1/2)*y+...
z4 = 1/8*(y*2^(1/2)*p^(1/2)+x*2^(1/2)*p^(1/2)+x^2*2^(1/2)*p^(1/2)*y+...
%end of getTriplesFamilyB

function point = gradFfamilyB(x,y,z,p)

%these are the xHx, yHy, and zHz computations
a = x*((1/2)*z*(-1+x)*(x+1)*(z*y^2+z-z^2*y*2^(1/2)*p^(1/2)
    -y*2^(1/2)*p^(1/2))/(x^2*y));
b = y*((1/2)*z*(y-1)*(y+1)*(z*x^2+z-x*z^2*2^(1/2)*p^(1/2)
    -x*2^(1/2)*p^(1/2))/(x*y^2));
c = z*((1/2)*(2*z*x^2-y*2^(1/2)*p^(1/2)-x*2^(1/2)*p^(1/2)
    +8*x*z*p*y+2*z-x*y^2*2^(1/2)*p^(1/2)+2*z*y^2
    -3*z^2*y*2^(1/2)*p^(1/2)-3*x*z^2*2^(1/2)*p^(1/2)
    +2*z*x^2*y^2+8*z^3*x*y-x^2*2^(1/2)*p^(1/2)*y
    -3*x*z^2*2^(1/2)*p^(1/2)*y^2
    -3*z^2*2^(1/2)*x^2*y*p^(1/2))/(x*y));

point(1) = real(a/c);
point(2) = real(b/c);
%end of getTriplesFamilyB
```

A.3 Simulation of QRW

Below is the Matlab code to generate the probabilities of of a given QRW, returned in a matrix.

```
function returnVal = QRWsim(N, U)
%N is number of steps, U is 4x4 unitary matrix of the QRW.

%These are the starting chirality amplitudes of the particle
%sum of the squares must be equal to 1
tright = 1/2;
tup = 1/2;
tdown = 1/2;
tleft = 1/2;

for t=1:N

    %chirality update
    right = (U(1,1)*tright + U(1,2)*tleft + U(1,3)*tup + U(1,4)*tdown);
    left = (U(2,1)*tright + U(2,2)*tleft + U(2,3)*tup + U(2,4)*tdown);
    up = (U(3,1)*tright + U(3,2)*tleft + U(3,3)*tup + U(3,4)*tdown);
    down = (U(4,1)*tright + U(4,2)*tleft + U(4,3)*tup + U(4,4)*tdown);

    %particle movement
    tup = [zeros(2*t-1,1) up zeros(2*t-1,1);zeros(2, 2*t+1)];
    tdown = [zeros(2, 2*t+1); zeros(2*t-1,1) down zeros(2*t-1,1)];
    tright= [zeros(2*t+1,2), [zeros(1,2*t-1); right; zeros(1,2*t-1)]];
    tleft = [[zeros(1,2*t-1); left; zeros(1,2*t-1)], zeros(2*t+1,2)];
end

%squaring amplitudes to get probabilities
aright = real(tright).^2 + imag(tright).^2;
aleft = real(tleft).^2 + imag(tleft).^2;
aup = real(tup).^2 + imag(tup).^2;
adown = real(tdown).^2 + imag(tdown).^2;

%adding up probabilities over chirality to get probability
%of particles location regardless of chirality
amp = aright + aleft + aup + adown;

returnVal = amp;
```

To generate a plot of locations based on probability, we do the following commands in Matlab.

```
EDU>> R = QRWsim(200,U);
```

```
EDU>> figure
EDU>> imagesc(-R, [-1/((150)^2), 0])
EDU>> colormap(gray)
```

The '150' in the third command can be adjusted to change the darkness of the plot.

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