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Analytic combinatorics in d variables: An overview

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Abstract. Let $F(\mathbf{Z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$ be a rational generating function in the d variables $Z_1, \ldots, Z_d$. Asymptotic formulae for the coefficients $a_{\mathbf{r}}$ may be obtained via Cauchy's integral formula in $\mathbb{C}^d$. Evaluation of this integral is not as straightforward as it is in the univariate case. This paper discusses geometric techniques that are needed for evaluation of these integrals and surveys classes of functions for which these techniques lead to explicit and effectively computable asymptotic formulae.

1. Introduction

A body of work in the last decade addresses the problem of estimating the coefficients of a multivariate generating function. The survey paper [**PW08**] is filled with examples and practical advice on how to extract asymptotics from such a generating function. It focuses on the theoretically easiest cases, wherein lie most known combinatorial examples. By contrast, the present overview is concerned with the theoretical structure of the enterprise and focuses on the boundaries of knowledge in the more difficult sub-cases. In particular, if we go beyond the *combinatorial* case (all coefficients are nonnegative real), as is necessary for instance with quantum random walks and with the diagonal applications in [**RW08**], then locating the dominating critical points can be much more difficult; see Section 1.4, and equation (2.4) and following. Central results from a number of papers are collected here. Proofs are included, sketched or omitted, according to the extent that they enhance understanding or give an alternative to the published argument. The context of the multivariate problem begins with a summary of the comparatively well understood univariate case.

1.1. Analytic combinatorics in one variable. Analytic combinatorics is the application of analytic methods to problems in combinatorial enumeration. This typically occurs as follows. A combinatorial class is defined whose size depends on a parameter $n = 0, 1, 2, 3, \ldots$. Let C_n denote the size of the n^{th} class. The description of the class, often recursive in nature, allows for the derivation of the

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generating function $F(z) := \sum_{n=0}^\infty a_n z^n$, where most often $a_n = C_n$ or $C_n/n!$. To apply analytic methods, the formal power series F must be convergent in some domain and the analytic properties of the function it represents must be understood, either because F is represented as some combination of elementary functions or because estimates on F or $|F|$ may somehow be obtained. Cauchy's integral formula expresses a_n exactly as an integral $(2\pi \mathrm{i})^{-1} \int z^{-n-1} F(z)\, \mathrm{d}z$. In order to evaluate the integral, complex contour methods must be brought to bear. The method of *singularity analysis*, described at length in the recent book [**FS09**], provides tools for analyzing this integral asymptotically as $n \to \infty$. The outcome depends on the behavior of F near its singularities of smallest modulus. If F is poorly behaved, for example failing to have any extension beyond its disk of convergence, circle methods such as Darboux' give asymptotic bounds on a_n. In the case of algebraic or logarithmic singularities, entire asymptotic developments may be carried out; see [**FO90**] for a description of how analytic information about F near its dominant singularity may be converted, nearly automatically, to asymptotic information about $\{a_n\}$. The subclass of rational functions is particularly simple, resulting in a limited number of types of asymptotic behavior: finite sums of terms $p_\gamma(n)\gamma^n$, where γ is a positive real number and p_γ is a polynomial (or, in the periodic case, a quasipolynomial).

1.2. Several variables. Consider now a generating function $F(Z_1, \ldots, Z_d) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$ in several variables, where $\mathbf{r}$ ranges over d-tuples of (usually nonnegative) integers and $\mathbf{Z}^{\mathbf{r}}$ stands for the monomial $Z_1^{r_1} \cdots Z_d^{r_d}$. Such a function arises in combinatorial applications from counting problems in which the class to be counted is naturally indexed by several parameters. Examples abound in which such generating functions turn out to be elementary functions; a long list may be found, for example, in [**PW08**]. Interesting univariate generating functions are at least of algebraic complexity, often transcendental. In the multivariate realm, interesting applications abound, with rational generating functions whose analyses are nontrivial. In fact a great proportion of combinatorial applications lead to rational functions or to no closed form at all. Indeed, the scarcity of compelling examples appears chiefly responsible for the slow development of multivariate generating function analysis outside the realm of rational generating functions. Furthermore, the main technical difficulties are already encountered with rational functions. In any case, the main thrust of multivariate analytic techniques to date is the rational case (though singularity analysis is sometimes possible for implicitly defined algebraic or D-finite generating functions), and this will be assumed until the last section of the present survey.

Let

$$F(\mathbf{Z}) = \frac{P(\mathbf{Z})}{Q(\mathbf{Z})} = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}} \tag{1.1}$$

be a d-variable rational generating function. Our objective is to estimate the coefficients $a_{\mathbf{r}}$ asymptotically. As in the univariate case, the coefficients $\{a_{\mathbf{r}}\}$ may be recovered from F via the multivariate Cauchy integral formula [**Hör90**, (2.2.3)]

$$a_{\mathbf{r}} = \frac{1}{(2\pi \mathrm{i})^d} \int_T \mathbf{Z}^{-\mathbf{r}} F(\mathbf{z}) \frac{\mathrm{d}\mathbf{Z}}{\mathbf{Z}}\,, \tag{1.2}$$

where the torus T is a product of sufficiently small circles about the origin in each coordinate and $\mathrm{d}\mathbf{Z}/\mathbf{Z}$ is $(Z_1 \cdots Z_d)^{-1}$ times the holomorphic volume form $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d$.

The purpose of this note is to give an overview of the analytic and geometric techniques necessary for the evaluation of (1.2). In more than one variable, the asymptotic analysis of generating functions is much less well understood than in the univariate case. Effective algorithms to produce asymptotics exist only for certain subclasses. Multivariate rational functions exhibit a wide range of asymptotic behaviors, which are not yet fully classified. Some of the complex contour methods that are necessary for the evaluation of this integral when $d = 1$ have higher-dimensional counterparts. These involve deforming the contour to pass through or near points of stationary phase on the pole variety $\mathcal{V} := \{\mathbf{Z} : Q(\mathbf{Z}) = 0\}$. Typically, asymptotics as $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}| \to \mathbf{r}_*$ will be determined by the geometry of $\mathcal{V}$ near a *dominating* point $\mathbf{Z}_*$ that depends on $\mathbf{r}_*$. The dominating point will be one of more points from a finite set of *critical* points for the log-linear function $-\sum_{j=1}^d (r_*)_j \log |Z_j|$ on $\mathcal{V}$. The easiest case is when $\mathbf{Z}_*$ is a smooth point of $\mathcal{V}$ and lies on the boundary of the domain of convergence of the power series $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$; such critical points are called *minimal* in the terminology of [**PW02, PW04**]. This case is analyzed in [**PW02**] via elementary methods. The case where $\mathcal{V}$ is a normal crossing in a neighborhood of $\mathbf{Z}_*$ is analyzed in [**PW04**] via elementary methods and in [**BP04**] via multivariate residues. When $\mathcal{V}$ has a singularity at $\mathbf{Z}_*$ that is not a local self-intersection, the analysis is more difficult. The subclass of products of powers of locally quadratic and locally linear divisors is analyzed in [**BP08**]; this class contains the generating functions arising in connection with some well known random tiling models [**CEP96, PS05**]. The elementary methods of [**PW02, PW04**] appear sometimes to be *ad hoc* but can be better understood in light of the apparatus introduced in later work, such as [**BP08**] and [**BBBP08**]. A second purpose of this note, therefore, is to re-cast the earlier analyses in a Morse-theoretic framework, thereby explaining the choices of reparametrization of the integrals and the forms of the results.

1.3. Notation. The following conventions are in place in order to achieve some consistency of notation and make the interpretations of variables visually obvious. The dimension is always d. Boldface is used for vectors and lightface for their coordinates; thus $\mathbf{Z} := (Z_1, \ldots, Z_d)$. A logarithmic change of variables is often required, in which case a corresponding lower case variable will be employed, for example $\mathbf{Z} = \exp(\mathbf{z}) := (\exp(z_1), \ldots, \exp(z_d))$; functions such as exp, log and absolute value, when applied to vectors, are taken coordinatewise. In the logarithmic coordinates, it is sometimes required to separate the real and imaginary parts of the vector $\mathbf{z}$; we shall denote these by $\mathbf{z} := \mathbf{x} + \mathrm{i}\mathbf{y}$. In the exponential space, we have no need for this and in low dimensions will sometimes use (X, Y, Z) in place of (Z_1, Z_2, Z_3). Unitized vectors will be denoted with a hat: $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$, where some norm is understood; a number of norms are useful depending on the application; instances are the euclidean norm, the L^1-norm and the pseudo-norm $|\mathbf{r}| := |r_d|$.

For asymptotics, the big-O, little-o and asymptotic equivalence notation $\sim$ will be employed; thus $f \sim g$ if and only if $f = (1 + o(1))g$. In the case of an asymptotic series development, $f \sim \sum_n b_n g_n$ will mean that for all N we have $\left| f - \sum_{n=0}^{N-1} b_n g_n \right| = O(g_N)$. This is slightly nonstandard because it allows some of the coefficients b_n to vanish, but we shall use it only when infinitely many are nonvanishing.

The most basic quantitative estimate on $\{a_\mathbf{r}\}$ is the exponential growth rate in a given direction. Define the rate in the direction $\mathbf{r}_*$ by

$$\beta(\mathbf{r}_*) := \lim |\mathbf{r}|^{-1} \log |a_\mathbf{r}| \tag{1.3}$$

if such a limit exists, where the limit is as $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}} \to \mathbf{r}_*$. One can force this to be well defined by taking a limsup instead of a limit. In fact there are a number of natural reasons, discussed in the next section, why one would not expect a limit to exist. In some cases, the limit will exist but fail to behave as expected for certain non-generic choices of $\mathbf{r}_*$. For this reason, we define a slightly more general limsup exponential growth rate by allowing $\hat{\mathbf{r}}$ to vary in a neighborhood $\mathcal{N}$ of $\mathbf{r}_*$ and taking the infimum over such neighborhoods:

$$\overline{\beta}(\mathbf{r}_*) := \inf_{\mathcal{N}} \limsup_{\mathbf{r}\to\infty,\, \hat{\mathbf{r}}\in\mathcal{N}} |\mathbf{r}|^{-1} \log |a_\mathbf{r}| \ . \tag{1.4}$$

EXAMPLE 1.1. The generating function $F(x, y) := (x-y)/(1-x-y)$ enumerates differences between consecutive binomial coefficients:

$$a_{ij} = \binom{i+j-1}{i-1} - \binom{i+j-1}{j-1} .$$

By symmetry, $a_{nn} = 0$ for all n, so that if $\mathbf{r}_*$ is the diagonal direction then $\beta(\mathbf{r}_*)$ exists and is equal to $-\infty$; whereas $\overline{\beta}(\mathbf{r}_*)$ is the logarithmic growth rate $\lim_{n\to\infty}(2n)^{-1} \log \binom{2n}{n} = \log 2$.

1.4. Organization of remainder of paper. Section 2 is concerned with computing the exponential rate. A function $\beta_Q(\mathbf{r}_*)$ is introduced that is always an upper bound for $\overline{\beta}$ (Proposition 2.2) and is often equal to $\overline{\beta}$. The formulation of β_Q and the dominating points $\mathbf{Z}_*$, as well as the proof of Proposition 2.2, are the central topics of Section 2.

Section 3 is concerned with the case where the dominating point $\mathbf{Z}_*$ is a smooth point of the variety $\mathcal{V}$. In this case explicit formulae are known for the leading term, and the entire asymptotic series is effectively computable. There are sometimes difficulties in selecting the dominating point from among a finite set of *critical points*, which we denote by $\mathsf{mincrit}$. Results are discussed in two special cases when $\mathcal{V}$ is everywhere smooth: the case where $d = 2$ and the *combinatorial* case where $a_\mathbf{r} \geqslant 0$. In the latter case, $\mathsf{mincrit}$ is always nonempty.

Section 4 catalogues a number of results that hold when $\mathbf{Z}_*$ is not a smooth point of $\mathcal{V}$. The next simplest geometry is that of a self-intersection or *multiple point*. This is discussed in Sections 4.1–4.3. After this, one might expect the next simplest case to be an algebraic curve ($d = 2$) with a cusp or other nore complicated singular point. However, as shown in [**BP08**], singularities in dimension 2 other than self-intersections are non-hyperbolic and cannot therefore contribute to the asymptotic expansion. Section 4.4 concentrates therefore on a three-dimensional example.

Returning to the problem of the exponential rate, Section 5 addresses the conjectured behavior of $\overline{\beta}$ in cases not covered by the results in the remainder of the paper. A modified version $\hat{\beta}_Q$ of β_Q is formulated that agrees with β_Q when the dominating critical point $\mathbf{Z}_*$ is minimal. Counterexamples from Sections 2–4 are catalogued, after which a weak converse to the upper bound in Proposition 2.2 (with β_Q replaced by $\hat{\beta}_Q$) is conjectured.

2. Exponential estimates

The crudest estimate of $a_{\mathbf{r}}$ that is still informative is the exponential rate of growth or decay. If we are unable to compute or estimate $\overline{\beta}$, then our quantitative understanding of $\{a_{\mathbf{r}}\}$ is certainly quite poor! In statistical mechanical models, $\beta(\mathbf{r}_*)$ has an interpretation as an *entropy function* (cf. [**Ell85**, Section II.4]), or *large deviation rate*. In combinatorial applications, β is the exponential growth rate of a *partition function*, this being a (weighted) sum over a combinatorial class. In this section, we discuss how to "read off" a rate function $\beta_Q(\mathbf{r}_*)$ from the denominator of $F = P/Q$, which is always to be an upper bound for $\overline{\beta}$ (Proposition 2.2) and is often equal to $\overline{\beta}$.

2.1. Multidimensional contour deformations. The integrand in (1.2) that we denote by $\omega := \mathbf{Z}^{-\mathbf{r}} F(\mathbf{Z})\, \mathrm{d}\mathbf{Z}/\mathbf{Z}$ is holomorphic on the domain

$$\mathcal{M} := \mathbb{C}^d \setminus (\mathcal{V} \cup \{\mathbf{Z} : Z_1 \cdots Z_d = 0\}) \, .$$

This is an open subset of $\mathbb{C}^d$, hence a real $(2d)$-manifold. Any holomorphic d-form has $\mathrm{d}\omega = 0$, from which it follows by Stokes' Theorem that $\int_{\mathcal{C}} \omega$ depends only on the homology class of $\mathcal{C}$ in $H_d(\mathcal{M})$. Intuitively, this says that any deformation of T within $\mathcal{M}$ leaves the integral unchanged; technically, homology is weaker than homotopy, which means that there are equivalent chains of integration not obtained via deformation, though these are not usually needed (this is briefly discussed in Section 5).

Define a height function $h = h_{\mathbf{r}_*} : \mathcal{M} \to \mathbb{R}$, depending on $\mathbf{r}_*$, as the dot product of $-\mathbf{r}_*$ with the coordinatewise log-modulus:

$$h(\mathbf{Z}) = -\mathbf{r}_* \cdot \log |\mathbf{Z}| \, .$$

Let $\mathcal{M}^a$ denote the set $\{\mathbf{Z} \in \mathcal{M} : h(\mathbf{Z}) \leqslant a\}$ of points up to height a and let $\iota = \iota_a$ denote the inclusion map of $\mathcal{M}^a$ into $\mathcal{M}$; for $b < a$, the homology group $H_d(\mathcal{M}^b)$ maps naturally into $H_d(\mathcal{M}^a)$ via ι_*. The following estimates for any $a > b$ are immediate.

$$\int_{\mathcal{C}} \omega = O(e^{a|\mathbf{r}|}), \quad \text{if } [\mathcal{C}] \in H_d(\mathcal{M}^a); \tag{2.1a}$$

$$\left| \int_{\mathcal{C}} \omega - \int_{\mathcal{D}} \omega \right| = O(e^{b|\mathbf{r}|}), \quad \text{if } [\mathcal{C}] = [\mathcal{D}] \in H_d(\mathcal{M}^a, \mathcal{M}^b) \, . \tag{2.1b}$$

We may interpret (2.1b) as saying that ω has well defined integrals on relative homology classes in $H_d(\mathcal{M}^a, \mathcal{M}^b)$, the value of the integral being taken to be an equivalence class under differences by $O(e^{b|\mathbf{r}|})$.

Our deformations will be guided by the following heuristic. The chief difficulty in estimating such an integral is that the integrand may be much bigger than the integral, with rapid oscillation leading to significant cancellation. To address these problems we therefore attempt to:

> Deform the chain of integration so as to minimize the maximum over the chain of the modulus of the integrand.

To obtain asymptotic estimates, we must do this simultaneously for many values $\mathbf{r}$. If $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}} \to \mathbf{r}_*$, then the exponential factor in the integral will be maximized where h is maximized. This suggests that we deform the contour so as to minimize $\sup_{\mathbf{Z} \in \mathcal{C}} h(\mathbf{Z})$. The deformations are constrained to lie in $\mathcal{M}$, that is, to avoid $\mathcal{V}$. Because of this, the minimizing contour is not achieved in $\mathcal{M}$ but is rather a limit of contours in $\mathcal{M}$, and touches $\mathcal{V}$ at one or more points $\mathbf{Z}_*$. A little calculus shows that $\mathbf{Z}_*$ must be a *stationary phase* point for h on $\mathcal{V}$, that is, $\mathrm{d}h$ restricted to $\mathcal{V}$ must vanish at $\mathbf{Z}_*$. At a stationary phase point, locally there is no cancellation due to oscillation, which justifies the prior assertion that minimizing the maximum modulus solves the oscillation problem as well. The remainder of the heuristic is that the minimized integral will be tractable. We shall see that this occurs in many families of cases, provided that we are careful with the interpretation of the integral on the limiting chain, which is not in $\mathcal{M}$.

2.2. Laurent series. It costs little and includes more applications if we extend the scope from power series to Laurent series. Formal Laurent series $\sum_{\mathbf{r} \in \mathbb{Z}^d} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$ are not as nice as formal power series because there is no well defined formal multiplication. However, for Laurent series expansions of rational functions, convergence will occur on certain domains, allowing formal operations to be defined by the corresponding analytic operations, and allowing analytic methods still to be used.

Corresponding to each rational function are a number of Laurent series, each convergent on a different domain. The following facts about Laurent series and amoebas of polynomials may be found in [**GKZ94**, Chapter 6]; for a complete proof of Cauchy's formula in a poly-annulus, see [**Ran86**] or [**Pem09a**, Section 8.2].

Let $\mathrm{ReLog}(\mathbf{Z}) := \log|\mathbf{Z}| = (\log|Z_1|, \ldots, \log|Z_d|)$ denote the coordinatewise log-modulus of $\mathbf{Z}$. If Q is any polynomial in d variables, let $\mathsf{amoeba}(Q)$ denote the image of its zero set $\mathcal{V}$ under ReLog. Two examples with $d = 2$ are given in Figure 1.

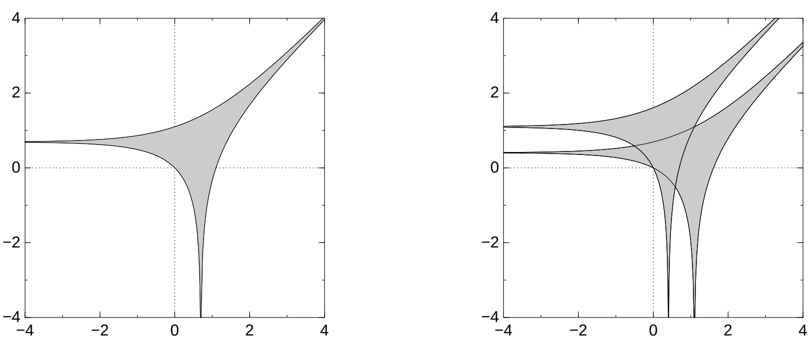

FIGURE 1. Two amoebae.
(a) $\mathsf{amoeba}(2 - X - Y)$. (b) $\mathsf{amoeba}((3 - X - 2Y)(3 - 2X - Y))$.

A general description of the amoeba and its relation to the various Laurent expansions for rational functions F with denominator Q is given by the following proposition.

PROPOSITION 2.1 ([**BP08**, Proposition 2.2]). *The connected components of $\mathbb{R}^d \setminus \mathsf{amoeba}(Q)$ are convex open sets. The components are in bijective correspondence with Laurent series expansions for $1/F$, as follows. For any Laurent series* *expansion of* $1/F$*, the open domain of convergence is precisely* $\text{ReLog}^{-1} B$ *where* B *is a component of* $\mathbb{R}^d \setminus \mathsf{amoeba}(Q)$*. Conversely, if* B *is such a component, a Laurent series* $1/F = \sum a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$ *convergent on* B *may be computed by the formula*

$$a_{\mathbf{r}} = \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathbf{T}} \mathbf{Z}^{-\mathbf{r}-\mathbf{1}} \frac{1}{F(\mathbf{Z})} \, \mathrm{d}\mathbf{Z} \, ,$$

where $\mathbf{T}$ *is the torus* $\text{ReLog}^{-1}(\mathbf{x})$ *for any* $\mathbf{x} \in B$*. Changing variables to* $\mathbf{Z} = \exp(\mathbf{z})$ *and* $\mathrm{d}\mathbf{Z} = \mathbf{Z} \, \mathrm{d}\mathbf{z}$ *gives*

$$a_{\mathbf{r}} = \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathbf{x}+\mathrm{i}\mathbf{t}} e^{-|\mathbf{r}|(\hat{\mathbf{r}} \cdot \mathbf{z})} \frac{1}{f(\mathbf{z})} \, \mathrm{d}\mathbf{z} \, , \tag{2.2}$$

where $f = F \circ \exp$ *and* $\mathbf{t}$ *is the torus* $\mathbb{R}^d/(2\pi\mathbb{Z})^d$*. We remark that the separation of* $\mathbf{r}$ *into* $|\mathbf{r}| \, \hat{\mathbf{r}}$ *will be convenient when we send* $\mathbf{r}$ *to* ∞ *with* $\hat{\mathbf{r}}$ *held roughly constant.*

The integral in (2.2) is of Fourier–Laplace type: $\int_{\mathcal{C}} e^{-\lambda\phi(\mathbf{z})} f(\mathbf{z}) \, \mathrm{d}\mathbf{z}$ for some phase function ϕ, amplitude function f and chain of integration $\mathcal{C}$. The term "Fourier–Laplace" is used because the distinction between Fourier-type integrals (ϕ is purely imaginary on $\mathcal{C}$) and Laplace-type integrals (ϕ is real and nonnegative on $\mathcal{C}$) vanishes when the chain $\mathcal{C}$ is deformed in complex d-space (see [**PW09**] for details). The coefficients $\{a_{\mathbf{r}}\}$ may be viewed as a kind of Fourier transform of the logarithmic generating function f. A rigorous version of this appears in [**BP08**, Section 6]. In the present paper we shall use this interpretation only to give a second viewpoint on various formulae. This is because of the considerable technical difficulties in dealing with nonconvergent Fourier integrals as well as with discretization of the Fourier parameter.

Given a component B of the complement of the amoeba of Q, and given a real unit vector $\mathbf{r}_*$, define

$$\beta_Q(\mathbf{r}_*) := \inf\{-\mathbf{r}_* \cdot \mathbf{x} : \mathbf{x} \in \overline{B}\}. \tag{2.3}$$

Unless the closure of B fails to be strictly convex, and as long as $-\mathbf{r}_* \cdot \mathbf{x}$ is bounded from below on B, there is a unique point of $\overline{B}$ at which this minimum is attained. This point $\mathbf{x}_*$ is called the *minimizing point* for $\mathbf{r}_*$ and lies on the common boundary of B and $\mathsf{amoeba}(Q)$. If we choose only contours of the form $\mathbf{x} + \mathrm{i}\mathbf{t}$ then h will be constant on our contour, and it is clear that the maximum height is minimized when $\mathbf{x} \to \mathbf{x}_*$. Sending $\mathbf{x} \to \mathbf{x}_*$ in (2.2) immediately implies the following proposition.

PROPOSITION 2.2. *Let* $F = P/Q$ *be a rational function, and let* $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$ *be the Laurent expansion of* F *corresponding to the component* B *of* $\mathsf{amoeba}(Q)^c$*. Define* $\beta_Q \in [-\infty, \infty)$ *by* (2.3)*. Then for any real unit vector* $\mathbf{r}_*$*,*

$$\overline{\beta}(\mathbf{r}_*) \leqslant \beta_Q(\mathbf{r}_*) \, .$$

REMARK 2.3. Computation of β_Q is semi-algebraic and hence effective; see, for instance, [**The02**, Section 2.2]).

2.3. Stratified spaces. Any algebraic variety admits a *Whitney stratification*. This is a partition into finitely many manifolds $\{S_\alpha\}$, called *strata*, satisfying two conditions. The first is that for distinct α, β, either S_α is disjoint from the closure of S_β or contained in it. The second is a condition on the limits of tangent spaces at points on the boundary of S_β; the reader is referred to [**PW09**, Definition 2.1] or [**GM88**, Section I.1.2] for a statement of this condition and its consequences.

Definition 2.4 (critical and minimal points). A smooth function f on a stratified space X is said to have a *critical* point at p if $\mathrm{d}f|_S(p) = 0$ where S is the stratum of X containing p. In other words, p must be a critical point for the restriction of f to S.

The set of critical points of each stratum is algebraic, with membership defined by the *critical point equations*. These say that $\mathbf{x} \in S$ and that $\nabla h(\mathbf{x})$ is orthogonal to the tangent space $T_{\mathbf{x}}(S)$. When S is a k-dimensional stratum and the ambient space has dimension n, there are $n-k$ equations for $\mathbf{x} \in S$ and k equations for $\nabla h(\mathbf{x}) \perp T_{\mathbf{x}}(S)$. Thus the set of critical points of S is zero-dimensional for any S. If S has complex structure, then $n = 2d$, $k = 2\ell$, and there are $d - \ell$ complex equations for $\mathbf{x} \in S$ and ℓ complex equations for $\nabla h(\mathbf{x}) \perp T_{\mathbf{x}}(S)$.

Given $F = P/Q$, a component B of $\mathsf{amoeba}(Q)^c$, a direction $\mathbf{r}_*$ and a minimizing point $\mathbf{x}_*$ as above, define the set of *minimal* critical points by

$$\mathsf{mincrit}(Q, \mathbf{r}_*) := \{\mathbf{Z}_* \in \mathcal{V} : \mathrm{ReLog}\, \mathbf{Z}_* = \mathbf{x}_* \text{ and } \mathbf{Z}_* \text{ is a critical point for } h_{\mathbf{r}_*} \text{ on } \mathcal{V}\}\,. \tag{2.4}$$

A consequence of Theorem 3.1 in the next section is that Proposition 2.2 is sharp when there is minimal critical point $\mathbf{Z}_*$ at which $\mathcal{V}$ is smooth.

Proposition 2.5. *Let $F = P/Q$ be a rational Laurent series and suppose there is a minimal critical point $\mathbf{Z}_*$ that is a smooth point of $\mathcal{V}$ with $P(\mathbf{Z}_*) \neq 0$. Then $\overline{\beta}(\mathbf{r}_*) = \beta_Q(\mathbf{r}_*)$.*

The following partial converse to this will be proved in Section 2.4 along with Theorem 2.8. Note that the computation of $\mathsf{mincrit}(\mathbf{r}_*)$ is algebraic and effective.

Proposition 2.6. *If $\mathsf{mincrit}(\mathbf{r}_*)$ is empty then $\overline{\beta}(\mathbf{r}_*) < \beta_Q(\mathbf{r})$.*

In the remaining cases, when $\mathsf{mincrit}(\mathbf{r}_*)$ contains no smooth point but is not empty, it can be difficult to tell whether $\overline{\beta} = \beta_Q$. In most cases this can be resolved by computing the normal cone $\mathbf{N}_* = \mathbf{N}_*(\mathbf{Z}_*)$ associated to each $\mathbf{Z}_* \in \mathsf{mincrit}$. A self-contained definition of this cone is too lengthy to give here, but the gist is as follows.

Definition 2.7 (normal cones). Let B be a component of $\mathsf{amoeba}(Q)^c$, let $\mathbf{x}_*$ be the minimizing point for $\mathbf{r}_*$ and let $\mathbf{Z}_* \in \mathsf{mincrit}(\mathbf{r}_*)$. Let $\mathbf{K} = \mathbf{K}(\mathbf{r}_*)$ denote the (geometric) tangent cone to B at $\mathbf{x}_*$, that is, $\mathbf{y} \in \mathbf{K}$ if and only if $\hat{\mathbf{y}}$ is the limit of normalized secants $(\mathbf{b} - \mathbf{x}_*)/\,|\mathbf{b} - \mathbf{x}_*|$. Denote by $\mathbf{N}_*(\mathbf{r}_*)$ the (outward) dual cone to $\mathbf{K}$, that is, the cone of vectors $\mathbf{v}$ such that $\mathbf{v} \cdot \mathbf{b} \leqslant 0$ for all $\mathbf{b} \in \mathbf{K}$. It is shown in [**BP08**, Definition 2.13] that for each $\mathbf{Z}_*$ there is a naturally defined cone $\mathbf{K}(\mathbf{Z}_*)$ that contains $\mathbf{K}(\mathbf{r}_*)$. Let $\mathbf{N}_*(\mathbf{Z}_*)$ denote the dual cone to $\mathbf{K}(\mathbf{Z}_*)$. Note that by duality, $\mathbf{N}_*(\mathbf{Z}_*) \subseteq \mathbf{N}_*(\mathbf{r}_*)$.

At the moment, the best known sufficient criterion for $\overline{\beta} < \beta_Q$ is given in the following result, proved in Section 2.4; see Section 5 for a discussion and conjecture as to how close this criterion is to being sharp.

Theorem 2.8 (upper bound). *Given $F, P, Q, B, \mathbf{r}_*$ and $\mathbf{x}_*$ as above, let $h = h_{\mathbf{r}_*}$ and $c := h(\mathbf{x}_*)$, and let $\mathcal{C}$ be the chain of integration in* (1.2). *Then*

(i) *There is an $\epsilon > 0$ such that the cycle $\mathcal{C}$ is homologous in $H_d(\mathcal{M}^{c+\epsilon}, \mathcal{M}^{c-\epsilon})$ to a sum of relative cycles $\mathcal{C}(\mathbf{Z}_*)$ supported in arbitrarily small neighborhoods of points $\mathbf{Z}_* \in \mathsf{mincrit}(\mathbf{r}_*)$ also satisfying $\mathbf{r}_* \in \mathbf{N}_*(\mathbf{Z}_*)$;*

(ii) *If* $\mathsf{mincrit}(\mathbf{r}_*)$ *is empty, or contains only points* $\mathbf{Z}_*$ *with* $\mathbf{r}_* \notin \mathbf{N}_*(\mathbf{Z}_*)$, *then there is an* $\epsilon > 0$ *and a neighborhood* $\mathcal{N}$ *of* $\mathbf{r}_*$ *such that*

$$|a_{\mathbf{r}}| = O\left(e^{(c-\epsilon)|\mathbf{r}|}\right),$$

as $\mathbf{r} \to \infty$ *with* $\hat{\mathbf{r}} \in \mathcal{N}$. *It follows in this case that*

$$\overline{\beta} < \beta_Q .$$

There are natural examples in which $\mathsf{mincrit}$ is empty; see for instance Example 3.6 below, or [**BP08**, Example 2.19]. On the other hand, nonnegativity of the coefficients $\{a_{\mathbf{r}}\}$ is sufficient to assure that part *(ii)* of Theorem 2.8 does not apply: $\mathsf{mincrit}$ is nonempty and contains a point $\mathbf{Z}_*$ with $\mathbf{r}_* \in \mathbf{N}_*(\mathbf{Z}_*)$.

PROPOSITION 2.9 (combinatorial case). *Suppose* $a_{\mathbf{r}} \geqslant 0$ *for all* $\mathbf{r}$. *Then for each* $\mathbf{r}_*$ *with unique minimizing* $\mathbf{x}_* \in \partial B$, *the real point* $\mathbf{Z}_* := \exp(\mathbf{x}_*)$ *is in* $\mathsf{mincrit}(\mathbf{r}_*)$ *and satisfies* $\mathbf{r}_* \in \mathbf{N}_*(\mathbf{Z}_*)$.

PROOF. Meromorphicity of F together with nonnegativity of F implies the presence of some pole of F on the torus $\mathrm{ReLog}^{-1}(\mathbf{x})$ for each $\mathbf{x} \in \partial B$; nonnegativity of the coefficients then implies that the positive real point $\mathbf{Z} := \exp(\mathbf{x})$ is in $\mathcal{V}$; see [**PW02**, Theorem 6.1] for further details on this step. We see from this that $\mathbf{Z}_* \in \mathcal{V}$, and in fact that the entire image of ∂B under the exponential map is in $\mathcal{V}$. It follows from the theory of hyperbolic functions that $\mathbf{K}(\mathbf{Z}_*) \subseteq \mathbf{K}(\mathbf{r}_*)$ and hence $\mathbf{N}_*(\mathbf{Z}_*) = \mathbf{N}_*(\mathbf{r}_*)$. It is automatic from the definition of $\mathbf{x}_*$ as a minimizing point for $\mathbf{r}_*$ that $\mathbf{r}_* \in \mathbf{N}_*(\mathbf{r}_*)$. Therefore, $\mathbf{r}_* \in \mathbf{N}_*(\mathbf{Z}_*)$, as desired. By [**BP08**, Proposition 2.22], this also implies that $\mathbf{Z}_* \in \mathsf{mincrit}(\mathbf{r}_*)$, finishing the proof. □

EXAMPLE 2.10 (large deviations). Let $\{p(\mathbf{r}) : \mathbf{r} \in \mathbb{Z}^{d-1}\}$ be a collection of nonnegative numbers summing to one. Assume that this probability distribution has finite moment generating function:

$$\phi(\mathbf{u}) := \sum_{\mathbf{r}} p(\mathbf{r}) e^{\mathbf{u}\cdot\mathbf{r}} < \infty, \qquad \text{for all } \mathbf{u} \in \mathbb{R}^{d-1} .$$

Let

$$F(\mathbf{Z}) := \frac{1}{1 - Z_d\, \phi(Z_1, \ldots, Z_{d-1})}$$

be the spacetime generating function for a random walk with steps governed by p. Thus, if $\mathbb{P}$ denotes the law of such a walk with partial sums $\{S_n\}$, and $\mathbf{Z}^{(\mathbf{r},n)}$ denotes $Z_1^{r_1} \cdots Z_{d-1}^{r_{d-1}} Z_d^n$, then

$$F(\mathbf{Z}) = \sum_{n=0}^{\infty} \sum_{\mathbf{r}} \mathbb{P}(S_n = \mathbf{r}) \mathbf{Z}^{(\mathbf{r},n)} .$$

The real surface $Q = 0$ is the graph over $\mathbb{R}^{d-1}$ of $1/\phi$; the region $\{\mathbf{Z} : Z_d < 1/\phi(Z_1, \ldots, Z_{d-1})\}$ is a component of the complement of the amoeba of Q; and each $\mathbf{Z} \in \partial B$ is a smooth point of $\mathcal{V}$ and is in $\mathsf{mincrit}(\mathbf{r}, 1)$, where $\mathbf{r}$ is the mean of the *tilted* distribution defined by

$$p_{\mathbf{Z}}(\mathbf{r}) = \frac{\mathbf{Z}^{\mathbf{r}} p(\mathbf{r})}{\sum_{\mathbf{r}'} \mathbf{Z}^{\mathbf{r}'} p(\mathbf{r}')} .$$

The preceding facts are not hard to show and may be found in [**Pem09b**]. It follows that for any $\mathbf{r}$ that is the mean of a tilted distribution $p_{\mathbf{Z}}$, the point $\mathbf{x} = \log \mathbf{Z}$ satisfies (2.3). Thus $\beta_Q(\mathbf{r}) = -\mathbf{r} \cdot \mathbf{x}$, and it follows from Proposition 2.5 that

$\overline{\beta}(\mathbf{r}) = -\mathbf{r} \cdot \mathbf{x}$. We may view this as a large deviation principle for sums of i.i.d. integer vectors with small-tailed distributions:

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left(\frac{S_n}{n} \in B(\mathbf{r}, \epsilon) \right) \to -\mathbf{r} \cdot \mathbf{x}$$

as $\epsilon \downarrow 0$, where $B(\mathbf{r}, \epsilon)$ is the ball of radius ϵ centered at $\mathbf{r}$. Subexponential decay occurs exactly when $\mathbf{Z} = \mathbf{1}$, corresponding to $\mathbf{r} = \nabla Q(\mathbf{1}) = \nabla \phi$, which we recognize to be the mean of the (untilted) distribution.

2.4. Proofs of criteria implying $\overline{\beta} < \beta_Q$. Theorem 2.8 and its corollary, Proposition 2.6, are proved in [**BP08**, Corollary 5.5]. The proof is via a direct construction of a homotopy between $\mathcal{C}$ and the local cycles in conclusion (i) of the theorem, the second conclusion and the proposition both following from the first conclusion. Of greater interest, however, is the Morse-theoretic argument which points the way to the proof. This argument does not appear in the published proof because it relies on the following conjecture, which has not been verified.

Conjecture 2.11 (compactification conjecture). There exists a compact space $\mathcal{V}^\dagger$ such that

(i) $\mathcal{V}$ embeds as a dense subset of $\mathcal{V}^\dagger$;
(ii) h extends to a continuous function mapping $\mathcal{V}^\dagger$ to the extended real line $[-\infty, \infty]$.

The conclusion of this conjecture is required as a hypothesis for the fundamental lemma of stratified Morse theory. It is also required that h be a Morse function, but Morse functions are generic, so this second requirement may be bypassed by taking a limit of Morse perturbations of h. Assuming this conjecture, Theorem 2.8 may be proved as follows.

Proof of Theorem 2.8 via Conjecture 2.11. The fundamental lemma of Morse theory states that the inclusion of $\mathcal{M}^b$ into $\mathcal{M}^a$ is a homotopy equivalence if there are no critical values in $[b, a]$. This homotopy equivalence deforms any chain in $\mathcal{M}^a$ into a homologous chain in $\mathcal{M}^b$. Thus we may lower the maximum height of $\mathcal{C}$ at least until a critical value for h is encountered. Let p be a critical point for h on $\mathcal{V}$ and let $c := h(p)$. Let D be a ball around p in $\mathcal{M}^c$ containing no other critical points, choose $\epsilon > 0$ small enough so that h has no critical values in $(c - \epsilon, c)$, and define the local space $\mathcal{M}^p_{\rm loc}$ to be the topological pair $(\mathcal{M}^{c-\epsilon} \cup D, \mathcal{M}^{c-\epsilon})$. A further consequence of the fundamental lemma is that $H_d(\mathcal{M}^c, \mathcal{M}^{c-\epsilon})$ is a direct sum of the groups $H_d(\mathcal{M}^p_{\rm loc})$. While the Morse-theoretic proof in [**GM88**] uses the gradient flow, these consequences are proved in [**BP08**] using a construction from [**ABG70**] involving hyperbolicity of Q at points on the boundary of $\mathsf{amoeba}(Q)$. In any case, the direct sum decomposition is exactly what is needed to finish the proof of Theorem 2.8. □

3. Smooth case

When $\mathsf{mincrit}$ is a single smooth point of $\mathcal{V}$ and a certain nondegeneracy assumption is satisfied, the form of the asymptotics for $\{a_\mathbf{r}\}$ is what is commonly called *Gaussian* or *Ornstein–Zernike*:

$$a_\mathbf{r} \sim C(\hat{\mathbf{r}}) \, |\mathbf{r}|^{(1-d)/2} \, \mathbf{Z}_*^{-\mathbf{r}} \; . \tag{3.1}$$

The nondegeneracy assumption is that the Gaussian curvature of $\log \mathcal{V}$ not vanish, which is the same as the nonvanishing of the Hessian determinant in (3.3).

3.1. Formula in coordinates. The following theorem identifies the function $C(\hat{\mathbf{r}})$ and extends to finitely many critical points. The theorem was first proved in [**PW02**, Theorem 3.5], under the extra assumption that the only points of $\mathcal{V}$ in $\mathrm{ReLog}^{-1}(\mathbf{x}_*)$ are critical. Their proof gave an explicit deformation of the chain of integration to a sum $\sum_{\mathbf{Z}_* \in \mathsf{mincrit}} \mathcal{C}(\mathbf{Z}_*)$ of relative classes in $H_d(\mathcal{M}_{\mathrm{loc}}^{\mathbf{Z}_*})$, though they did not use this terminology. The extra assumption was removed in [**BP08**], via an existence proof which replaced the elementary deformations.

Theorem 3.1. *Let $F = P/Q$ be a d-variate rational Laurent series corresponding to the component B of $\mathsf{amoeba}(Q)^c$, and let $\mathbf{x}_*$ be the minimizing point for $h = h_{\mathbf{r}_*}$. Suppose that $\mathsf{mincrit}(Q, \mathbf{r}_*)$ is nonempty and that at every $\mathbf{Z}_* \in \mathsf{mincrit}$, both the gradient of Q and the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{Z}_*$ are nonvanishing. Then there are relative homology classes $\mathcal{C}(\mathbf{Z}_*) \in H_d(\mathcal{M}_{\mathrm{loc}}^{\mathbf{Z}_*})$ such that*

$$a_{\mathbf{r}} = \sum_{\mathbf{Z}_* \in \mathsf{mincrit}} \int_{\mathcal{C}(\mathbf{Z}_*)} \omega \,, \tag{3.2}$$

where the equality is of equivalence classes up to $O(e^{(\beta_Q - \epsilon)|\mathbf{r}|})$, as in (2.1b). *At each point $\mathbf{Z}_* \in \mathsf{mincrit}$, if P and $\partial Q/\partial Z_d$ are both nonzero, the corresponding summand of* (3.2) *is given asymptotically by*

$$\int_{\mathcal{C}(\mathbf{Z}_*)} \omega \ \sim \ \Phi(\mathbf{Z}^*) := (2\pi r_d)^{(1-d)/2} (\mathbf{Z}_*)^{-\mathbf{r}} \mathcal{H}^{-1/2} \frac{P(\mathbf{Z}_*)}{(z_d \, \partial Q/\partial z_d)(\mathbf{Z}_*)} \,. \tag{3.3}$$

Here $\mathcal{H}$ is the determinant of the Hessian matrix of the parametrization of $\log \mathcal{V}$ by the coordinates $(z_1, \ldots, z_{d-1})$ near the point $\mathbf{z}_* := \log \mathbf{Z}_*$.

Remark 3.2. The decomposition (3.2) holds whether or not P or $\mathcal{H}$ vanishes. In fact, when $\mathcal{H}$ does not vanish, the corresponding summand of (3.2) may be expanded in an asymptotic series is descending powers of r_d:

$$\int_{\mathcal{C}(\mathbf{Z}_*)} \omega \ \sim \ \mathbf{Z}_*^{-\mathbf{r}} \sum_{n=0}^{\infty} b_n(\mathbf{Z}_*) r_d^{(1-d)/2 - n} \,.$$

When $P(\mathbf{Z}_*)$ is also nonzero, then b_0 is nonzero so it is the leading term and agrees with (3.3):

$$b_0 = (2\pi)^{(1-d)/2} \mathcal{H}^{-1/2} \frac{P(\mathbf{Z}_*)}{(\partial Q/\partial z_d)(\mathbf{Z}_*)} \,.$$

If $\mathcal{H}$ vanishes, an asymptotic expansion exists in decreasing fractional powers of r_d and possibly $\log r_d$; see [**Var77**].

A familiar example from [**PW02, PW08**] gives a concrete illustration of Theorem 3.1.

Example 3.3. Let $F = 1/(1 - x - y - xy)$ be the generating function for the Delannoy numbers. This example is worked in [**PW08**, Section 4.2]. The variety $\mathcal{V}$ where $Q := 1 - x - y - xy = 0$ is smooth. In two variables, letting $\mathbf{r} := (r, s)$, the critical point equations are

$$Q = 0\,,$$
$$sx\frac{\partial Q}{\partial x} - ry\frac{\partial Q}{\partial y} = 0\,.$$

Plugging this into a Gröbner basis package with $Q = 1 - x - y - xy$ yields precisely two solutions:

$$\mathbf{Z}_\pm := \left(\frac{\pm\sqrt{r^2+s^2} - s}{r}, \frac{\pm\sqrt{r^2+s^2} - r}{s} \right), \tag{3.4}$$

where the same sign choice is taken in both coordinates. The positive point is easily shown to be minimal (directly, or via Proposition 2.9), so $\mathbf{x}_* = \mathrm{ReLog}\,\mathbf{Z}_+$ and $\mathsf{mincrit}(\mathbf{r}_*) = \{\mathbf{Z}_+\}$. We have $\beta_Q = -\hat{\mathbf{r}} \cdot \mathrm{ReLog}\,\mathbf{Z}_+$ and

$$a_{\mathbf{r}} \sim (2\pi)^{-1/2} \sqrt{\frac{rs}{2\pi\sqrt{r^2+s^2}(r+s-\sqrt{r^2+s^2})^2}} \cdot \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r}.$$

Here the initial factor is computed by plugging in the values $\mathbf{Z}_\pm$ in (3.4) for $\mathbf{Z}_*$ on the right-hand side of (3.3), and simplifying. Nonvanishing of this quantity verifies the hypothesis of nonvanishing curvature of $\log \mathcal{V}$ at the point $\mathbf{z}_* := \log|\mathbf{Z}_*|$.

Proof of Proposition 2.5. At a smooth $\mathbf{Z}_* \in \mathsf{mincrit}$ we have an asymptotic series for $\int_{\mathcal{C}(\mathbf{Z}_*)} \omega$, either of the form (3.3) or the more general form in the subsequent remark. If there are more points in $\mathsf{mincrit}$, the corresponding summands in (3.2) will have different phases and therefore will not be able to cancel the contribution from $\mathbf{Z}_*$, except along a sublattice. $\square$

Example 3.4 (local large deviations and CLT). This example shows why the asymptotics in Theorem 3.1 are known as Gaussian. Continuing Example 2.10, we suppose the random walk to be aperiodic. Theorem 3.1 gives the asymptotic value of $\mathbb{P}(S_n = \mathbf{v})$. Letting $\mathbf{r} := \mathbf{v}/n$, after solving for $\mathbf{Z}$ and $\mathbf{x}$ we obtain

$$\mathbb{P}(S_n = \mathbf{v}) \sim (2\pi)^{-d/2} K^{-1/2}(\mathbf{r})\; n^{-d/2} e^{-n\beta_Q(\mathbf{r})}\,,$$

where K is the determinant of the covariance matrix for the tilted distribution $p_{\mathbf{Z}}$. This estimate is uniform as long as $\mathbf{r}$ stays within a compact subset of the set of tilted means, which is just the interior of the convex hull of the support of p.

Let $\boldsymbol{\mu}$ be the mean of the distribution p. If $|\mathbf{v} - n\boldsymbol{\mu}| = o(n)$ then the determinants of the tilted covariance matrices are all $K_0 + o(1)$, where K_0 is the determinant of the untilted covariance matrix. In this regime, therefore, $\mathbb{P}(S_n = \mathbf{v})$ is proportional to $e^{-n\beta_Q(\mathbf{r})}$. The function β_Q reaches its maximum of zero at $\boldsymbol{\mu}$. Letting H denote the quadratic Taylor term, we then have

$$\begin{aligned} n\beta_Q\left(\frac{\mathbf{v}}{n}\right) &= n\left[H\left(\frac{\mathbf{v}}{n} - \boldsymbol{\mu}\right) + O\left(\left|\frac{|\mathbf{v}|}{n} - \boldsymbol{\mu}\right|\right)^3\right] \\ &= \frac{H(\mathbf{v} - n\boldsymbol{\mu})}{n} + O\left(\frac{|\mathbf{v} - n\boldsymbol{\mu}|^3}{n^2}\right). \end{aligned}$$

Therefore, as long as $|\mathbf{v} - n\boldsymbol{\mu}|$ is $o(n^{2/3})$, we have a uniform local central limit estimate

$$\mathbb{P}(S_n = \mathbf{v}) \sim C n^{-d/2} e^{-H(\mathbf{v} - n\boldsymbol{\mu})}\,, \tag{3.5}$$

where the quadratic Taylor term H is represented by the inverse of the covariance matrix and the normalizing constant C is given by $(2\pi)^{-d/2}K_0^{-1/2}$.

3.2. Coordinate-free formula. In Theorem 3.1, it is sufficient that any partial derivative of Q not vanish: in the formula (3.3) for Φ, z_d may then be replaced by z_j for any j such that $\partial Q/\partial Z_j \neq 0$. Although the explicit coordinate choice makes (3.3) useful for computing, this observation prompts us to rewrite the quantity in (3.3) in a more canonical way. The Hessian determinant looks like, and is, a curvature. To avoid discussing Gaussian curvature in any case beyond that of a real hypersurface, we give the coordinate-free formula only in a special case, arising in the study of quantum random walks. The important feature of this generating function is that $\log \mathcal{V}$ has a large intersection (co-dimension 1) with $i\mathbb{R}^d$. More specifically, the following *torality* hypothesis is satisfied (see [**BBBP08**, Proposition 2.1]):

$$\mathbf{Z} \in \mathcal{V} \text{ and } |Z_1| = \cdots = |Z_{d-1}| = 1 \quad \Longrightarrow \quad |Z_d| = 1 \,. \tag{3.6}$$

This next result is stated and proved in [**BBBP08**, Theorem 3.3].

THEOREM 3.5. *Suppose* $\mathbf{x}_* = \mathbf{0}$ *and* $\mathsf{mincrit}$ *is non-empty and that the torality hypothesis* (3.6) *is satisfied. Then, with* $|\cdot|$ *denoting the euclidean norm,*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi\,|\mathbf{r}|}\right)^{d/2} \sum_{\mathbf{Z}\in\mathsf{mincrit}} \mathbf{Z}^{-\mathbf{r}} \frac{P(\mathbf{Z})}{|\nabla_{\log} Q(\mathbf{Z})|} \frac{1}{\sqrt{\mathcal{K}(\mathbf{Z})}}\, e^{-i\pi\tau(\mathbf{Z})/4} + O\left(|\mathbf{r}|^{(-1-d)/2}\right) .$$

Here, $\nabla_{\log}$ *is the logarithmic gradient* $(Z_1 Q_1, \ldots, Z_d Q_d)$ *and* $\tau(\mathbf{Z})$ *is the difference between the numbers of positive and negative eigenvalues of the Hessian matrix. The estimate is uniform as* $\mathbf{r}_*$ *varies over compact subsets of the set of unit vectors* $\hat{\mathbf{r}}$ *for which* $\mathcal{K} \neq 0$ *and* $\mathsf{mincrit}(\hat{\mathbf{r}})$ *is nonempty.*

EXAMPLE 3.6. A *quantum random walk* (QRW) on $\mathbb{Z}^d$ with unitary coin U was defined in [**ADZ93**], where U is any matrix in the unitary group of rank $2d$. (See also [**ABN$^+$01, Kem05**].) Starting with a single particle at the origin, let $a(\mathbf{r}, n)$ denote the amplitude of finding the particle at position $\mathbf{r}$ at time n (technically, one must also fix the starting and ending chiralities $(i, j) \in \{1, \ldots, 2d\}^2$, which will be assumed, but not explained). The spacetime generating function is

$$F(\mathbf{x}, y) := \sum_{n\geqslant 0,\, \mathbf{r}\in\mathbb{Z}^d} a(\mathbf{r}, n)\, \mathbf{x}^{\mathbf{r}} y^n \,.$$

It is shown in [**BBBP08**, equation (2.2)] that $F = P/Q$ with

$$Q = \det\left(I - y\, M(\mathbf{x})\, U\right) ,$$

where $M(\mathbf{x})$ is the diagonal matrix of order $2d$ with entries $x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}$. The torality hypothesis (3.6) is verified as [**BBBP08**, Proposition 2.1].

The origin is on the common boundary of $\mathsf{amoeba}(Q)$ and a component B of $\mathsf{amoeba}(Q)^c$. For any $(\mathbf{r}_*, 1)$ not outwardly normal to a support hyperplane of B at $\mathbf{0}$, the origin is not a minimizing point, whence the amplitudes $a(\mathbf{r}, n)$ decay exponentially as $n \to \infty$ with $\mathbf{r}/n \to \mathbf{r}_*$. When $(\mathbf{r}_*, 1)$ is inside the normal cone to B at $\mathbf{0}$, it may be verified in a number of cases that $\mathcal{V}$ is smooth; see [**BBBP08**, Section 4] for several families of smooth QRW generating functions in dimension $d = 2$. It then follows from Proposition 2.5 and Theorem 2.8 that there is exponential decay in direction $\mathbf{r}^*$ if and only if $\mathbf{r}^*$ is not the lognormal direction to any point $\mathbf{Z}_*$ of $\mathcal{V}_1 := \mathcal{V} \cap \mathrm{ReLog}^{-1}(\mathbf{0})$. When the amplitudes do not decay exponentially and $\mathcal{K} \neq 0$, the summands Φ in (3.3) are of order $n^{-d/2}$ with magnitudes proportional to the $-1/2$-power of the curvature of $\log \mathcal{V}_1$ at $\mathbf{z}_* := \log \mathbf{Z}_*$. The curvature $\mathcal{K}$ vanishes on a co-dimension 1 set including not only the boundary of the asymptotically feasible region, but also certain interior curves.

Typically, there is more than one summand, with the phases of the summands related in a complicated way. This results in the Moiré patterns visible in Figure 2. Note that the set of directions of non-exponential decay is not always convex. On the other hand, the normal cone to B is the dual to the tangent cone, hence convex. We conclude that there are directions $\mathbf{r}_*$ for which $\mathbf{0}$ is the minimizing point, but for which $\mathsf{mincrit}$ is empty.

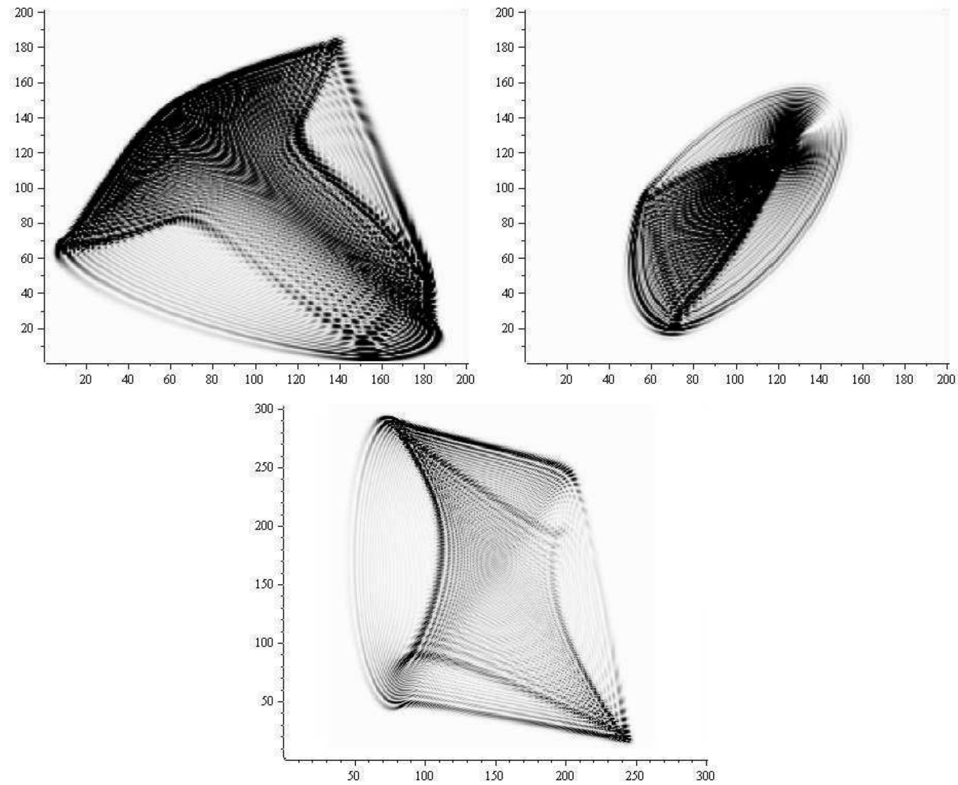

FIGURE 2. Intensity plot of squared amplitudes for three QRWs at time $n = 200$; already the asymptotic behavior is clearly visible.

3.3. No minimal points. The statement and the original proof of Theorem 3.1 require the dominating point(s) $\mathbf{Z}_*$ to be minimal. In order to determine asymptotics in directions $\mathbf{r}_*$ for which $\mathsf{mincrit}(\mathbf{r}_*)$ is empty, it is useful to employ a residue form in place of the univariate residues employed in [**PW02**]; this difference is somewhat cosmetic, but still quite helpful.

Define the *residue form* of a meromorphic form $\omega := (P/Q)\, d\mathbf{Z}$ on $\mathcal{V} := \{Q = 0\}$ by

$$\mathrm{Res}(\omega) := \iota^* \eta \,,$$

where η is any solution to

$$dQ \wedge \eta = P\, d\mathbf{Z}$$

and ι is the inclusion of $\mathcal{V}$ into $\mathbb{C}^d$. It is shown in [**DeV10**, Proposition 2.6] that such an η exists and that $\iota^*\eta$ is independent of the choice of η. The form $\text{Res}(\omega)$ is also known in [**AGZV88**, Chapter 7] as the *Gel'fand–Leray* form of Q.

Lemma 3.7 (Cauchy–Leray Residue Theorem). *Any transverse intersection of $\mathcal{V}$ with a homotopy between $\mathcal{C}$ and infinity has the same homology class $\psi \in H_d(\mathcal{V}, \mathcal{V}^\lambda)$ as long as λ is less than all critical values of h. For this class ψ,*

$$\int_{\mathcal{C}} \omega = 2\pi \mathrm{i} \int_{\psi} \text{Res}(\omega)\,. \tag{3.7}$$

Proof. See [**DeV10**, Theorem 2.8]). □

This result may be coupled with a formalization of the saddle point method that parallels the construction in the proof of Theorem 2.8.

Lemma 3.8 (saddle point method). *Let $\mathcal{V}$ be a complex d-manifold and h a proper Morse function on $\mathcal{V}$ which is the real part of a complex analytic function. Let $\mathcal{C}$ be any d-cycle on $\mathcal{V}$. Then there is a unique critical value c and cycle $\mathcal{C}' \in \mathcal{V}^c$ such that:*

(i) *$\mathcal{C}$ projects to zero in $H_d(\mathcal{V}, \mathcal{V}^{c+\epsilon})$;*
(ii) *$[\mathcal{C}] = [\mathcal{C}']$ in any $H_d(\mathcal{V}, \mathcal{V}^{c-\epsilon})$;*
(iii) *$\mathcal{C}' = \sum_p \mathcal{C}_p$, where p runs over some subset of the critical points at height c and $\mathcal{C}_p$ is diffeomorphic to a d-ball.*

Remarks. (1) We call the set of p appearing in the sum the *contributing critical points* and the height c the *minimax height.* (2) If h has isolated critical points then the Morse property is not actually needed: h is a limit of Morse perturbations h_ϵ and any weak limit of the resulting cycles $\mathcal{C}'_\epsilon$ satisfies (i) and (ii); the third conclusion must be generalized to allow a wedge of balls. (3) When $\mathcal{V}$ is not smooth but is smooth in a neighborhood of $\mathsf{mincrit}$, the same construction gives a relative cycle $\psi \in H_d(\mathcal{V}^c, \mathcal{V}^{c-\epsilon})$ where c is the common height of the points in $\mathsf{mincrit}$; this is good enough to produce asymptotics.

Proof. The first two facts are quite general. If

$$\varnothing = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$$

is any filtration and the homology dimension of all spaces is d, then for any $\sigma \in H_d(X_n)$ there is a unique least j such that σ is homologous to a cycle supported on X_j, and if $j > 0$ then σ is nonzero in $H_d(X_j, X_{j-1})$. Choosing $X_j := \mathcal{V}^{c_j}$ for successive critical values c_j of h and applying the fundamental lemma of Morse theory proves (i) and (ii).

Being the real part of a complex analytic function, h has critical points of index d only. Ordinary Morse theory tells us that the homology of the pair $(\mathcal{V}^c, \mathcal{V}^{c-\epsilon})$ is a free abelian group generated by $(D_p, \partial D_p)$ where D_p is a d-ball centered at p and p runs through critical points of height c. Thus $\mathcal{C}'$ has a cycle representative that is a sum of arbitrarily small d-balls in $\mathcal{V}^c$ localized to critical points of height c, proving (iii). □

In this way, we fulfill the promise, made at the end of Section 2.1, to find a limiting minimax chain which will no longer lie in $\mathcal{M}$. To each chain $\psi \in \mathcal{V}$ there corresponds a chain $\tilde{\psi} \in \mathcal{M}$ defined to be the product (in some local coordinates) of ψ with a small circle around the origin in the complementary complex one-space to $\mathcal{V}$. The saddle point deformation of ψ corresponds to the quasi-local representation $\tilde{\psi}$ of $\mathcal{C}$ in the proof of Theorem 2.8. The residue identity (3.7) in these coordinates is an obvious consequence of the ordinary residue theorem, the contribution from Morse theory being the existence of a cycle of the form $\tilde{\psi}$ in the homology class of $\mathcal{C}$. Putting together the previous two lemmas directly implies the following result.

THEOREM 3.9. *Suppose $\mathcal{V}$ is smooth and $h = h_{\mathbf{r}_*}$ is proper and Morse. Then there is a set Ξ of critical points for h on $\mathcal{V}$ at some common height c, along with topological $(d-1)$-balls $\mathcal{B}(\mathbf{Z}_*) \subseteq \mathcal{V}$ on which h is maximized at $\mathbf{Z}_*$, such that*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi \mathrm{i}}\right)^d \sum_{\mathbf{Z}_* \in \Xi} \int_{\mathcal{B}(\mathbf{Z}_*)} \omega$$

up to a difference of $O(e^{(c-\epsilon)|\mathbf{r}|})$ for some $\epsilon > 0$. Asymptotic series expansions for the summands in decreasing fractional powers of $|\mathbf{r}|$ are computable.

REMARKS. This shows that $a_{\mathbf{r}}$ may be estimated up to $O(e^{(c-\epsilon)|\mathbf{r}|})$ by a sum of terms $\sum_{\mathbf{Z}_* \in \Xi} \Phi(\mathbf{Z}_*)$, with Φ as defined by (3.3). This greatly generalizes Theorem 3.1 because it allows the contributing critical points to be non-minimal. When $d-1 \geqslant 2$ the computation of the expansions and even the leading term can be somewhat complicated. It is shown in [**Var77**] how to compute the leading exponent from the Newton diagram at the singularity. When $d-1=1$, the only degenerate possibilities are $h(\mathbf{Z}_* + u) \sim Cu^k$ for some $k \geqslant 3$, which lead to formulae analogous to (3.3) but with the exponent $(1-d)/2$ replaced by $-1/k$, the Hessian determinant replaced by the derivative of order k, and the power of 2π replaced by a value of the Gamma function.

EXAMPLE 3.10. The paper [**DeV10**] considers the generating function

$$F(X,Y) := 2X^2Y \frac{2X^5Y^2 - 3X^3Y + X + 2X^2Y - 1}{X^5Y^2 + 2X^2Y - 2X^3Y + 4Y + X - 2},$$

which is reverse-engineered so that its diagonal counts bi-colored supertrees. When $\mathbf{r}_*$ is the diagonal direction, there are precisely three critical points; none of these is minimal, but it is shown by direct homotopy methods that $\mathbf{Z}_* := (2, 1/8)$ is the unique contributing critical point. Near $\mathbf{Z}_*$, the behavior of h is quartic rather than quadratic, a double degeneracy coming from the merging of three distinct saddles that re-appear in any perturbation. Normally this would produce a factor of $|\mathbf{r}|^{-1/4}$ but the numerator vanishes to degree one at $\mathbf{Z}_*$ leading instead to a factor $n^{-5/4}$ and the asymptotic estimate

$$a_{n,n} \sim \frac{4^n}{8\Gamma(3/4)\, n^{5/4}}.$$

Theorem 3.9 and its generalizations as discussed in the subsequent remark reduce the problem of estimating coefficient asymptotics in the smooth case to identification of the contributing set Ξ. The good news is that even without this last step, the estimation problem is solved modulo a choice from among a finite set of possible estimates. The bad news is that we have no general method for determining Ξ. Example 3.10, for example, was handled by *ad hoc* methods. The one case where we appear to have an effective procedure for determining Ξ is in some work in progress on the case $d = 2$. Briefly, in this case one proceeds by computing the Morse complex as a finite multigraph. The generators for $H_1(\mathcal{V})$ given by this cell complex are precisely the one-dimensional saddle point contours descending from each critical point. Resolution of the cycle $\mathcal{C}$ in this basis is done by counting intersections with a dual basis, consisting of ascending contours from each critical point: the steepest ascent contours are replaced by polygonal approximations, whose intersection number with the special cycle $\mathcal{C}$ are computable from the combinatorics of these arcs, each of which connects a critical point to a pole ($x = 0$ or $y = 0$) of the height function.

4. Non-smooth case

Let $\mathbf{Z} \in \mathcal{V} = \{Q = 0\}$. After smooth points, the simplest local geometry $\mathcal{V}$ can have near $\mathbf{Z}$ is to be a union of smooth divisors intersecting transversally; such a critical point is called a *multiple point*. Multiple points consume the bulk of this section because this is the case in which the most is known.

4.1. Multiple points. There are a number of ways to formulate the definition of a multiple point, the most direct and geometric of which is modeled after [**PW04**, Definition 2.1].

Definition 4.1 (multiple point). The point $\mathbf{Z} \in \mathcal{V}$ is a multiple point if and only if there exist analytic functions $v_1, \dots, v_n$ and ϕ defined on a neighborhood of $(Z_1, \dots, Z_{d-1})$ in $\mathbb{C}^{d-1}$, and positive integers $k_1, \dots, k_n$, such that

$$F(\mathbf{Z}') = \frac{\phi(\mathbf{Z}')}{\prod_{j=1}^{n}(1 - Z'_d\, v_j(Z'_1, \dots, Z'_{d-1}))^{k_j}}\,, \tag{4.1}$$

the equality being one of meromorphic functions in a neighborhood of $\mathbf{Z}$, and such that

(i) $Z_d\, v_j(Z_1, \dots, Z_d) = 1$ for all $1 \leqslant j \leqslant n$;
(ii) $Z'_d / v_j(Z'_1, \dots, Z'_{d-1}) = 1$ for some j if and only if $\mathbf{Z}' \in \mathcal{V}$;
(iii) any set of at most d of the vectors $\{\nabla v_j(\mathbf{Z}) : 1 \leqslant j \leqslant n\}$ is linearly independent.

Remarks. (*i*) A smooth point is a multiple point. (*ii*) The reason we allow multiplicities ($k_j \geqslant 2$) but not tangencies ($\nabla v_i(\mathbf{Z}) \parallel \nabla v_j(\mathbf{Z})$) is to ensure genericity of intersections. In particular, $\bigcap_{j=1}^{n}\{\mathcal{V}_j\}$ will be a manifold and $\mathcal{V}$ will have a local product structure.

We let $\mathcal{V}_j := \{Z'_d v_j(Z'_1, \dots, Z'_{d-1}) = 1\}$, so that $\mathcal{V}_1, \dots, \mathcal{V}_n$ are locally smooth varieties parametrized by $Z'_d = u_j(Z'_1, \dots, Z'_{d-1}) := 1/v_j(Z'_1, \dots, Z'_{d-1})$. The geometric formulation is probably the most intuitive, but there is also an algebraic formulation that allows us to compute more effectively whether $\mathbf{Z}$ is a multiple point. To check (*i*) and (*ii*), we check whether Q factors completely in the local ring of germs of analytic functions at $\mathbf{Z}$. Supposing this to be true, we may write $Q(\mathbf{Z}') = \prod_{j=1}^{n} g_j^{k_j}$ where $g_j = Z'_d - u_j(Z'_1, \dots, Z'_{d-1})$. If $n \geqslant d$, the transversality assumption is equivalent to the assertion that the dimension of $\mathbb{C}[\mathbf{Z}]/I$ as a complex vector space is one, where I is the ideal of the local ring generated by $u_1, \dots, u_n$.

If $n < d$, it is probably easiest to check linear independence directly. Apparatus for doing these computations may be found in [**CLO98**, Chapter 4]. The following example illustrates and underscores that being a multiple point is a local property.

Example 4.2. A generating function is given in [**PW04**, Example 1.3] that counts winning plays in a game with two time parameters and two coins with different biases. The denominator of the generating function is

$$Q := (1 - \frac{1}{3}X - \frac{2}{3}Y)(1 - \frac{2}{3}X - \frac{1}{3}Y)\,.$$

Here, Q is reducible in $\mathbb{C}[X,Y]$ as well as factoring locally, and the pole set $\mathcal{V}$ is the union of the two divisors, in this case two complex lines intersecting at the point $(1,1)$. The lines intersect transversally, so this point is a multiple point.

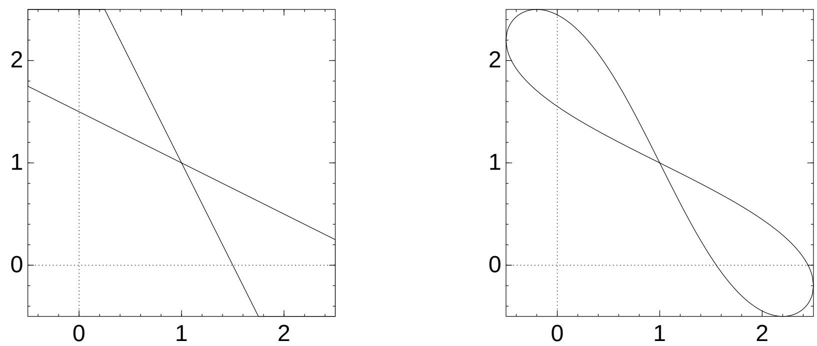

Figure 3. Zero sets of Q and $\tilde{Q}$.

Perturbing the polynomial gives a new polynomial

$$\tilde{Q} := 19 - 20X - 20Y + 5X^2 + 14XY + 5Y^2 - 2X^2Y - 2XY^2 + X^2Y^2\,,$$

the zero set of which, a rational curve, is drawn in [**PW04**, Example 3.2]. The real part of $\mathcal{V}$ looks like a lemniscate (a tilted figure eight) with a double point at $(1,1)$. Although $\tilde{Q}$ is irreducible in the polynomial ring, it factors in the local ring, and the divisors intersect transversally.

When the denominator of F factors, even locally, a condition on the numerator is needed to ensure we are analyzing the correct denominator. Define the lognormal vectors to the divisors g_j at $\mathbf{Z}_*$ by

$$\mathfrak{n}_j := \left(Z_1 \frac{\partial v_j}{\partial Z_1}, \ldots, Z_{d-1} \frac{\partial v_j}{\partial Z_{d-1}}, Z_d \right) (\mathbf{Z}_*)\,.$$

Definition 4.3 (partial fraction ideal). Let $Q = \phi \cdot \prod_{j=1}^n g_j^{k_j}$ locally near $\mathbf{Z}_*$. Say that a subset $S \subseteq \{1, \ldots, n\}$ is *small* if the vectors $\{\mathfrak{n}_j : j \in S\}$ span a proper subspace $V_S \subseteq \mathbb{R}^d$. Denote by $\mathcal{J} = \mathcal{J}(\mathbf{Z})$ the ideal in the local ring at $\mathbf{Z}$ generated by all products $\prod_{j=1}^n g_j^{\ell_j}$ such that there is a small set S with $\ell_j = k_j$ for $j \notin S$.

Equivalently, $P \in \mathcal{J}$ if and only if P/Q has a partial fraction expansion $\sum_S F_S$ in which the denominator of F_S is a product over $j \in S$ of powers of g_j and S is small. The relevance of this is that $\mathbf{x}_* := \text{ReLog}(\mathbf{Z}_*)$ is not a minimizing point for F_S unless $\mathbf{r}_* \in V_S$. If we let G denote the union over small S of the proper subspaces V_S, we see that if $P \in \mathcal{J}$, then $|\mathbf{Z}_*|^{\mathbf{r}} a_{\mathbf{r}}$ is exponentially small when $|\mathbf{r}| \to \infty$ with $\mathbf{r}_*$ in a compact subset of G^c.

Example 4.4. Let $F = P/Q$ and $\tilde{F} = P/\tilde{Q}$ with $P = X - Y$ and $Q, \tilde{Q}$ as in Example 4.2. In the former case, there is a global partial fraction expansion

$$F = \frac{3}{Q_1} - \frac{3}{Q_2} := \frac{3}{1-(2/3)X-(1/3)Y} - \frac{3}{1-(2/3)Y-(1/3)X}\,.$$

Clearly, $\overline{\beta}_Q = \max \overline{\beta}_{Q_1} - \overline{\beta}_{Q_2}$, and when $\mathbf{r}_*$ is the diagonal, for instance, we have $\overline{\beta} < 0 = \beta_Q$. In the second case, near $\mathbf{Z}_*$ there is a partial fraction expansion into a sum of meromorphic functions

$$F = \frac{P_1}{g_1} + \frac{P_2}{g_2}\,,$$

and again $\overline{\beta}(\mathbf{r}_*) < 0 = \beta_Q(\mathbf{r}_*)$. It is worth noting that these exceptions are in some sense trivialities, which will be prevented by the hypothesis $P \notin \mathcal{J}$.

4.2. Minimal multiple points and a piecewise polynomial formula when $n \geqslant d$. Suppose that the multiple point $\mathbf{Z}_*$ is minimal, meaning that $\mathbf{x}_* := \mathrm{ReLog}(\mathbf{Z}_*) \in \partial B$. Characterization of the dual cone is reasonably explicit in this case. The hyperplane normal to each lognormal vector $\mathfrak{n}_j$ at $\mathbf{x}_*$ is a support hyperplane to B at $\mathbf{x}_*$, so there is a well defined outward direction (away from B); choosing outward normals and taking the positive gives the cone $\mathbf{N}_*(\mathbf{Z}_*)$. In Theorem 4.7, a formula will be given for the contribution to the Cauchy integral from the critical point $\mathbf{Z}_*$ when $\hat{\mathbf{r}} \to \mathbf{r}_* \in \mathbf{N}_*(\mathbf{Z}_*)$. The contribution is denoted by $\Phi(\mathbf{r})$ in analogy to Theorem 3.1. When $P \notin \mathcal{J}(\mathbf{Z})$, the quantity $|\Phi(\mathbf{r})\mathbf{Z}_*^{-\mathbf{r}}|$ does not decay exponentially, again in analogy with Theorem 3.1. The existence of such a formula implies that Proposition 2.5 holds with "smooth point" replaced by "minimal point". We then have the following analogue of Proposition 2.9.

Proposition 4.5 (combinatorial case for multiple points). *Supposing $a_{\mathbf{r}} \geqslant 0$ for all $\mathbf{r}$, let $\mathbf{r}_*$ be any vector with unique minimizer $\mathbf{x}_*$. If all critical points are multiple points then $\overline{\beta}(\mathbf{r}_*) = \beta_Q(\mathbf{r}_*)$, provided that $P \notin \mathcal{J}(\exp(\mathbf{x}_*))$.*

Proof. Fix $\mathbf{r}_*$. As in Proposition 2.9, the point $\mathbf{Z}_* := \exp(\mathbf{x}_*)$ is minimal. Locally, $\mathsf{amoeba}(Q)$ is the union of the $\mathrm{ReLog}[\mathcal{V}_j]$, therefore the span of all the lognormal vectors is equal to the dual cone to $\mathbf{K}(\mathbf{r}_*)$, hence contains $\mathbf{r}_*$. It follows from formula (4.3) in Theorem 4.7 below that $\overline{\beta}(\mathbf{r}_*) = \beta_Q(\mathbf{r}_*)$. ☐

We now turn to asymptotic formulae for the contribution to $a_{\mathbf{r}}$ from a minimal multiple point $\mathbf{Z}_*$. As one might expect from part (*iii*) of the definition of multiple points, there are two somewhat different cases, depending on whether n or d is greater. We consider the case $n \geqslant d$ in somewhat greater detail because the formulae are nicer. Along with the assumptions on multiple points, the inequality $n \geqslant d$ implies that $\{\mathbf{Z}_*\}$ is a zero-dimensional stratum and that the cone $\mathbf{N}_*$ has nonempty interior. The surprising fact here is that for $\hat{\mathbf{r}} \to \mathbf{r}_* \in \mathbf{N}_*(\mathbf{Z}_*)$, the quantity $a_{\mathbf{r}}$ is nearly piecewise polynomial. This was observed already in [**Pem00**, Theorem 3.1], where the following qualitative description was given.

Theorem 4.6. *Let $F = P/Q$ and let $\mathbf{Z}_*$ be a multiple point with $n \geqslant d$. There is a finite vector space $\mathcal{W}$ of polynomials and an algebraic set G of positive* *codimension in* $\mathbb{R}^d$ *such that*

$$a_{\mathbf{r}} = \mathbf{Z}^{-\mathbf{r}}(\eta(\mathbf{r}) + E(\mathbf{r}))\,,$$

where $\eta \in \mathcal{W}$ *depends on* P *and* E *decays exponentially in* $\mathbf{r}$*, uniformly as* $\mathbf{r}$ *varies over compact connected subsets of* $\mathbf{N}_* \setminus G$.

The set G is identified there as the same set $G = \bigcup_S V_S$ discussed subsequently to the definition of the partial fraction ideal. The complement of G may be disconnected, in which case η may be equal to different polynomials on the different components, and is best described as a (piecewise) spline. As long as η is not the zero polynomial, it gives the asymptotic behavior of $a_{\mathbf{r}}$ on that component. As shown in Theorem 4.7 below, $\eta = 0$ if and only if $P \in \mathfrak{J}(\mathbf{Z}_*)$, the "if" direction following already from [**Pem00**]. The proof of Theorem 4.6 given in [**Pem00**] is based on finite-dimensionality of the space of shifts of the array $\{a_{\mathbf{r}}\}$ modulo exponentially decaying terms, and is not constructive. This leaves the question of an explicit description of η. This is given by the following result, which combines the published result [**PW04**, Theorem 3.6] with the equivalent but more canonical formula given in [**BP04**]. The formula is visually simplest when $\mathbf{Z}_* = (1, \ldots, 1)$, so this is given first as a special case.

THEOREM 4.7. *Let* $F = P/Q$*, let* B *be a component of* $\mathsf{amoeba}(Q)^c$*, and let* $\mathbf{r}_*$ *have unique minimal point* $\mathbf{x}_*$.

(i) *Suppose that* $\mathsf{mincrit}(\mathbf{r}_*) = \{\mathbf{Z}_*\}$ *where* $\mathbf{Z}_*$ *is a multiple point with local factorization* $f := F \circ \exp = \phi / \prod_{j=1}^n g_j^{k_j}$. *Let* $D := \sum_{j=1}^n k_j$. *If* $P \notin \mathfrak{J}(\mathbf{Z}_*)$ *then*

$$a_{\mathbf{r}} \sim \mathbf{Z}_*^{-\mathbf{r}}\, \phi(\mathbf{Z}_*)\, \theta(\mathbf{r})\,, \tag{4.2}$$

where $\theta(\mathbf{r}_*) = \theta_Z(\mathbf{r}_*)$ *is the density of the image of Lebesgue measure under the map* $\Psi\colon (\mathbb{R}^+)^D \to \mathbf{N}_*$ *which takes* k_j *of the standard basis vectors to the lognormal vector* $\mathfrak{n}_j$.

(ii) *If* $\mathsf{mincrit}(\mathbf{r}_*)$ *is the union of a set* Ξ *of multiple points with a common value of* D *and a set of points* $\mathbf{Z}$ *for which* $F(\mathbf{Z}+\mathbf{Z}') = O(|\mathbf{Z}'|^{1-D})$*, then*

$$a_{\mathbf{r}} = \sum_{\mathbf{Z} \in \Xi} \mathbf{Z}^{-\mathbf{r}}\, \phi(\mathbf{Z})\theta_{\mathbf{Z}}(\mathbf{r}) + O(|\mathbf{r}|^{D-1-d})\,, \tag{4.3}$$

where $\theta_{\mathbf{Z}}$ *is the polynomial appearing on the right-hand side of* (4.2)*, defined in terms of the local product decomposition at* $\mathbf{Z}$.

The formula (4.3) *gives an asymptotic estimate for* $a_{\mathbf{r}}$ (*possibly on a sublattice, in case of periodicity*)*, as long as* $P \notin \mathfrak{J}(\mathbf{Z})$ *for some* $\mathbf{Z} \in \Xi$.

SKETCH OF PROOF. Let $\mathcal{B}$ be the collection of sub-multisets of $\{1, \ldots, n\}$ for which the corresponding subset of $\{\mathfrak{n}_j : 1 \leqslant j \leqslant n\}$ is independent and each $\mathfrak{n}_j$ a number of times $\ell_j \leqslant k_j$. A local partial fraction decomposition allows us to write f as

$$\sum_{S \in \mathcal{B}} \frac{\phi_S}{\prod g_j^{\ell_j}}\,, \tag{4.4}$$

where $\ell_j = 0$ if $j \notin S$. For each summand, integrating by parts $D - d$ times gives

$$\int e^{-\mathbf{r}\cdot\mathbf{z}} \frac{\phi_S(\mathbf{z})}{\prod g_j^{\ell_j - 1}}\, \mathrm{d}\mathbf{z} = \left[\frac{\partial}{\partial r_1}^{\ell_1} \cdots \frac{\partial}{\partial r_d}^{\ell_d}\right] \int e^{-\mathbf{r}\cdot\mathbf{z}} \frac{\phi_S(\mathbf{z})}{\prod_{j \in S} g_j}\, \mathrm{d}\mathbf{z}\,. \tag{4.5}$$

Having reduced to the case $n = d$, a number of methods suffice for us to show that the last integral is asymptotic to $e^{-\mathbf{r}\cdot\mathbf{z}}J^{-1}$ when $\mathbf{r}$ is in the positive hull of $\{\mathfrak{n}_j : j \in S\}$ and J is the determinant of $\{\mathfrak{n}_j : j \in S\}$; see for example [**BP04**, Section 5] or the original proof of Theorem 3.3 in [**PW04**]. Evaluating derivatives shows that (4.5) is asymptotic to

$$e^{-\mathbf{r}\cdot\mathbf{z}}\mathbf{r}^{\ell-\mathbf{1}}J^{-1}.$$

Summing over $S \in \mathcal{B}$ gives an expression for (4.4). It is not hard to show by induction that this expression is equal to the spline described on the right-hand side of (4.2).

To prove (4.3), one localizes to neighborhoods of each $\mathbf{Z}_* \in \mathsf{mincrit}$; this is carried out in [**PW04**, Section 4] in the case when $\mathcal{V} \cap \mathrm{ReLog}^{-1}(\mathbf{x}_*)$ contains only critical points; the general localization may be found in [**BP08**, Section 5]. If every $\mathbf{Z}_* \in \mathsf{mincrit}$ is a multiple point, the result now follows by summing (4.2) over $\mathbf{Z}_*$. If there are critical points of another type, then an upper estimate is required on their contribution. This may be found in [**BP08**, Lemma 5.9]. □

Example 4.8 (Example 4.2, continued). Let

$$F = \frac{P}{Q} = \frac{1}{(1 - \frac{1}{3}X - \frac{2}{3}Y)(1 - \frac{2}{3}X - \frac{1}{3}Y)}.$$

The variety $\mathcal{V}$ is the union of two complex lines, meeting at the single point $(1, 1)$. This point is a zero-dimensional stratum; each punctured complex line is a stratum as well. The boundary of B is the union of the two smooth curves $e^x + 2e^y = 3$ and $2e^x + e^y = 3$, intersecting at $(0, 0)$ (see Figure 4; cf. Figure 1(b)).

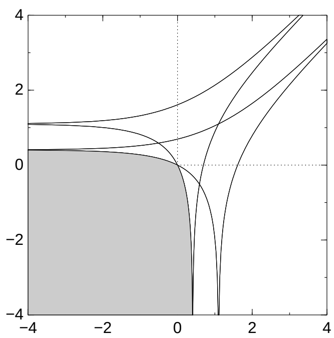

Figure 4. B is the unbounded shaded region.

When $r/s \in [1/2, 2]$, the minimizing point is at the origin. Because $a_{rs} \geqslant 0$, we know that the positive point $\mathbf{Z}_* = (1, 1) = \exp(\mathbf{x}_*)$ is in $\mathsf{mincrit}$, and it is easy to verify that $\mathsf{mincrit}(r, s) = \{\mathbf{Z}_*\}$ for $1/2 \leqslant r/s \leqslant 2$. Plugging into Theorem 4.7 and evaluating the Hessian determinant we find that θ is a constant because $n = d$, and that $a_{rs} = 1/9 + o(1)$. When $r/s \notin [1/2, 2]$, we have $\mathbf{x}_* \neq (0, 0)$. Here, $\beta_Q < 0$ and a_{rs} decays exponentially; all points other than $(1, 1)$ are smooth, so we may use Theorem 3.1 to evaluate $a_{\mathbf{r}}$ asymptotically.

4.3. Non-minimal points and the case $n < d$. Suppose $\mathbf{Z}_*$ is a multiple point and that f is factored locally as $\phi / \prod_{j=1}^n g_j^{k_j}$. The point $\mathbf{Z}_*$ lies in some stratum of $\mathcal{V}$, which, under the transversality assumptions, can be taken to be $\mathcal{V}_0 := \bigcap_{j=1}^n \mathcal{V}_j$. Let δ denote the dimension of $\mathcal{V}_0$. In the next paragraph we justify writing the Cauchy integral (1.2) as an iterated integral

$$\Phi_{\mathbf{Z}_*}(\mathbf{r}) := \left(\frac{1}{2\pi \mathrm{i}}\right)^d \int_B \left[\int_\alpha \mathbf{Z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{Z}) \, \mathrm{d}\mathbf{Z}_\perp\right] \mathrm{d}\mathbf{Z}_\parallel \,, \tag{4.6}$$

where B is a ball of dimension δ in $\mathcal{V}_0$ on which h is strictly maximized at $\mathbf{Z}_*$, and α is some cycle supported on a $(d-\delta)$-dimensional complex plane through $\mathbf{Z}_*$ which is transverse to $\mathcal{V}_0$. This justification pulls in some Morse-theoretic facts which may be skipped on first reading.

Thom's Isotopy Lemma ([**Mat70**], quoted in [**GM88**, Section I.1.4])) asserts that the pair $(\mathbb{C}^d, \mathcal{V})$ is locally a product of $\mathcal{V}_0$ with a pair $(\mathbb{C}^{d-\delta}, X)$, where X is the intersection of $\mathcal{V}$ with a complex space of dimension $d-\delta$ through $\mathbf{Z}_*$ which is transverse to $\mathcal{V}_0$. Recall from the end of Section 2 the decomposition of the chain of integration $\mathcal{C}$ into relative homology classes in $H_d(\mathcal{M}_{\text{loc}}^{\mathbf{Z}_*})$. In general, representability of $\mathcal{C}$ as the sum of these relies on Conjecture 2.11, but in the cases considered in this section (in which $\mathbf{Z}_*$ is minimal or all factors g_j are linear) this is known. In any case, these local homology groups are identified in [**GM88**] as isomorphic to $H_{d-\delta}(\mathbb{C}^{d-\delta}, \mathbb{C}^{d-\delta} \setminus X)$, the latter pair being topologically equivalent to the cone $\mathbb{C}^{d-\delta} \setminus X$. The isomorphism takes a $(d-\delta)$-cycle α to $\alpha \times (B_\delta, \partial B_\delta)$. Here B_δ is a ball of dimension δ about $\mathbf{Z}_*$ in $\mathcal{V}_0$; the tangent space to the complex d-manifold $\mathcal{V}_0$ has a real d-dimensional subspace along which h is maximized at $\mathbf{Z}_*$; and taking B_δ tangent to this assures that $h < h(\mathbf{Z}_*) - \epsilon$ on ∂B_δ.

In general, when $\mathbf{Z}_*$ is not minimal, we have at present no way of knowing for which points $\mathbf{Z}_*$ the integral (4.6) contributes to $\mathcal{C}$ with nonzero coefficient, hence contributing to the estimate for $a_{\mathbf{r}}$. In the special case where $F = P/Q$ and Q is the product of linear factors, a combinatorial method is given in [**BP04**] for writing the chain of integration in the Cauchy integral as a sum of cycles $\alpha \times \beta$, where β is a global cycle that projects to $(B_\delta, \partial B_\delta)$ in $\mathcal{M}^{h(\mathbf{Z}_*)-\epsilon}$. The other well understood case is when $\mathbf{Z}_*$ is minimal. In this case, the local cycle contributes to the Cauchy integral whenever $\mathbf{x}_*$ is a minimizing point for $\mathbf{r}_*$. In either of these two cases, an asymptotic estimate for $a_{\mathbf{r}}$ is obtained by summing (4.6) over those $\mathbf{Z}_*$ known to contribute to the Cauchy integral. The following theorem then rests on evaluation of (4.6). To make sense of the final result, note that $\{\mathfrak{n}_j\}$ span the complex linear space normal to the tangent space to $\mathcal{V}_0$ at $\mathbf{Z}_*$, and that the definition of θ in Theorem 4.7 may be extended by taking it to be the density of $\Psi(\mathrm{d}\mathbf{x})$ with respect to $(d-\delta)$-dimensional Lebesgue measure on the normal space.

THEOREM 4.9. *Let $F = P/Q = \phi / \prod_j g_j^{k_j}$ near a multiple point $\mathbf{Z}_*$. Let $y_1, \ldots, y_d$ be unitary local coordinates in which $\mathcal{V}_0$ is the set $y_{\delta+1} = \cdots = y_d = 0$, and for $\mathbf{Z} \in \mathcal{V}_0$, define $\Lambda(\mathbf{Z})$ to be the Hessian determinant of the function $h_{\mathbf{r}_*}$ restricted to $\mathcal{V}_0$:*

$$\Lambda(\mathbf{Z}) := \left| \frac{\partial^2 h_{\mathbf{r}_*}}{\partial y_i \partial y_j} \right|_{1 \leqslant i,j \leqslant \delta} .$$

If $\phi(\mathbf{Z}_*)$ and $\Lambda(\mathbf{Z}_*)$ are nonvanishing then the quantity $\Phi(\mathbf{r})$ in (4.6) *may be evaluated asymptotically to yield*

$$\Phi(\mathbf{r}) \sim (2\pi)^{-\delta/2}\, \Lambda(\mathbf{Z}_*)^{-1/2}\, \mathbf{Z}_*^{-\mathbf{r}}\, \phi(\mathbf{Z}_*)\, \theta(\mathbf{r})\, .$$

PROOF. The inner integral is of the form evaluated in Theorem 4.7. At any point $\mathbf{Z} \in \mathcal{V}_0$, the lognormal vectors $\{\mathfrak{n}_j\}$ defined there span the complex linear space orthogonal to the tangent space to $\mathcal{V}_0$. Equation 4.2 therefore evaluates the inner integral asymptotically as $\mathbf{Z}_*^{-\mathbf{r}}\phi(\mathbf{Z}_*)\theta(\mathbf{r})$. The function $A(\mathbf{Z}) := \theta(\mathbf{r})\phi(\mathbf{Z})$ does not vanish at $\mathbf{Z}_*$, hence the outer integral is a saddle point integral of the type $\int_{\mathcal{B}} \mathbf{Z}^{-\mathbf{r}} A(\mathbf{Z})\, \mathrm{d}\mathbf{Z}$. Standard integrating techniques (see, e.g., [**PW09**, Theorem 2.3]) then show that the integral is asymptotic to $(2\pi)^{-\delta/2}\, A(\mathbf{r})\, \mathbf{Z}_*^{-\mathbf{r}} \Lambda(\mathbf{Z}_*)^{-1/2}$, as desired. □

4.4. More complicated geometries. When $\mathbf{Z}_*$ is not a smooth point and is not in a stratum which is locally an intersection of normally crossing smooth divisors, residue theory alone does not suffice to estimate $a_{\mathbf{r}}$. The analysis of this case is the most intricate and recent. The rather long preprint [**BP08**] is devoted to a subcase of this in which $\mathcal{V}$ is locally a quadratic, with the same geometry as the cone $\{z^2 - x^2 - y^2 = 0\}$. This section will be limited to a quick sketch of the derivation of results in this case, beginning with an example of such a generating function.

EXAMPLE 4.10. The generating function for *creation rates* in domino tilings of the Aztec Diamond is given by

$$F(x,y,z) = \frac{1}{1 - (x + x^{-1} + y + y^{-1})\dfrac{z}{2} + z^2}\, . \tag{4.7}$$

The precise meaning of creation rates is not important here; they are differences of the quantities of primary interest, namely the *placement probabilities* that tell the likelihood of the square (i,j) in the order-n diamond being covered by a domino in a particular orientation. (The placement probability generating function is similar to the creation rate generating function but has an extra factor in the denominator.) It is not hard to verify that Q, the denominator of F, has a singularity at $(1,1,1)$ which is geometrically a cone. The tangent cone to Q at $(1,1,1)$ is $z^2 - (1/2)(x^2+y^2)$, which is also the tangent cone to the complement of $\mathsf{amoeba}(Q)$ at $(0,0,0)$, and its dual is the cone $\mathbf{N}_* := \{(r,s,t) : t \geqslant 0, t^2 \geqslant 2(r^2+s^2)\}$.

We consider asymptotics whose dominating point is some fixed $\mathbf{Z}_*$ at which $\mathcal{V}$ has nontrivial local geometry. We assume that $\mathbf{r}_* \in \mathbf{N}_*$, the dual to the tangent cone to B at $\mathbf{x}_* := \mathrm{ReLog}\, \mathbf{Z}_*$ (in the absence of this, $e^{\mathbf{r}\cdot\mathbf{x}} a_{\mathbf{r}}$ will be exponentially small). The Cauchy integral in logarithmic coordinates (2.2) is a generalized Fourier transform of $f = F \circ \exp$, which has a singularity of the same type. What we find in the end is that $a_{\mathbf{r}}$ is estimated by $\hat{f}(\mathbf{r})$, the generalized Fourier transform of f evaluated at $\mathbf{r}$. The Fourier transform of f is estimated asymptotically by the Fourier transform of its leading homogeneous term, which is the cone $\alpha(x,y,z) := z^2 - (1/2)(x^2+y^2)$. The Fourier transform of a homogeneous quadratic is the dual quadratic $\alpha^*(r,s,t)$ which in this case is given by $t^2 - 2(r^2+s^2)$. Thus we have, for example, the following result.

THEOREM 4.11 ([**BP08**, Theorem 4.2]). *Let $\{a_{\mathbf{r}}\}$ be the coefficients of the series for the Aztec Diamond creation generating function* (4.7)*, with $\mathbf{r} := (r, s, t)$. Then*

$$a_{rst} \sim \frac{4}{\pi}(t^2 - 2r^2 - 2s^2)^{-1/2},$$

uniformly as $\mathbf{r} \to \infty$ and $\hat{\mathbf{r}}$ varies over compact subsets of the set $\{r^2 + s^2 < t^2/2\}$.

SKETCH OF PROOF. Begin with the Cauchy integral (2.2) in logarithmic coordinates. The function $f = F \circ \exp$ is meromorphic. Expand it as p/q and let $q = \alpha \cdot (1 + \rho)$, where α is the homogeneous quadratic $z^2 - (1/2)(x^2 + y^2)$ and $\rho(\mathbf{x}) = O(|\mathbf{x}|^3)$. We may then expand $1/q$ as a series

$$\frac{1}{q} = \frac{1}{\alpha}\left(1 - \frac{\rho}{\alpha} + \frac{\rho^2}{\alpha^2} + \cdots\right),$$

where $(\rho/\alpha)^j = O(|\mathbf{x}|^j)$ on compact cones avoiding the cone $\alpha = 0$. Expanding the numerator p into monomials, we arrive at a double series

$$f = \sum_{\mathbf{m},j} \frac{\mathbf{x}^{\mathbf{m}}}{\alpha^{1+j}}.$$

Provided the remainders obey the necessary estimate, we may integrate this term by term; the theorem requires only the leading integral and remainder estimate.

The integral

$$\int_{\mathbf{x}+i\mathbb{R}^d} e^{-\lambda(\mathbf{r}\cdot\mathbf{z})} \frac{1}{\alpha^s}\, d\mathbf{z}$$

is studied in [**ABG70**]. Although this integral is not convergent, it defines a *distribution* or *generalized function* in the sense of [**GS64**]. As a distribution, this is precisely what M. Riesz [**Rie49**] identified as the generalized function $C(\alpha^*)^{s-d/2}$, where the constant C is the product of values of the Gamma function, slightly misquoted in [**ABG70**, Equation (4.20)]. To get from the generalized function to an estimate for the particular integral (2.2), and to replace the limits of the integral as well by $\mathbf{x} + i\mathbb{R}^d$, one requires a deformation of $\mathbf{x} + i\mathbb{R}^d$ to a set where $\mathbf{r} \cdot \mathbf{x}$ is uniformly positive, i.e., $\mathbf{r} \cdot \mathbf{x} \geqslant \epsilon\,|\mathbf{x}|$. This is another Morse-theoretic result, proved in [**ABG70**] for hyperbolic linear differential operators and adapted in [**BP08**, Theorem 5.8]. □

5. Summary, a counterexample, and a conjecture

On the subject of the exponential growth rate, we have seen that $\overline{\beta}(\mathbf{r}_*)$ is at most $\beta_Q(\mathbf{r}_*) := -\mathbf{r}_* \cdot \mathbf{x}_*$ where $\mathbf{x}_*$ is the minimizing point. A list of known reasons why $\overline{\beta}$ (or β) can be strictly less than β_Q is as follows.

- (i) Periodicity. For example, if $\beta(\mathbf{r}_*)$ exists for F and is finite, then β will fail to exist for $F(\mathbf{Z}) + F(-\mathbf{Z})$, the coefficients of which are twice those of F when $\mathbf{r}$ is even, and zero when $\mathbf{r}$ is odd.
- (ii) mincrit is empty. We saw in Theorem 2.8 that this implies $\overline{\beta} < \beta_Q$.
- (iii) mincrit is nonempty, but $P(\mathbf{Z}_*)$ vanishes on mincrit.
- (iv) mincrit is nonempty, but at every $\mathbf{Z}_* \in$ mincrit on which $P(\mathbf{Z}_*)$ is nonvanishing, the normal cone $\mathbf{N}_*$ fails to contain $\mathbf{r}_*$.

However, if there is no topological obstruction to moving the chain of integration further down, then the minimax height c is less than $-\mathbf{r}_* \cdot \mathbf{x}_*$ and is in fact equal to $-\mathbf{r}_* \cdot \mathbf{x}'$, where $\mathbf{x}' = \operatorname{ReLog} \mathbf{Z}'$ and $\mathbf{Z}'$ is a contributing critical point. If there is a contributing critical point of height c for which $P \notin \mathcal{J}$, then we define $\hat{\beta}_Q(\mathbf{r}) = c$. Otherwise, the partial fraction expansion tells us that the integrand is the sum of pieces whose domains of holomorphy are greater than that of the original integrand. By treating each separately, the chain of integration may be pushed below c to some new minimax height. We continue in this manner until we reach a point $\mathbf{Z}_*$ for which $\mathcal{C}$ has nonzero projection in $H_d(\mathcal{M}_{\text{loc}}^{\mathbf{Z}_*})$ and $P \notin \mathcal{J}(\mathbf{Z}_*)$. In this way, we may define our best guess for the exponential rate as

$$\hat{\beta}_Q(\mathbf{r}_*) := -\mathbf{r}_* \cdot \mathbf{x}_*$$

where $\mathbf{x}_* = \operatorname{ReLog} \mathbf{Z}_*$ for this critical point $\mathbf{Z}_*$ which maximizes height among all contributing critical points with $P \notin \mathcal{J}(\mathbf{Z}_*)$. A reasonable conjecture would seem to be that $\overline{\beta}(\mathbf{r}_*) = \hat{\beta}_Q$, but there is a counterexample (not yet published) as follows. The trivariate generating function $F = P/Q$ for the so-called *fortress* variant of the Aztec Diamond tiling model has a non-smooth point at $(1,1,1)$. The normal cone $\mathbf{N}_*$ dual to $\mathbf{K}(1,1,1)$ may be divided into an inner region A and an outer region B such that for $\mathbf{r}_* \in B$, asymptotics are not exponential, whereas for $\mathbf{r}_* \in A$, asymptotics decay exponentially. For $\mathbf{r}_* \in B$, the magnitude of $a_{\mathbf{r}}$ does not decay exponentially as $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}} \to \mathbf{r}_*$. By the contrapositive of Theorem 2.8, since $\mathsf{mincrit}(\mathbf{r}_*) = \{(1,1,1)\}$, we conclude that $\mathbf{r}_* \in \mathbf{N}_*(\mathbf{r}_*)$ for $\mathbf{r}_* \in B$. It follows that this holds for $\mathbf{r}_* \in A$ as well. Exponential decay for $\mathbf{r}_* \in A$ then shoots down the conjecture. We are left with the following weakened version of the conjecture for minimal points, based on the belief that the fortress counterexample is the only kind that can arise and that it must happen on the interior of $\mathbf{N}_*(\mathbf{r}_*)$.

CONJECTURE 5.1 (weak sharpness of $\hat{\beta}_Q$). Suppose $\mathbf{Z}_*$ is a minimal critical point for some $\mathbf{r}_*$. Suppose that $P \notin \mathcal{J}(\mathbf{Z}_*)$, and let $\mathbf{x}_* := \operatorname{ReLog}(\mathbf{Z}_*)$. Then $\overline{\beta}(\mathbf{r}') = -\mathbf{r}' \cdot \mathbf{x}_*$ for $\mathbf{r}'$ in some set A whose convex hull is $\mathbf{N}_*(\mathbf{Z}_*)$.

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