Analytic Combinatorics in Several Variables

Mark C. Wilson
Department of Computer Science
University of Auckland

AEC Summer School, Hagenberg, 2014-08-(18-22)

Lecture I

Motivation, review, overview

Table of Contents

Recommended books for univariate asymptotics

- H. Wilf, *generatingfunctionology*, `http://www.math.upenn.edu/~wilf/DownldGF.html`

Recommended books for univariate asymptotics

- H. Wilf, *generatingfunctionology*, http://www.math.upenn.edu/~wilf/DownldGF.html
- M. Kauers and P. Paule, *The Concrete Tetrahedron*, www.risc.jku.at/people/mkauers/publications/kauers11h.pdf.

Recommended books for univariate asymptotics

- H. Wilf, *generatingfunctionology*, http://www.math.upenn.edu/~wilf/DownldGF.html
- M. Kauers and P. Paule, *The Concrete Tetrahedron*, www.risc.jku.at/people/mkauers/publications/kauers11h.pdf.
- P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, http://algo.inria.fr/flajolet/Publications/AnaCombi/anacombi.html

Recommended books for univariate asymptotics

- H. Wilf, *generatingfunctionology*, http://www.math.upenn.edu/~wilf/DownldGF.html
- M. Kauers and P. Paule, *The Concrete Tetrahedron*, www.risc.jku.at/people/mkauers/publications/kauers11h.pdf.
- P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, http://algo.inria.fr/flajolet/Publications/AnaCombi/anacombi.html
- A. Odlyzko, *Asymptotic Enumeration Methods*, www.dtc.umn.edu/~odlyzko/doc/enumeration.html..

Main references for all lectures

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. `https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/`

Main references for all lectures

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. `https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/`
- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.

Main references for all lectures

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. `https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/`
- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.
- Sage implementations by Alex Raichev: `https://github.com/araichev/amgf`.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:
  (i) Motivation, review of univariate case, overview of results.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:
  (i) Motivation, review of univariate case, overview of results.
  (ii) Smooth points in dimension 2.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:
  (i) Motivation, review of univariate case, overview of results.
  (ii) Smooth points in dimension 2.
  (iii) Higher dimensions, multiple points.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:
  (i) Motivation, review of univariate case, overview of results.
  (ii) Smooth points in dimension 2.
  (iii) Higher dimensions, multiple points.
  (iv) Computational issues.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:
  (i) Motivation, review of univariate case, overview of results.
  (ii) Smooth points in dimension 2.
  (iii) Higher dimensions, multiple points.
  (iv) Computational issues.
  (v) Extensions and open problems.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:
  - (i) Motivation, review of univariate case, overview of results.
  - (ii) Smooth points in dimension 2.
  - (iii) Higher dimensions, multiple points.
  - (iv) Computational issues.
  - (v) Extensions and open problems.
- Exercises are of varying levels of difficulty. We can discuss some in the problem sessions. Those marked (C) involve probably publishable research, for which I am seeking collaborators, and should be accessible to PhD students.

Table of Contents

Lecture 1: Overview

- In one variable, starting with a sequence $a_{\mathbf{r}}$ of interest, we form its generating function $F(\mathbf{z})$. Cauchy's integral theorem allows us to express $a_{\mathbf{r}}$ as an integral. The exponential growth rate of $a_{\mathbf{r}}$ is determined by the location of a dominant singularity $\mathbf{z}_*$ of F. More precise estimates depend on the local geometry of the singular set $\mathcal{V}$ of F near $\mathbf{z}_*$.

Lecture 1: Overview

- In one variable, starting with a sequence $a_{\mathbf{r}}$ of interest, we form its generating function $F(\mathbf{z})$. Cauchy's integral theorem allows us to express $a_{\mathbf{r}}$ as an integral. The exponential growth rate of $a_{\mathbf{r}}$ is determined by the location of a dominant singularity $\mathbf{z}_*$ of F. More precise estimates depend on the local geometry of the singular set $\mathcal{V}$ of F near $\mathbf{z}_*$.
- In the multivariate case, all the above is still true. However, we need to specify the direction in which we want asymptotics; we then need to worry about uniformity; the definition of "dominant" is a little different; the local geometry of $\mathcal{V}$ can be much nastier; the local analysis is more complicated.

Standing assumptions

- Unless otherwise specified, the following hold throughout.

Standing assumptions

- Unless otherwise specified, the following hold throughout.
- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.

Standing assumptions

- Unless otherwise specified, the following hold throughout.
- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.

Standing assumptions

- Unless otherwise specified, the following hold throughout.
- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The generating function (GF) is the formal power series $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$.

Standing assumptions

- Unless otherwise specified, the following hold throughout.
- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The generating function (GF) is the formal power series $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$.
- The combinatorial case: all $a_{\mathbf{r}} \geq 0$.

Standing assumptions

- Unless otherwise specified, the following hold throughout.
- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The generating function (GF) is the formal power series $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$.
- The combinatorial case: all $a_{\mathbf{r}} \geq 0$.
- The aperiodic case: $a_{\mathbf{r}}$ is not supported on a proper sublattice of $\mathbb{N}^d$.

From sequence to generating function and back

- Most sequences of interest satisfy recurrences. We analyse them using the GF. Sequence operations correspond to algebraic operations on power series (e.g. $a_n \leftrightarrow F(z)$ implies $na_n \leftrightarrow zF'(z)$).

From sequence to generating function and back

- Most sequences of interest satisfy recurrences. We analyse them using the GF. Sequence operations correspond to algebraic operations on power series (e.g. $a_n \leftrightarrow F(z)$ implies $na_n \leftrightarrow zF'(z)$).
- The GF can often be determined by translating the recurrence into a functional equation for F, then solving it.

From sequence to generating function and back

- Most sequences of interest satisfy recurrences. We analyse them using the GF. Sequence operations correspond to algebraic operations on power series (e.g. $a_n \leftrightarrow F(z)$ implies $na_n \leftrightarrow zF'(z)$).
- The GF can often be determined by translating the recurrence into a functional equation for F, then solving it.
- Example: (Fibonacci)

$$\begin{aligned} a_r &= a_{r-1} + a_{r-2} \quad \text{if } r \geq 2 \\ a_0 &= 0, a_1 = 1 \end{aligned}$$

automatically yields $F(z) = z/(1 - z - z^2)$.

From sequence to generating function and back

- Most sequences of interest satisfy recurrences. We analyse them using the GF. Sequence operations correspond to algebraic operations on power series (e.g. $a_n \leftrightarrow F(z)$ implies $na_n \leftrightarrow zF'(z)$).
- The GF can often be determined by translating the recurrence into a functional equation for F, then solving it.
- Example: (Fibonacci)

$$\begin{aligned} a_r &= a_{r-1} + a_{r-2} \qquad \text{if } r \geq 2 \\ a_0 &= 0, a_1 = 1 \end{aligned}$$

  automatically yields $F(z) = z/(1 - z - z^2)$.
- Our focus this week is on the next step: deriving a formula (usually asymptotic approximation) for a_r, given a nice representation of F. This is coefficient extraction.

Univariate case: exponential growth rate

- Let U be the open disc of convergence of F, having radius ρ, $\partial\,\mathrm{U}$ its boundary.

Univariate case: exponential growth rate

- Let U be the open disc of convergence of F, having radius ρ, $\partial\,\mathrm{U}$ its boundary.
- Let C_R denote the circle of radius R centred at 0. If $R < \rho$ then by Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i} \int_C z^{-n-1} F(z)\, dz.$$

Univariate case: exponential growth rate

- Let U be the open disc of convergence of F, having radius ρ, $\partial\,\mathrm{U}$ its boundary.
- Let C_R denote the circle of radius R centred at 0. If $R < \rho$ then by Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i}\int_C z^{-n-1}F(z)\,dz.$$

- This directly yields $|a_n| \leq (2\pi i R)KR^{-n-1}/(2\pi i) = KR^{-n}$.

Univariate case: exponential growth rate

- Let U be the open disc of convergence of F, having radius ρ, $\partial\,\mathrm{U}$ its boundary.
- Let C_R denote the circle of radius R centred at 0. If $R < \rho$ then by Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i}\int_C z^{-n-1} F(z)\, dz.$$

- This directly yields $|a_n| \leq (2\pi i R) K R^{-n-1}/(2\pi i) = K R^{-n}$.
- Letting $R \to \rho_-$ shows that the exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{n} \log |a_n| = -\log \rho.$$

Univariate case: exponential growth rate

- Let U be the open disc of convergence of F, having radius ρ, $\partial\,\mathrm{U}$ its boundary.
- Let C_R denote the circle of radius R centred at 0. If $R < \rho$ then by Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i} \int_C z^{-n-1} F(z)\, dz.$$

- This directly yields $|a_n| \leq (2\pi i R) K R^{-n-1}/(2\pi i) = K R^{-n}$.
- Letting $R \to \rho_-$ shows that the exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{n} \log |a_n| = -\log \rho.$$

- Suppose that $\rho < \infty$. Then (Vivanti-Pringsheim) $z = \rho$ is a singularity of F, and is the only singularity of F on $\partial\,\mathrm{U}$.

Univariate case: exponential growth rate

- Let U be the open disc of convergence of F, having radius ρ, $\partial\,\mathrm{U}$ its boundary.
- Let C_R denote the circle of radius R centred at 0. If $R < \rho$ then by Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i} \int_C z^{-n-1} F(z)\, dz.$$

- This directly yields $|a_n| \leq (2\pi i R) K R^{-n-1}/(2\pi i) = K R^{-n}$.
- Letting $R \to \rho_-$ shows that the exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{n} \log |a_n| = -\log \rho.$$

- Suppose that $\rho < \infty$. Then (Vivanti-Pringsheim) $z = \rho$ is a singularity of F, and is the only singularity of F on $\partial\,\mathrm{U}$.
- Further analysis depends on the type of singularity.

From singularities to asymptotic expansions

There are standard methods for dealing with each type of singularity, all relying on choosing appropriate contours of integration. The most common:

- if ρ is a pole, use the residue theorem;

From singularities to asymptotic expansions

There are standard methods for dealing with each type of singularity, all relying on choosing appropriate contours of integration. The most common:

- if ρ is a pole, use the residue theorem;
- if F is rational, can also use partial fraction decomposition;

From singularities to asymptotic expansions

There are standard methods for dealing with each type of singularity, all relying on choosing appropriate contours of integration. The most common:

- if ρ is a pole, use the residue theorem;
- if F is rational, can also use partial fraction decomposition;
- if ρ is algebraic/logarithmic, use singularity analysis (Flajolet-Odlyzko 1990);

From singularities to asymptotic expansions

There are standard methods for dealing with each type of singularity, all relying on choosing appropriate contours of integration. The most common:

- if ρ is a pole, use the residue theorem;
- if F is rational, can also use partial fraction decomposition;
- if ρ is algebraic/logarithmic, use singularity analysis (Flajolet-Odlyzko 1990);
- if ρ is essential, use the saddle point method.

Example (Univariate pole: derangements)

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$, so $a_r = O(1)$.

Example (Univariate pole: derangements)

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$, so $a_r = O(1)$.
- Using a circle of radius $1+\varepsilon$ we obtain, by Cauchy's residue theorem,

$$a_r = \frac{1}{2\pi i} \int_{C_{1+\varepsilon}} z^{-r-1} F(z)\, dz - \operatorname{Res}(z^{-r-1}F(z); z = 1).$$

Example (Univariate pole: derangements)

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$, so $a_r = O(1)$.
- Using a circle of radius $1+\varepsilon$ we obtain, by Cauchy's residue theorem,

$$a_r = \frac{1}{2\pi i}\int_{C_{1+\varepsilon}} z^{-r-1}F(z)\,dz - \mathrm{Res}(z^{-r-1}F(z); z=1).$$

- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.

Example (Univariate pole: derangements)

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$, so $a_r = O(1)$.
- Using a circle of radius $1 + \varepsilon$ we obtain, by Cauchy's residue theorem,

$$a_r = \frac{1}{2\pi i} \int_{C_{1+\varepsilon}} z^{-r-1} F(z)\, dz - \mathrm{Res}(z^{-r-1}F(z); z = 1).$$

- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.
- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.

Example (Univariate pole: derangements)

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$, so $a_r = O(1)$.
- Using a circle of radius $1+\varepsilon$ we obtain, by Cauchy's residue theorem,

$$a_r = \frac{1}{2\pi i} \int_{C_{1+\varepsilon}} z^{-r-1} F(z)\, dz - \mathrm{Res}(z^{-r-1}F(z); z = 1).$$

- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.
- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.
- Since there are no more poles, we can push the contour of integration to ∞ in this case, so the error in the approximation decays faster than any exponential function of r.

Univariate rational functions: general solution

- Given a rational function $p(z)/q(z)$ with $q(0) = 1$, factor it as $q(z) = \prod_i (1 - \phi_i z)^{n_i}$ with all ϕ_i distinct.

Univariate rational functions: general solution

- Given a rational function $p(z)/q(z)$ with $q(0) = 1$, factor it as $q(z) = \prod_i (1 - \phi_i z)^{n_i}$ with all ϕ_i distinct.
- Use partial fractions to expand

$$F(z) = \sum_i \sum_{j=1}^{n_i} \frac{c_{ij}}{(1 - \phi_i z)^j}.$$

Univariate rational functions: general solution

- Given a rational function $p(z)/q(z)$ with $q(0) = 1$, factor it as $q(z) = \prod_i (1 - \phi_i z)^{n_i}$ with all ϕ_i distinct.
- Use partial fractions to expand

$$F(z) = \sum_i \sum_{j=1}^{n_i} \frac{c_{ij}}{(1 - \phi_i z)^j}.$$

- This allows for an exact formula, and restricting to the largest ϕ_i (corresponding to the minimal zeros of q) gives the asymptotic expansion.

Univariate rational functions: general solution

- Given a rational function $p(z)/q(z)$ with $q(0) = 1$, factor it as $q(z) = \prod_i (1 - \phi_i z)^{n_i}$ with all ϕ_i distinct.
- Use partial fractions to expand

$$F(z) = \sum_i \sum_{j=1}^{n_i} \frac{c_{ij}}{(1 - \phi_i z)^j}.$$

- This allows for an exact formula, and restricting to the largest ϕ_i (corresponding to the minimal zeros of q) gives the asymptotic expansion.
- For example, Fibonacci yields $a_r \sim 5^{-1/2}[(1 + \sqrt{5})/2]^r$.

Univariate rational functions: general solution

- Given a rational function $p(z)/q(z)$ with $q(0) = 1$, factor it as $q(z) = \prod_i (1 - \phi_i z)^{n_i}$ with all ϕ_i distinct.
- Use partial fractions to expand

$$F(z) = \sum_i \sum_{j=1}^{n_i} \frac{c_{ij}}{(1 - \phi_i z)^j}.$$

- This allows for an exact formula, and restricting to the largest ϕ_i (corresponding to the minimal zeros of q) gives the asymptotic expansion.
- For example, Fibonacci yields $a_r \sim 5^{-1/2}[(1 + \sqrt{5})/2]^r$.
- Repeated roots provide polynomial correction to the exponential factor. For example, $1/(1 - 2z)^3 = \sum_r \binom{r+2}{2} 2^r z^r$.

Example (Essential singularity: saddle point method)

- Here $F(z) = \exp(z)$. The Cauchy integral formula on a circle C_R of radius R gives $a_n \leq F(R)/R^n$.

Example (Essential singularity: saddle point method)

- Here $F(z) = \exp(z)$. The Cauchy integral formula on a circle C_R of radius R gives $a_n \leq F(R)/R^n$.
- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.

Example (Essential singularity: saddle point method)

- Here $F(z) = \exp(z)$. The Cauchy integral formula on a circle C_R of radius R gives $a_n \leq F(R)/R^n$.
- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.
- The integral over C_n has most mass near $z = n$, so that

$$
\begin{aligned}
a_n &= \frac{F(n)}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) \frac{F(ne^{i\theta})}{F(n)} \, d\theta \\
&\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp\left(-in\theta + \log F(ne^{i\theta}) - \log F(n)\right) \, d\theta.
\end{aligned}
$$

Example (Saddle point example continued)

- The Maclaurin expansion yields

$$-in\theta + \log F(ne^{i\theta}) - \log F(n) = -n\theta^2/2 + O(n\theta^3).$$

Example (Saddle point example continued)

- The Maclaurin expansion yields

$$-in\theta + \log F(ne^{i\theta}) - \log F(n) = -n\theta^2/2 + O(n\theta^3).$$

- This gives, with $b_n = 2\pi n^n e^{-n} a_n$,

$$b_n \approx \int_{-\varepsilon}^{\varepsilon} \exp(-n\theta^2/2)\, d\theta \approx \int_{-\infty}^{\infty} \exp(-n\theta^2/2)\, d\theta = \sqrt{2\pi/n}.$$

Example (Saddle point example continued)

- The Maclaurin expansion yields

$$-in\theta + \log F(ne^{i\theta}) - \log F(n) = -n\theta^2/2 + O(n\theta^3).$$

- This gives, with $b_n = 2\pi n^n e^{-n} a_n$,

$$b_n \approx \int_{-\varepsilon}^{\varepsilon} \exp(-n\theta^2/2)\, d\theta \approx \int_{-\infty}^{\infty} \exp(-n\theta^2/2)\, d\theta = \sqrt{2\pi/n}.$$

- This recaptures Stirling's approximation, since $n! = 1/a_n$:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

Multivariate asymptotics — some quotations

- (Bender 1974) "Practically nothing is known about asymptotics for recursions in two variables even when a GF is available. Techniques for obtaining asymptotics from bivariate GFs would be quite useful."

Multivariate asymptotics — some quotations

- (Bender 1974) "Practically nothing is known about asymptotics for recursions in two variables even when a GF is available. Techniques for obtaining asymptotics from bivariate GFs would be quite useful."
- (Odlyzko 1995) "A major difficulty in estimating the coefficients of mvGFs is that the geometry of the problem is far more difficult. . . . Even rational multivariate functions are not easy to deal with."

Multivariate asymptotics — some quotations

- (Bender 1974) "Practically nothing is known about asymptotics for recursions in two variables even when a GF is available. Techniques for obtaining asymptotics from bivariate GFs would be quite useful."
- (Odlyzko 1995) "A major difficulty in estimating the coefficients of mvGFs is that the geometry of the problem is far more difficult. . . . Even rational multivariate functions are not easy to deal with."
- (Flajolet/Sedgewick 2009) "Roughly, we regard here a bivariate GF as a collection of univariate GFs . . . ."

Finding multivariate GFs

- Unlike the univariate case, a constant coefficient linear recursion need not yield a rational function. This occurs, for example, in lattice walks where steps go forward in some dimensions and backward in others.

Finding multivariate GFs

- Unlike the univariate case, a constant coefficient linear recursion need not yield a rational function. This occurs, for example, in lattice walks where steps go forward in some dimensions and backward in others.
- The kernel method (see Chapter 2.3) is often useful for dealing with these cases.

Finding multivariate GFs

- Unlike the univariate case, a constant coefficient linear recursion need not yield a rational function. This occurs, for example, in lattice walks where steps go forward in some dimensions and backward in others.
- The kernel method (see Chapter 2.3) is often useful for dealing with these cases.
- Linear recursions with polynomial coefficients yield linear PDEs, which can be hard to solve, certainly harder than the ODEs in the univariate case.

Finding multivariate GFs

- Unlike the univariate case, a constant coefficient linear recursion need not yield a rational function. This occurs, for example, in lattice walks where steps go forward in some dimensions and backward in others.
- The kernel method (see Chapter 2.3) is often useful for dealing with these cases.
- Linear recursions with polynomial coefficients yield linear PDEs, which can be hard to solve, certainly harder than the ODEs in the univariate case.
- We will not deal with this issue in these lectures - we assume that the GF is given in explicit form (say rational or algebraic) and concentrate on extraction of Maclaurin coefficients.

Diagonal method

- Suppose that $d = 2$ and we want asymptotics from $F(z, w)$ on the diagonal $r = s$.

Diagonal method

- Suppose that $d = 2$ and we want asymptotics from $F(z, w)$ on the diagonal $r = s$.
- The diagonal GF is $F_{1,1}(x) = \sum_n a_{nn} x^n$.

Diagonal method

- Suppose that $d = 2$ and we want asymptotics from $F(z, w)$ on the diagonal $r = s$.
- The diagonal GF is $F_{1,1}(x) = \sum_n a_{nn} x^n$.
- We can compute, for some circle γ_x around $t = 0$,

$$
\begin{aligned}
F_{1,1}(x) &= [t^0] F(x/t, t) \\
&= \frac{1}{2\pi i} \int_{\gamma_x} \frac{F(x/t, t)}{t} \, dt \\
&= \sum_k \mathrm{Res}(F(x/t, t)/t; t = s_k(x))
\end{aligned}
$$

where the $s_k(x)$ are the singularities satisfying $\lim_{x \to 0} s_k(x) = 0$.

Diagonal method

- Suppose that $d = 2$ and we want asymptotics from $F(z, w)$ on the diagonal $r = s$.
- The diagonal GF is $F_{1,1}(x) = \sum_n a_{nn} x^n$.
- We can compute, for some circle γ_x around $t = 0$,

$$\begin{aligned} F_{1,1}(x) &= [t^0]F(x/t, t) \\ &= \frac{1}{2\pi i} \int_{\gamma_x} \frac{F(x/t, t)}{t}\, dt \\ &= \sum_k \mathrm{Res}(F(x/t, t)/t; t = s_k(x)) \end{aligned}$$

  where the $s_k(x)$ are the singularities satisfying $\lim_{x \to 0} s_k(x) = 0$.
- If F is rational, then $F_{1,1}$ is algebraic.

Example (Delannoy lattice walks)

- Consider walks in $\mathbb{Z}^2$, starting from $(0,0)$, with steps in $\{(1,0),(0,1),(1,1)\}$ (Delannoy walks).

Example (Delannoy lattice walks)

- Consider walks in $\mathbb{Z}^2$, starting from $(0,0)$, with steps in $\{(1,0),(0,1),(1,1)\}$ (Delannoy walks).
- Here $F(x,y) = (1-x-y-xy)^{-1}$.

Example (Delannoy lattice walks)

- Consider walks in $\mathbb{Z}^2$, starting from $(0,0)$, with steps in $\{(1,0),(0,1),(1,1)\}$ (Delannoy walks).
- Here $F(x,y) = (1 - x - y - xy)^{-1}$.
- This corresponds to the recurrence $a_{rs} = a_{r,s-1} + a_{r-1,s} + a_{r-1,s-1}$.

Example (Delannoy lattice walks)

- Consider walks in $\mathbb{Z}^2$, starting from $(0,0)$, with steps in $\{(1,0),(0,1),(1,1)\}$ (Delannoy walks).
- Here $F(x,y) = (1-x-y-xy)^{-1}$.
- This corresponds to the recurrence $a_{rs} = a_{r,s-1} + a_{r-1,s} + a_{r-1,s-1}$.
- How to compute a_{rs} for large r, s?

Example (Delannoy lattice walks)

- Consider walks in $\mathbb{Z}^2$, starting from $(0,0)$, with steps in $\{(1,0),(0,1),(1,1)\}$ (Delannoy walks).
- Here $F(x,y) = (1-x-y-xy)^{-1}$.
- This corresponds to the recurrence $a_{rs} = a_{r,s-1} + a_{r-1,s} + a_{r-1,s-1}$.
- How to compute a_{rs} for large r, s?
- For example, what does $a_{7n,5n}$ look like as $n \to \infty$?

Use the diagonal method?

- We could try to compute the diagonal GF $F_{pq}(z) := \sum_{n \geq 0} a_{pn,qn} z^n$ as above.

Use the diagonal method?

- We could try to compute the diagonal GF $F_{pq}(z) := \sum_{n\geq 0} a_{pn,qn} z^n$ as above.
- This would work fairly well for $p = q = 1$, but is generally a bad idea (see Chapter 13.1):

Use the diagonal method?

- We could try to compute the diagonal GF $F_{pq}(z) := \sum_{n \geq 0} a_{pn,qn} z^n$ as above.
- This would work fairly well for $p = q = 1$, but is generally a bad idea (see Chapter 13.1):
  - The computational complexity increases rapidly with $p + q$.

Use the diagonal method?

- We could try to compute the diagonal GF $F_{pq}(z) := \sum_{n\geq 0} a_{pn,qn} z^n$ as above.
- This would work fairly well for $p = q = 1$, but is generally a bad idea (see Chapter 13.1):
  - The computational complexity increases rapidly with $p + q$.
  - We can't handle irrational diagonals, or derive uniform asymptotics (if p/q changes slightly, what do we do?).

Use the diagonal method?

- We could try to compute the diagonal GF $F_{pq}(z) := \sum_{n \geq 0} a_{pn,qn} z^n$ as above.
- This would work fairly well for $p = q = 1$, but is generally a bad idea (see Chapter 13.1):
  - The computational complexity increases rapidly with $p + q$.
  - We can't handle irrational diagonals, or derive uniform asymptotics (if p/q changes slightly, what do we do?).
  - If $d > 2$, diagonals will not be algebraic in general, even if F is rational.

Use the diagonal method?

- We could try to compute the diagonal GF $F_{pq}(z) := \sum_{n \geq 0} a_{pn,qn} z^n$ as above.
- This would work fairly well for $p = q = 1$, but is generally a bad idea (see Chapter 13.1):
  - The computational complexity increases rapidly with $p + q$.
  - We can't handle irrational diagonals, or derive uniform asymptotics (if p/q changes slightly, what do we do?).
  - If $d > 2$, diagonals will not be algebraic in general, even if F is rational.
  - Fancier methods exist (based on holonomic or D-finite theory), but again computational complexity is a major obstacle.

Our plan

- Thoroughly investigate asymptotic coefficient extraction, starting with meromorphic $F(\mathbf{z}) := F(z_1, \ldots, z_d)$ (pole singularities).

Our plan

- Thoroughly investigate asymptotic coefficient extraction, starting with meromorphic $F(\mathbf{z}) := F(z_1, \ldots, z_d)$ (pole singularities).
- Directly generalize the $d = 1$ analysis for poles.

Our plan

- Thoroughly investigate asymptotic coefficient extraction, starting with meromorphic $F(\mathbf{z}) := F(z_1, \ldots, z_d)$ (pole singularities).
- Directly generalize the $d = 1$ analysis for poles.
- Use the Cauchy Integral Formula in dimension d.

Our plan

- Thoroughly investigate asymptotic coefficient extraction, starting with meromorphic $F(\mathbf{z}) := F(z_1, \ldots, z_d)$ (pole singularities).
- Directly generalize the $d = 1$ analysis for poles.
- Use the Cauchy Integral Formula in dimension d.
- Use residue analysis to derive asymptotics.

Our plan

- Thoroughly investigate asymptotic coefficient extraction, starting with meromorphic $F(\mathbf{z}) := F(z_1, \ldots, z_d)$ (pole singularities).
- Directly generalize the $d = 1$ analysis for poles.
- Use the Cauchy Integral Formula in dimension d.
- Use residue analysis to derive asymptotics.
- Amazingly little was known even about rational F in 2 variables. We aimed to create a general theory.

Some difficulties when $d > 1$

- Asymptotics:

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points, and may self-intersect;

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points, and may self-intersect;
  - real dimension of contour is d, that of $\mathcal{V}$ is $2d - 2$, so less room to avoid each other;

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points, and may self-intersect;
  - real dimension of contour is d, that of $\mathcal{V}$ is $2d - 2$, so less room to avoid each other;
  - topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated;

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points, and may self-intersect;
  - real dimension of contour is d, that of $\mathcal{V}$ is $2d - 2$, so less room to avoid each other;
  - topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated;
- Analysis: the (Leray) residue formula is much harder to use.

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\overline{\mathbf{r}}$, we may restrict to a dominant point $\mathbf{z}_*(\overline{\mathbf{r}})$ lying in the positive orthant. (*)

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\overline{\mathbf{r}}$, we may restrict to a dominant point $\mathbf{z}_*(\overline{\mathbf{r}})$ lying in the positive orthant. (*)
- There is an asymptotic expansion $\mathrm{formula}(\mathbf{z}_*)$ for $a_{\mathbf{r}}$, where $\mathrm{formula}(\mathbf{z}_*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\overline{\mathbf{r}}$, we may restrict to a dominant point $\mathbf{z}_*(\overline{\mathbf{r}})$ lying in the positive orthant. (*)
- There is an asymptotic expansion $\mathrm{formula}(\mathbf{z}_*)$ for $a_{\mathbf{r}}$, where $\mathrm{formula}(\mathbf{z}_*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.
- This yields
$$a_{\mathbf{r}} \sim \mathrm{formula}(\mathbf{z}_*)$$
where the expansion is uniform on compact subsets of directions, provided the geometry does not change.

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\overline{\mathbf{r}}$, we may restrict to a dominant point $\mathbf{z}_*(\overline{\mathbf{r}})$ lying in the positive orthant. (*)
- There is an asymptotic expansion $\mathrm{formula}(\mathbf{z}_*)$ for $a_{\mathbf{r}}$, where $\mathrm{formula}(\mathbf{z}_*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.
- This yields

$$a_{\mathbf{r}} \sim \mathrm{formula}(\mathbf{z}_*)$$

  where the expansion is uniform on compact subsets of directions, provided the geometry does not change.
- The set $\mathrm{crit}(\overline{\mathbf{r}})$ is computable via symbolic algebra.

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\overline{\mathbf{r}}$, we may restrict to a dominant point $\mathbf{z}_*(\overline{\mathbf{r}})$ lying in the positive orthant. (*)
- There is an asymptotic expansion $\mathrm{formula}(\mathbf{z}_*)$ for $a_{\mathbf{r}}$, where $\mathrm{formula}(\mathbf{z}_*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.
- This yields

$$a_{\mathbf{r}} \sim \mathrm{formula}(\mathbf{z}_*)$$

  where the expansion is uniform on compact subsets of directions, provided the geometry does not change.
- The set $\mathrm{crit}(\overline{\mathbf{r}})$ is computable via symbolic algebra.
- To determine the dominant point requires a little more work, but usually not much. (*)

Obvious questions

- Can we always find asymptotics in a given direction in this way?

Obvious questions

- Can we always find asymptotics in a given direction in this way?
- How do we find the dominant point?

Obvious questions

- Can we always find asymptotics in a given direction in this way?
- How do we find the dominant point?
- How easy is it to carry out all the computations?

Obvious questions

- Can we always find asymptotics in a given direction in this way?
- How do we find the dominant point?
- How easy is it to carry out all the computations?
- What about higher order terms in the expansions?

Obvious questions

- Can we always find asymptotics in a given direction in this way?
- How do we find the dominant point?
- How easy is it to carry out all the computations?
- What about higher order terms in the expansions?
- How does our method compare with others?

Obvious questions

- Can we always find asymptotics in a given direction in this way?
- How do we find the dominant point?
- How easy is it to carry out all the computations?
- What about higher order terms in the expansions?
- How does our method compare with others?
- How does it all work? (I want to see the details)

Exercises: finding GFs

- Find (a defining equation for) the GF for the sequence (a_n) defined by $a_0 = 0; a_n = n + (2/n)\sum_{0\le k<n} a_k$ for $n \geq 1$.

Exercises: finding GFs

- Find (a defining equation for) the GF for the sequence (a_n) defined by $a_0 = 0; a_n = n + (2/n)\sum_{0\leq k<n} a_k$ for $n \geq 1$.
- (C) Find an explicit form for the GF of the sequence given by

$$p(n,j) = \frac{2n-1-j}{2n-1}p(n-1,j) + \frac{j-1}{2n-1}p(n-1,j-1)$$

with initial condition $p(1,2) = 1$.

Exercises: finding GFs

- Find (a defining equation for) the GF for the sequence (a_n) defined by $a_0 = 0; a_n = n + (2/n)\sum_{0\le k<n} a_k$ for $n \ge 1$.
- (C) Find an explicit form for the GF of the sequence given by

$$p(n, j) = \frac{2n-1-j}{2n-1} p(n-1, j) + \frac{j-1}{2n-1} p(n-1, j-1)$$

  with initial condition $p(1, 2) = 1$.
- Express the GF for the sequence given by the recursion

$$f(r, s) = f(r-1, s) + f(r, s-1) - \frac{(r+s-1)}{(r+s)} f(r-1, s-1)$$

$$f(0, s) = 1, f(r, 0) = 1$$

  as explicitly as you can.

Exercises: diagonal method

- Find (by hand) a closed form for the GF for the leading diagonal in the Delannoy case (that is, compute $F_{1,1}$).

Exercises: diagonal method

- Find (by hand) a closed form for the GF for the leading diagonal in the Delannoy case (that is, compute $F_{1,1}$).
- Repeat this for $F_{2,1}$.

Exercises: diagonal method

- Find (by hand) a closed form for the GF for the leading diagonal in the Delannoy case (that is, compute $F_{1,1}$).
- Repeat this for $F_{2,1}$.
- Challenge for D-finiteness experts: for Delannoy walks, what is the largest $p+q$ (where $\gcd\{p,q\}=1$) for which you can compute an asymptotic approximation of $a_{pn,qn}$, with an error of less than 0.01% when $n=10$?

Lecture II

Smooth points in dimension 2

Lecture 2: Overview

- If the dominant singularity is a smooth point of $\mathcal{V}$, the local geometry is simple. In the generic case, the local analysis is also straightforward. We can derive explicit results that apply to a huge number of applications. In dimension 2, these are even more explicit.

Lecture 2: Overview

- If the dominant singularity is a smooth point of $\mathcal{V}$, the local geometry is simple. In the generic case, the local analysis is also straightforward. We can derive explicit results that apply to a huge number of applications. In dimension 2, these are even more explicit.
- We first consider the case where the dominant singularity is strictly minimal, meaning that F is analytic on the open polydisc D defined by z_*, which is the only singularity on $\overline{D}$. In this case we can use univariate residue theory accompanied by elementary deformations of the contour of integration.

Table of Contents

Proof of generic smooth point formula — overview

- A point $\mathbf{z}$ of $\mathcal{V}$ is smooth if $\nabla H(\mathbf{z}) \neq 0$ ($\mathbf{z}$ is a simple pole).

Proof of generic smooth point formula — overview

- A point $\mathbf{z}$ of $\mathcal{V}$ is smooth if $\nabla H(\mathbf{z}) \neq 0$ ($\mathbf{z}$ is a simple pole).
- At a smooth point, we can reduce to an iterated integral where the inner integral is 1-dimensional.

Proof of generic smooth point formula — overview

- A point $\mathbf{z}$ of $\mathcal{V}$ is smooth if $\nabla H(\mathbf{z}) \neq 0$ ($\mathbf{z}$ is a simple pole).
- At a smooth point, we can reduce to an iterated integral where the inner integral is 1-dimensional.
- We can use univariate residue theory to approximate the inner integral.

Proof of generic smooth point formula — overview

- A point $\mathbf{z}$ of $\mathcal{V}$ is smooth if $\nabla H(\mathbf{z}) \neq 0$ ($\mathbf{z}$ is a simple pole).
- At a smooth point, we can reduce to an iterated integral where the inner integral is 1-dimensional.
- We can use univariate residue theory to approximate the inner integral.
- It remains then to integrate over the remaining $d-1$ variables.

Proof of generic smooth point formula — overview

- A point $\mathbf{z}$ of $\mathcal{V}$ is smooth if $\nabla H(\mathbf{z}) \neq 0$ ($\mathbf{z}$ is a simple pole).
- At a smooth point, we can reduce to an iterated integral where the inner integral is 1-dimensional.
- We can use univariate residue theory to approximate the inner integral.
- It remains then to integrate over the remaining $d-1$ variables.
- The first and last steps are unnecessary in the univariate case.

Proof of generic smooth point formula — overview

- A point $\mathbf{z}$ of $\mathcal{V}$ is smooth if $\nabla H(\mathbf{z}) \neq 0$ ($\mathbf{z}$ is a simple pole).
- At a smooth point, we can reduce to an iterated integral where the inner integral is 1-dimensional.
- We can use univariate residue theory to approximate the inner integral.
- It remains then to integrate over the remaining $d-1$ variables.
- The first and last steps are unnecessary in the univariate case.
- We focus here on the $d-1=1$ case but everything works in general dimension.

Reduction step 1: localization

- Suppose that (z_*, w_*) is a smooth strictly minimal pole with nonzero coordinates, and let $\rho = |z_*|, \sigma = |w_*|$. Let C_a denote the circle of radius a centred at 0.

Reduction step 1: localization

- Suppose that (z_*, w_*) is a smooth strictly minimal pole with nonzero coordinates, and let $\rho = |z_*|, \sigma = |w_*|$. Let C_a denote the circle of radius a centred at 0.
- By Cauchy, for small $\delta > 0$,

$$a_{rs} = (2\pi i)^{-2} \int_{C_\rho} z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

Reduction step 1: localization

- Suppose that (z_*, w_*) is a smooth strictly minimal pole with nonzero coordinates, and let $\rho = |z_*|, \sigma = |w_*|$. Let C_a denote the circle of radius a centred at 0.
- By Cauchy, for small $\delta > 0$,

$$a_{rs} = (2\pi i)^{-2} \int_{C_\rho} z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

- The inner integral is small away from z_*, so that for some small neighbourhood N of z_* in C_ρ,

$$a_{rs} \approx I := (2\pi i)^{-2} \int_{N} z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

Reduction step 1: localization

- Suppose that (z_*, w_*) is a smooth strictly minimal pole with nonzero coordinates, and let $\rho = |z_*|, \sigma = |w_*|$. Let C_a denote the circle of radius a centred at 0.
- By Cauchy, for small $\delta > 0$,

$$a_{rs} = (2\pi i)^{-2} \int_{C_\rho} z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

- The inner integral is small away from z_*, so that for some small neighbourhood N of z_* in C_ρ,

$$a_{rs} \approx I := (2\pi i)^{-2} \int_{N} z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

- Note that this is because of strict minimality: off N, the function $F(z, \cdot)$ has radius of convergence greater than σ, and compactness allows us to do everything uniformly.

Reduction step 2: residue

- By smoothness, there is a local parametrization $w = g(z) := 1/v(z)$ near z_*.

Reduction step 2: residue

- By smoothness, there is a local parametrization $w = g(z) := 1/v(z)$ near z_*.
- If δ is small enough, the function $w \mapsto F(z,w)/w$ has a unique pole in the annulus $\sigma - \partial \leq |w| \leq \sigma + \delta$. Let $\Psi(z)$ be the residue there.

Reduction step 2: residue

- By smoothness, there is a local parametrization $w = g(z) := 1/v(z)$ near z_*.
- If δ is small enough, the function $w \mapsto F(z,w)/w$ has a unique pole in the annulus $\sigma - \partial \leq |w| \leq \sigma + \delta$. Let $\Psi(z)$ be the residue there.
- By Cauchy,

$$I = I' + (2\pi i)^{-1} v(z)^s \Psi(z),$$

where

$$I' := (2\pi i)^{-2} \int_N z^{-r} \int_{C_{\sigma+\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

Reduction step 2: residue

- By smoothness, there is a local parametrization $w = g(z) := 1/v(z)$ near z_*.
- If δ is small enough, the function $w \mapsto F(z,w)/w$ has a unique pole in the annulus $\sigma - \partial \leq |w| \leq \sigma + \delta$. Let $\Psi(z)$ be the residue there.
- By Cauchy,

$$I = I' + (2\pi i)^{-1} v(z)^s \Psi(z),$$

where

$$I' := (2\pi i)^{-2} \int_N z^{-r} \int_{C_{\sigma+\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

- Clearly $|z_*^r I'| \to 0$, and hence

$$a_{rs} \approx (2\pi i)^{-1} \int_N z^{-r} v(z)^s \Psi(z) \, dz.$$

Reduction step 3: Fourier-Laplace integral

- We make the substitution

$$
\begin{aligned}
f(\theta) &= -\log \frac{v(z_* e^{i\theta})}{v(z_*)} + i\frac{r\theta}{s} \\
A(\theta) &= \Psi(z_* \exp(i\theta)).
\end{aligned}
$$

Reduction step 3: Fourier-Laplace integral

- We make the substitution

$$
\begin{aligned}
f(\theta) &= -\log \frac{v(z_* e^{i\theta})}{v(z_*)} + i\frac{r\theta}{s} \\
A(\theta) &= \Psi(z_* \exp(i\theta)).
\end{aligned}
$$

- This yields

$$
a_{rs} \sim \frac{1}{2\pi} z_*^{-r} w_*^{-s} \int_D \exp(-s f(\theta)) A(\theta)\, d\theta
$$

where D is a small neighbourhood of $0 \in \mathbb{R}$.

Fourier-Laplace integrals

- We have been led to asymptotic ($\lambda >> 0$) analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\theta)} A(\theta)\, d\theta$$

where:

Fourier-Laplace integrals

- We have been led to asymptotic ($\lambda >> 0$) analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\theta)} A(\theta)\, d\theta$$

where:

  - $\mathbf{0} \in D, f(\mathbf{0}) = 0$.

Fourier-Laplace integrals

- We have been led to asymptotic ($\lambda >> 0$) analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\theta)} A(\theta)\, d\theta$$

where:

  - $\mathbf{0} \in D, f(\mathbf{0}) = 0$.
  - $\mathrm{Re}\, f \geq 0$; the phase f and amplitude A are analytic.

Fourier-Laplace integrals

- We have been led to asymptotic ($\lambda >> 0$) analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\theta)} A(\theta)\, d\theta$$

where:

- $\mathbf{0} \in D, f(\mathbf{0}) = 0$.
- $\operatorname{Re} f \geq 0$; the phase f and amplitude A are analytic.
- D is a neighbourhood of 0.

Fourier-Laplace integrals

- We have been led to asymptotic ($\lambda >> 0$) analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\theta)} A(\theta)\, d\theta$$

  where:
  - $\mathbf{0} \in D, f(\mathbf{0}) = 0$.
  - $\operatorname{Re} f \geq 0$; the phase f and amplitude A are analytic.
  - D is a neighbourhood of 0.
- Such integrals are well known in many areas including mathematical physics. Potential difficulties in analysis: interplay between exponential and oscillatory decay of f, degeneracy of f, boundary issues.

Laplace approximation to Fourier-Laplace integrals

- Integration by parts shows that unless $f'(0) = 0$, $I(\lambda)$ is rapidly decreasing (except for boundary terms).

Laplace approximation to Fourier-Laplace integrals

- Integration by parts shows that unless $f'(0) = 0$, $I(\lambda)$ is rapidly decreasing (except for boundary terms).
- If 0 is an isolated stationary point and the boundary terms can be neglected, then we have a good chance of computing an asymptotic expansion for the integral.

Laplace approximation to Fourier-Laplace integrals

- Integration by parts shows that unless $f'(0) = 0$, $I(\lambda)$ is rapidly decreasing (except for boundary terms).
- If 0 is an isolated stationary point and the boundary terms can be neglected, then we have a good chance of computing an asymptotic expansion for the integral.
- If furthermore $f''(0) \neq 0$ (the nondegeneracy condition), we have the nicest formula: the standard Laplace approximation for the leading term is

$$I(\lambda) \sim A(0)\sqrt{\frac{2\pi}{\lambda f''(0)}}.$$

Our specific F-L integral

- Note that
$$f'(0) = -i\left(\frac{z_* v'(z_*)}{v(z_*)} - \frac{r}{s}\right).$$

Our specific F-L integral

- Note that
$$f'(0) = -i\left(\frac{z_* v'(z_*)}{v(z_*)} - \frac{r}{s}\right).$$
- Thus if $\alpha := r/s$ and $\alpha \neq zv'(z_*)/v(z_*)$, our "reduction" is of no use, whereas when $\alpha = zv'(z_*)/v(z_*)$ (critical point equation), we definitely get a result of order $|z_*|^{-r}|w_*|^{-s}$ as $r \to \infty$ with $r/s = \alpha$.

Our specific F-L integral

- Note that
$$f'(0) = -i\left(\frac{z_* v'(z_*)}{v(z_*)} - \frac{r}{s}\right).$$
- Thus if $\alpha := r/s$ and $\alpha \neq zv'(z_*)/v(z_*)$, our "reduction" is of no use, whereas when $\alpha = zv'(z_*)/v(z_*)$ (critical point equation), we definitely get a result of order $|z_*|^{-r}|w_*|^{-s}$ as $r \to \infty$ with $r/s = \alpha$.
- Furthermore
$$f''(0) = \frac{z_*^2 v''(z_*)}{v(z_*)} + \frac{z_* v'(z_*)}{v(z_*)} - \left(\frac{z_* v'(z_*)}{v(z_*)}\right)^2.$$

Our specific F-L integral

- Note that
$$f'(0) = -i\left(\frac{z_* v'(z_*)}{v(z_*)} - \frac{r}{s}\right).$$
- Thus if $\alpha := r/s$ and $\alpha \neq zv'(z_*)/v(z_*)$, our "reduction" is of no use, whereas when $\alpha = zv'(z_*)/v(z_*)$ (critical point equation), we definitely get a result of order $|z_*|^{-r}|w_*|^{-s}$ as $r \to \infty$ with $r/s = \alpha$.
- Furthermore
$$f''(0) = \frac{z_*^2 v''(z_*)}{v(z_*)} + \frac{z_* v'(z_*)}{v(z_*)} - \left(\frac{z_* v'(z_*)}{v(z_*)}\right)^2.$$
- So given (z_*, w_*), for this value of α we can derive asymptotics using the Laplace approximation as above.

Converting back to the original data

- We have made several reductions and obtained an asymptotic approximation for a_{rs}, in terms of derived data.

Converting back to the original data

- We have made several reductions and obtained an asymptotic approximation for a_{rs}, in terms of derived data.
- The derivatives of f can be be expressed in terms of derivatives of H by using the chain rule and solving equations.

Converting back to the original data

- We have made several reductions and obtained an asymptotic approximation for a_{rs}, in terms of derived data.
- The derivatives of f can be be expressed in terms of derivatives of H by using the chain rule and solving equations.
- Substituting at the point $\theta = 0$ and solving yields

$$
\begin{aligned}
f'(0) &= i\frac{r}{s} - i\frac{zH_z}{wH_w} \\
f''(0) &= Q := -(wH_w)^2 zH_z - wH_w(zH_z)^2 - (wH_w)^2 z^2 H_{zz} \\
&\quad - (zH_z)^2 w^2 H_{ww} + zwH_zH_wH_{zw}.
\end{aligned}
$$

where these are evaluated at (z_*, w_*).

Converting back to the original data

- We have made several reductions and obtained an asymptotic approximation for a_{rs}, in terms of derived data.
- The derivatives of f can be be expressed in terms of derivatives of H by using the chain rule and solving equations.
- Substituting at the point $\theta = 0$ and solving yields

$$f'(0) = i\frac{r}{s} - i\frac{zH_z}{wH_w}$$

$$f''(0) = Q := -(wH_w)^2 zH_z - wH_w(zH_z)^2 - (wH_w)^2 z^2 H_{zz} - (zH_z)^2 w^2 H_{ww} + zwH_z H_w H_{zw}.$$

  where these are evaluated at (z_*, w_*).
- The residue can also be computed in terms of H. We can now put everything together to give an explicit formula in terms of original data.

Generic smooth point asymptotics in dimension 2

- Suppose that $F = G/H$ has a strictly minimal simple pole at $\mathbf{p} = (z^*, w^*)$.
  If $Q(\mathbf{p}) \neq 0$, then when $s \to \infty$ with $(rwH_w - szH_z)_{|\mathbf{p}} = 0$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{2\pi}}\sqrt{\frac{-wH_w(\mathbf{p})}{sQ(\mathbf{p})}} + O(s^{-3/2})\right].$$

  The apparent lack of symmetry is illusory, since $wH_w/s = zH_z/r$ at $\mathbf{p}$.

Generic smooth point asymptotics in dimension 2

- Suppose that $F = G/H$ has a strictly minimal simple pole at $\mathbf{p} = (z^*, w^*)$.
  If $Q(\mathbf{p}) \neq 0$, then when $s \to \infty$ with $(rwH_w - szH_z)_{|\mathbf{p}} = 0$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{2\pi}}\sqrt{\frac{-wH_w(\mathbf{p})}{sQ(\mathbf{p})}} + O(s^{-3/2})\right].$$

  The apparent lack of symmetry is illusory, since $wH_w/s = zH_z/r$ at $\mathbf{p}$.
- This, the simplest multivariate case, already covers hugely many applications.

Generic smooth point asymptotics in dimension 2

- Suppose that $F = G/H$ has a strictly minimal simple pole at $\mathbf{p} = (z^*, w^*)$.
  If $Q(\mathbf{p}) \neq 0$, then when $s \to \infty$ with $(rwH_w - szH_z)_{|\mathbf{p}} = 0$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{2\pi}}\sqrt{\frac{-wH_w(\mathbf{p})}{sQ(\mathbf{p})}} + O(s^{-3/2})\right].$$

  The apparent lack of symmetry is illusory, since $wH_w/s = zH_z/r$ at $\mathbf{p}$.
- This, the simplest multivariate case, already covers hugely many applications.
- Here $\mathbf{p}$ is given, which specifies the only direction in which we can say anything useful. But we can vary $\mathbf{p}$ and obtain asymptotics that are uniform in the direction.

Table of Contents

Important special case: Riordan arrays

- A Riordan array is a bivariate sequence with GF of the form

$$F(x, y) = \frac{\phi(x)}{1 - yv(x)}.$$

Important special case: Riordan arrays

- A Riordan array is a bivariate sequence with GF of the form
$$F(x,y) = \frac{\phi(x)}{1 - yv(x)}.$$
- Examples include: Pascal, Catalan, Motzkin, Schröder, etc, triangles; sums of IID random variables; many plane lattice walk models.

Important special case: Riordan arrays

- A Riordan array is a bivariate sequence with GF of the form
$$F(x,y) = \frac{\phi(x)}{1 - yv(x)}.$$
- Examples include: Pascal, Catalan, Motzkin, Schröder, etc, triangles; sums of IID random variables; many plane lattice walk models.
- In this case, if we define
$$\mu(x) := xv'(x)/v(x)$$
$$\sigma^2(x) := x^2 v''(x)/v(x) + \mu(x) - \mu(x)^2$$
the previous formula boils down (under extra assumptions) to
$$a_{rs} \sim (x_*)^{-r} v(x_*)^s \frac{\phi(x_*)}{\sqrt{2\pi s \sigma^2(x_*)}}$$
where x_* satisfies $\mu(x_*) = r/s$.

Example (Delannoy walks)

- Recall that $F(x, y) = (1 - x - y - xy)^{-1}$. This is Riordan with $\phi(x) = (1 - x)^{-1}$ and $v(x) = (1 + x)/(1 - x)$. Here $\mathcal{V}$ is globally smooth and for each (r, s) there is a unique solution to $\mu(x) = r/s$.

Example (Delannoy walks)

- Recall that $F(x, y) = (1 - x - y - xy)^{-1}$. This is Riordan with $\phi(x) = (1 - x)^{-1}$ and $v(x) = (1 + x)/(1 - x)$. Here $\mathcal{V}$ is globally smooth and for each (r, s) there is a unique solution to $\mu(x) = r/s$.
- Solving, and using the formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{r}{\Delta - s}\right]^r \left[\frac{s}{\Delta - r}\right]^s \sqrt{\frac{rs}{2\pi\Delta(r + s - \Delta)^2}}.$$

where $\Delta = \sqrt{r^2 + s^2}$.

Example (Delannoy walks)

- Recall that $F(x, y) = (1 - x - y - xy)^{-1}$. This is Riordan with $\phi(x) = (1 - x)^{-1}$ and $v(x) = (1 + x)/(1 - x)$. Here $\mathcal{V}$ is globally smooth and for each (r, s) there is a unique solution to $\mu(x) = r/s$.
- Solving, and using the formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{r}{\Delta - s}\right]^r \left[\frac{s}{\Delta - r}\right]^s \sqrt{\frac{rs}{2\pi\Delta(r + s - \Delta)^2}}.$$

where $\Delta = \sqrt{r^2 + s^2}$.
- Extracting the diagonal is now easy: $a_{7n,5n} \sim AC^n n^{-1/2}$ where $A \approx 0.236839621050264, \qquad C \approx 30952.9770838817.$

Example (Delannoy walks)

- Recall that $F(x, y) = (1 - x - y - xy)^{-1}$. This is Riordan with $\phi(x) = (1 - x)^{-1}$ and $v(x) = (1 + x)/(1 - x)$. Here $\mathcal{V}$ is globally smooth and for each (r, s) there is a unique solution to $\mu(x) = r/s$.
- Solving, and using the formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{r}{\Delta - s}\right]^r \left[\frac{s}{\Delta - r}\right]^s \sqrt{\frac{rs}{2\pi\Delta(r + s - \Delta)^2}}.$$

  where $\Delta = \sqrt{r^2 + s^2}$.
- Extracting the diagonal is now easy: $a_{7n,5n} \sim AC^n n^{-1/2}$ where $A \approx 0.236839621050264, \qquad C \approx 30952.9770838817.$
- Compare Panholzer-Prodinger, Bull. Aust. Math. Soc. 2012.

Example (Equal numbers of parts)

- What is the probability that two independently and uniformly chosen elements of a combinatorial class have the same number of parts, k, given that they have the same total size n?

Example (Equal numbers of parts)

- What is the probability that two independently and uniformly chosen elements of a combinatorial class have the same number of parts, k, given that they have the same total size n?
- Compare Banderier-Hitczenko, Discrete Mathematics 2012.

Example (Equal numbers of parts)

- What is the probability that two independently and uniformly chosen elements of a combinatorial class have the same number of parts, k, given that they have the same total size n?
- Compare Banderier-Hitczenko, Discrete Mathematics 2012.
- If (a_{nk}) is Riordan defined by ϕ, v, then the numerator is

$$b_n := \sum_k a_{nk}^2 = [t^n u^n] \frac{\phi(t)\phi(u)}{1 - v(t)v(u)}$$

Example (Equal numbers of parts)

- What is the probability that two independently and uniformly chosen elements of a combinatorial class have the same number of parts, k, given that they have the same total size n?
- Compare Banderier-Hitczenko, Discrete Mathematics 2012.
- If (a_{nk}) is Riordan defined by ϕ, v, then the numerator is

$$b_n := \sum_k a_{nk}^2 = [t^n u^n] \frac{\phi(t)\phi(u)}{1 - v(t)v(u)}$$

- Aside: this formula gives interesting sum of squares identities..

Example (Equal numbers of parts continued)

- The smooth point formula applies, provided $\lim_{x \to \rho_-} v(x) > 1$, where ρ is the radius of convergence of v. This is the supercritical case.

Example (Equal numbers of parts continued)

- The smooth point formula applies, provided $\lim_{x\to\rho_-} v(x) > 1$, where ρ is the radius of convergence of v. This is the supercritical case.
- In the supercritical case, let c be the positive root of $v(x) = 1$. Then

$$b_n \sim c^{-2n} \frac{\phi(c)^2}{\sqrt{4\pi\mu_v(c)\sigma_v^2(c)}} n^{-1/2}.$$

Example (Equal numbers of parts continued)

- The smooth point formula applies, provided $\lim_{x \to \rho_-} v(x) > 1$, where ρ is the radius of convergence of v. This is the supercritical case.
- In the supercritical case, let c be the positive root of $v(x) = 1$. Then

$$b_n \sim c^{-2n} \frac{\phi(c)^2}{\sqrt{4\pi \mu_v(c) \sigma_v^2(c)}} n^{-1/2}.$$

- Aside: we can proceed analogously for arbitrary $d \geq 2$.

Example (Equal numbers of parts continued)

- The smooth point formula applies, provided $\lim_{x\to\rho_-} v(x) > 1$, where ρ is the radius of convergence of v. This is the supercritical case.
- In the supercritical case, let c be the positive root of $v(x) = 1$. Then

$$b_n \sim c^{-2n} \frac{\phi(c)^2}{\sqrt{4\pi\mu_v(c)\sigma_v^2(c)}} n^{-1/2}.$$

- Aside: we can proceed analogously for arbitrary $d \geq 2$.
- See M.C. Wilson, *Diagonal asymptotics for products of combinatorial classes*, Combinatorics, Probability and Computing (Flajolet memorial issue).

Example (Polyominoes)

- A horizontally convex polyomino (HCP) is a union of cells $[a, a+1] \times [b, b+1]$ in the two-dimensional integer lattice such that the interior of the figure is connected and every row is connected.

Example (Polyominoes)

- A horizontally convex polyomino (HCP) is a union of cells $[a, a+1] \times [b, b+1]$ in the two-dimensional integer lattice such that the interior of the figure is connected and every row is connected.
- The GF for horizontally convex polyominoes ($k =$ rows, $n =$ squares) is

$$
\begin{aligned}
F(x, y) &= \sum_{n,k} a_{nk} x^n y^k \\
&= \frac{xy(1-x)^3}{(1-x)^4 - xy(1 - x - x^2 + x^3 + x^2 y)}.
\end{aligned}
$$

Example (Polyominoes continued)

- Here $\mathcal{V}$ is smooth everywhere except $(1, 0)$, which cannot contribute to asymptotics except when $s = 0$, so we ignore that.

Example (Polyominoes continued)

- Here $\mathcal{V}$ is smooth everywhere except $(1, 0)$, which cannot contribute to asymptotics except when $s = 0$, so we ignore that.
- For each direction with $0 < \lambda := k/n \leq 1$, there are 4 critical points. Finding the dominant one symbolically is a little tricky. It lies in the first quadrant and there is a unique such point.

Example (Polyominoes continued)

- Here $\mathcal{V}$ is smooth everywhere except $(1, 0)$, which cannot contribute to asymptotics except when $s = 0$, so we ignore that.
- For each direction with $0 < \lambda := k/n \leq 1$, there are 4 critical points. Finding the dominant one symbolically is a little tricky. It lies in the first quadrant and there is a unique such point.
- The x and y-coordinates of the dominant point are each given by a quartic (with coefficients that are polynomial in λ). Thus they are algebraic, but complicated to express.

Example (Polyominoes continued)

- Here $\mathcal{V}$ is smooth everywhere except $(1, 0)$, which cannot contribute to asymptotics except when $s = 0$, so we ignore that.
- For each direction with $0 < \lambda := k/n \leq 1$, there are 4 critical points. Finding the dominant one symbolically is a little tricky. It lies in the first quadrant and there is a unique such point.
- The x and y-coordinates of the dominant point are each given by a quartic (with coefficients that are polynomial in λ). Thus they are algebraic, but complicated to express.
- For each λ we can solve numerically if desired. The general asymptotic shape is clear from the smooth point formula.

Example (Polyominoes continued)

- Here $\mathcal{V}$ is smooth everywhere except $(1, 0)$, which cannot contribute to asymptotics except when $s = 0$, so we ignore that.
- For each direction with $0 < \lambda := k/n \leq 1$, there are 4 critical points. Finding the dominant one symbolically is a little tricky. It lies in the first quadrant and there is a unique such point.
- The x and y-coordinates of the dominant point are each given by a quartic (with coefficients that are polynomial in λ). Thus they are algebraic, but complicated to express.
- For each λ we can solve numerically if desired. The general asymptotic shape is clear from the smooth point formula.
- More on this example in Lecture 4.

Example (Symmetric Eulerian numbers)

- Let $a_{rs}/(r!s!)$ be the number of permutations of the set $[r+s+1] := \{1, 2, \ldots, r+s+1\}$ with precisely r descents.

Example (Symmetric Eulerian numbers)

- Let $a_{rs}/(r!s!)$ be the number of permutations of the set $[r+s+1] := \{1, 2, \ldots, r+s+1\}$ with precisely r descents.
- The exponential GF is

$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} = \frac{(e^x - e^y)/(x-y)}{(xe^y - ye^x)/(x-y)}$$

Example (Symmetric Eulerian numbers)

- Let $a_{rs}/(r!s!)$ be the number of permutations of the set $[r+s+1] := \{1, 2, \ldots, r+s+1\}$ with precisely r descents.
- The exponential GF is

$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} = \frac{(e^x - e^y)/(x-y)}{(xe^y - ye^x)/(x-y)}$$

- Here $\mathcal{V}$ is globally smooth. The dominant point for $r = s$ is $(1,1)$ and for other directions it is given by

$$\begin{gathered}(1-x)s = (y-1)r \\ xe^y = ye^x.\end{gathered}$$

Example (Symmetric Eulerian numbers)

- Let $a_{rs}/(r!s!)$ be the number of permutations of the set $[r+s+1] := \{1, 2, \ldots, r+s+1\}$ with precisely r descents.
- The exponential GF is

$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} = \frac{(e^x - e^y)/(x-y)}{(xe^y - ye^x)/(x-y)}$$

- Here $\mathcal{V}$ is globally smooth. The dominant point for $r = s$ is $(1,1)$ and for other directions it is given by

$$\begin{gathered}(1-x)s = (y-1)r \\ xe^y = ye^x.\end{gathered}$$

- The smooth point formula gives the asymptotic form, and for a fixed direction we can solve numerically.

Exercises: 2D smooth points

- Write down explicitly the Fourier-Laplace integral for the Delannoy example. Is it obvious that $f'(0) = 0$ from this representation?

Exercises: 2D smooth points

- Write down explicitly the Fourier-Laplace integral for the Delannoy example. Is it obvious that $f'(0) = 0$ from this representation?
- What extra assumptions on ϕ and v are required in order for the smooth point analysis to apply to a Riordan array, and for which directions does our method yield asymptotics?

Exercises: 2D smooth points

- Write down explicitly the Fourier-Laplace integral for the Delannoy example. Is it obvious that $f'(0) = 0$ from this representation?
- What extra assumptions on ϕ and v are required in order for the smooth point analysis to apply to a Riordan array, and for which directions does our method yield asymptotics?
- Given an equation of the form $f(z) = z\phi(f(z))$ where $f(x) = \sum_n a_n z^n$, use the Lagrange Inversion Formula to show that
$$na_n = [x^n y^n] \frac{y}{1 - x\phi(y)}.$$
and hence derive first order asymptotics for a_n. When is the approximation valid?

Exercises: 2D smooth points

- Write down explicitly the Fourier-Laplace integral for the Delannoy example. Is it obvious that $f'(0) = 0$ from this representation?
- What extra assumptions on ϕ and v are required in order for the smooth point analysis to apply to a Riordan array, and for which directions does our method yield asymptotics?
- Given an equation of the form $f(z) = z\phi(f(z))$ where $f(x) = \sum_n a_n z^n$, use the Lagrange Inversion Formula to show that
$$na_n = [x^n y^n] \frac{y}{1 - x\phi(y)}.$$
and hence derive first order asymptotics for a_n. When is the approximation valid?
- (C) Use the formula for b_n above to systematically derive identities involving sums of squares that are not in OEIS.

Lecture III

Higher dimensions, other geometries

Lecture 3: Overview

- We can generalize the smooth point analysis to the case of multiple points. In higher dimensions, there is a nice geometric interpretation in terms of convex geometry of the logarithmic domain of convergence.

Lecture 3: Overview

- We can generalize the smooth point analysis to the case of multiple points. In higher dimensions, there is a nice geometric interpretation in terms of convex geometry of the logarithmic domain of convergence.
- We derive explicit formulae for multiple points. The residue computations can be done in terms of residue forms, which enables us to derive stronger results.

Table of Contents

Higher dimensions

- The smooth point argument from the previous lecture generalizes directly to dimension d.

Higher dimensions

- The smooth point argument from the previous lecture generalizes directly to dimension d.
- The difference is that the ensuing Fourier-Laplace integral is in dimension $d-1$.

Higher dimensions

- The smooth point argument from the previous lecture generalizes directly to dimension d.
- The difference is that the ensuing Fourier-Laplace integral is in dimension $d-1$.
- There is a generalization of the Laplace approximation, namely

$$I(\lambda) \sim A(\mathbf{0})\sqrt{\frac{1}{\lambda \det \frac{Q}{2\pi}}}.$$

Higher dimensions

- The smooth point argument from the previous lecture generalizes directly to dimension d.
- The difference is that the ensuing Fourier-Laplace integral is in dimension $d-1$.
- There is a generalization of the Laplace approximation, namely

$$I(\lambda) \sim A(\mathbf{0})\sqrt{\frac{1}{\lambda \det \frac{\mathrm{Q}}{2\pi}}}.$$

- There are technical issues involved in proving this, because the phase f is neither purely real nor purely imaginary. See Chapter 5.

Smooth formulae for general d

- $\mathbf{z}$ is a critical point for $\mathbf{r}$ iff

$$\nabla_{\log} H(\mathbf{z}) := (z_1 H_1, \ldots, z_d H_d) \text{ is parallel to } \mathbf{r}.$$

Smooth formulae for general d

- $\mathbf{z}$ is a critical point for $\mathbf{r}$ iff

$$\nabla_{\log} H(\mathbf{z}) := (z_1 H_1, \ldots, z_d H_d) \text{ is parallel to } \mathbf{r}.$$

- When $\mathbf{z}_*$ is a critical point for $\mathbf{r}$, then, with Q denoting the Hessian of the derived function f in the Fourier-Laplace integral, k any coordinate where $H_k := \partial H / \partial z_k \neq 0$:

$$a_{\mathbf{r}} \sim {\mathbf{z}_*}^{-\mathbf{r}} \frac{1}{\sqrt{\det 2\pi \, \mathrm{Q}(\mathbf{r})}} \frac{G(\mathbf{z})}{z_k H_k(\mathbf{z}_*)} r_k^{(1-d)/2}.$$

Smooth formulae for general d

- $\mathbf{z}$ is a critical point for $\mathbf{r}$ iff

$$\nabla_{\log} H(\mathbf{z}) := (z_1 H_1, \ldots, z_d H_d) \text{ is parallel to } \mathbf{r}.$$

- When $\mathbf{z}_*$ is a critical point for $\mathbf{r}$, then, with Q denoting the Hessian of the derived function f in the Fourier-Laplace integral, k any coordinate where $H_k := \partial H / \partial z_k \neq 0$:

$$a_{\mathbf{r}} \sim {\mathbf{z}_*}^{-\mathbf{r}} \frac{1}{\sqrt{\det 2\pi\, \mathrm{Q}(\mathbf{r})}} \frac{G(\mathbf{z})}{z_k H_k(\mathbf{z}_*)} r_k^{(1-d)/2}.$$

- This specializes when $d = 2$ to the previous formula.

Example (Alignments)

- A $(d, r_1, \ldots, r_d)$-alignment is a d-row binary matrix with jth row sum r_j and no zero columns.

Example (Alignments)

- A $(d, r_1, \ldots, r_d)$-alignment is a d-row binary matrix with jth row sum r_j and no zero columns.
- These have applications to bioinformatics.

Example (Alignments)

- A $(d, r_1, \dots, r_d)$-alignment is a d-row binary matrix with jth row sum r_j and no zero columns.
- These have applications to bioinformatics.
- The generating function for the number of $(d, \cdot)$-alignments is

$$F(\mathbf{z}) = \sum a(r_1, \dots, r_d)\mathbf{z}^{\mathbf{r}} = \frac{1}{2 - \prod_{i=1}^{d}(1 + z_i)}.$$

Example (Alignments)

- A $(d, r_1, \ldots, r_d)$-alignment is a d-row binary matrix with jth row sum r_j and no zero columns.
- These have applications to bioinformatics.
- The generating function for the number of $(d, \cdot)$-alignments is

$$F(\mathbf{z}) = \sum a(r_1, \ldots, r_d)\mathbf{z}^{\mathbf{r}} = \frac{1}{2 - \prod_{i=1}^{d}(1 + z_i)}.$$

- Our hypotheses are satisfied: smooth, combinatorial, aperiodic. For each $\overline{\mathbf{r}}$, there is a dominant point in the positive orthant.

Example (Alignments continued)

- For the diagonal direction we have $\mathbf{z}_*(\bar{\mathbf{1}}) = (2^{1/d} - 1)\mathbf{1}$ (by symmetry), so the number of "square" alignments satisfies

$$a(n, n \ldots, n) \sim (2^{1/d} - 1)^{-dn} \frac{1}{(2^{1/d} - 1) 2^{(d^2-1)/2d} \sqrt{d(\pi n)^{d-1}}}$$

Example (Alignments continued)

- For the diagonal direction we have $\mathbf{z}_*(\bar{\mathbf{1}}) = (2^{1/d} - 1)\mathbf{1}$ (by symmetry), so the number of "square" alignments satisfies

$$a(n, n \ldots, n) \sim (2^{1/d} - 1)^{-dn} \frac{1}{(2^{1/d} - 1) 2^{(d^2-1)/2d} \sqrt{d(\pi n)^{d-1}}}$$

- Confirms a result of Griggs, Hanlon, Odlyzko & Waterman, *Graphs and Combinatorics* 1990, with less work, and extends to generalized alignments.

Table of Contents

Logarithmic domain

- Recall U is the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence.

Logarithmic domain

- Recall U is the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence.
- This is convex with boundary $\log \mathcal{V} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in \mathcal{V}\}$.

Logarithmic domain

- Recall U is the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence.
- This is convex with boundary $\log \mathcal{V} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in \mathcal{V}\}$.
- Each point $\mathbf{x}_*$ of $\log \mathcal{V}$ yields a minimal point $\mathbf{z}_* := \exp(\mathbf{x}_*)$ of $\mathcal{V}$, lying in the positive orthant.

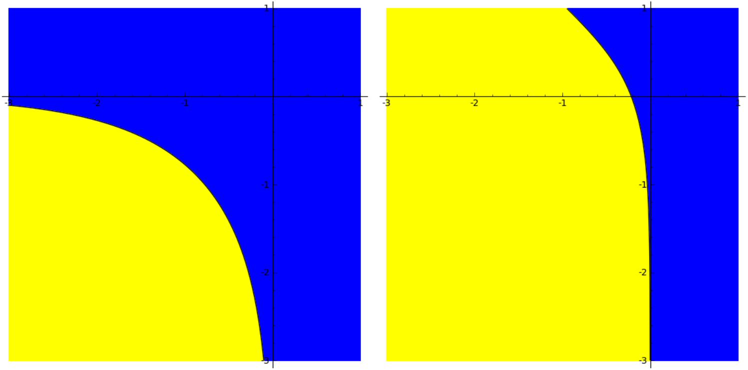
Logarithmic domain

- Recall U is the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence.
- This is convex with boundary $\log \mathcal{V} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in \mathcal{V}\}$.
- Each point $\mathbf{x}_*$ of $\log \mathcal{V}$ yields a minimal point $\mathbf{z}_* := \exp(\mathbf{x}_*)$ of $\mathcal{V}$, lying in the positive orthant.
- The cone spanned by normals to supporting hyperplanes at $\mathbf{x}^* \in \log \mathcal{V}$ we denote by $\mathrm{K}(\mathbf{z}_*)$.

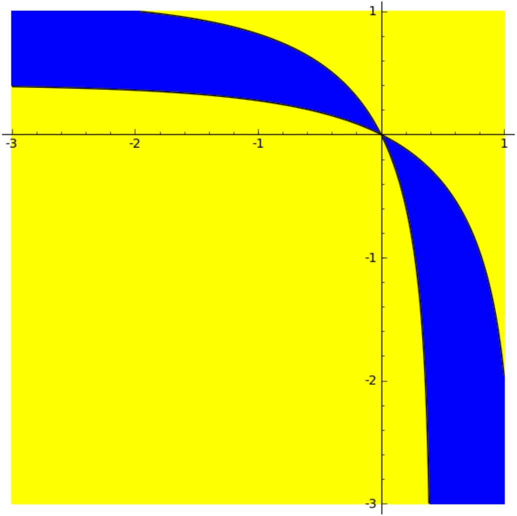
Logarithmic domain

- Recall U is the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence.
- This is convex with boundary $\log \mathcal{V} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in \mathcal{V}\}$.
- Each point $\mathbf{x}_*$ of $\log \mathcal{V}$ yields a minimal point $\mathbf{z}_* := \exp(\mathbf{x}_*)$ of $\mathcal{V}$, lying in the positive orthant.
- The cone spanned by normals to supporting hyperplanes at $\mathbf{x}^* \in \log \mathcal{V}$ we denote by $\mathrm{K}(\mathbf{z}_*)$.
- If $\mathbf{z}_*$ is smooth, this is a single ray determined by the image of $\mathbf{z}_*$ under the logarithmic Gauss map $\nabla_{\log} H$.

$\log \mathsf{U}$ for smooth Delannoy and polyomino examples



$\log \mathrm{U}$ for nonsmooth example



Interpretation of smooth asymptotic formula

- The stationary point of the F-L integral for direction $\mathbf{r}$ corresponds to a critical point of $\mathcal{V}$ that lies on $\partial\,\mathrm{U}$.

Interpretation of smooth asymptotic formula

- The stationary point of the F-L integral for direction $\mathbf{r}$ corresponds to a critical point of $\mathcal{V}$ that lies on $\partial\,\mathrm{U}$.
- The dominant point $\mathbf{z}$ in the first orthant is $\exp(\mathbf{x})$, where the outward normal to $\log \mathrm{U}$ at $\mathbf{x}$ is parallel to $\mathbf{r}$.

Interpretation of smooth asymptotic formula

- The stationary point of the F-L integral for direction $\mathbf{r}$ corresponds to a critical point of $\mathcal{V}$ that lies on $\partial\,\mathrm{U}$.
- The dominant point $\mathbf{z}$ in the first orthant is $\exp(\mathbf{x})$, where the outward normal to $\log \mathrm{U}$ at $\mathbf{x}$ is parallel to $\mathbf{r}$.
- If $\mathcal{V}$ is smooth everywhere, then asymptotics in all directions are supplied by such points.

Interpretation of smooth asymptotic formula

- The stationary point of the F-L integral for direction $\mathbf{r}$ corresponds to a critical point of $\mathcal{V}$ that lies on $\partial\,\mathrm{U}$.
- The dominant point $\mathbf{z}$ in the first orthant is $\exp(\mathbf{x})$, where the outward normal to $\log \mathrm{U}$ at $\mathbf{x}$ is parallel to $\mathbf{r}$.
- If $\mathcal{V}$ is smooth everywhere, then asymptotics in all directions are supplied by such points.
- The quantity Q is essentially the Gaussian curvature of $\log \mathcal{V}$.

Alternative smooth point formula

- $$a_{\mathbf{r}} \sim {\mathbf{z}_*}^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2}\kappa(\mathbf{z}_*)}} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|}$$

  where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.

Nonsmooth points

- Arbitrarily complicated singularities are possible. We should be satisfied with a general procedure rather than a formula. Today we discuss multiple points.

Nonsmooth points

- Arbitrarily complicated singularities are possible. We should be satisfied with a general procedure rather than a formula. Today we discuss multiple points.
- The point $\mathbf{z} \in \mathcal{V}$ is a multiple point if every small neighbourhood of $\mathbf{z}$ in $\mathcal{V}$ is the union of finitely many smooth hypersurfaces.

Nonsmooth points

- Arbitrarily complicated singularities are possible. We should be satisfied with a general procedure rather than a formula. Today we discuss multiple points.
- The point $\mathbf{z} \in \mathcal{V}$ is a multiple point if every small neighbourhood of $\mathbf{z}$ in $\mathcal{V}$ is the union of finitely many smooth hypersurfaces.
- We have good results when the intersection of these sheets is transverse.

Nonsmooth points

- Arbitrarily complicated singularities are possible. We should be satisfied with a general procedure rather than a formula. Today we discuss multiple points.
- The point $\mathbf{z} \in \mathcal{V}$ is a multiple point if every small neighbourhood of $\mathbf{z}$ in $\mathcal{V}$ is the union of finitely many smooth hypersurfaces.
- We have good results when the intersection of these sheets is transverse.
- For multiple points that are not transverse, we also have results.

Nonsmooth points

- Arbitrarily complicated singularities are possible. We should be satisfied with a general procedure rather than a formula. Today we discuss multiple points.
- The point $\mathbf{z} \in \mathcal{V}$ is a multiple point if every small neighbourhood of $\mathbf{z}$ in $\mathcal{V}$ is the union of finitely many smooth hypersurfaces.
- We have good results when the intersection of these sheets is transverse.
- For multiple points that are not transverse, we also have results.
- We also have some results for cone points (Chapter 11, very difficult, not presented this week).

Table of Contents

A generalization of the smooth argument works

- We can follow the same reduction steps as in the smooth case. Step 1 (localization) is the same.

A generalization of the smooth argument works

- We can follow the same reduction steps as in the smooth case. Step 1 (localization) is the same.
- Step 2 (residue): there are n poles in the annulus, and we need to express the residue sum somehow (the individual residues are not integrable). A trick allows us to do this via an integral over a simplex.

A generalization of the smooth argument works

- We can follow the same reduction steps as in the smooth case. Step 1 (localization) is the same.
- Step 2 (residue): there are n poles in the annulus, and we need to express the residue sum somehow (the individual residues are not integrable). A trick allows us to do this via an integral over a simplex.
- Step 3 (Fourier-Laplace integral): the resulting integral is more complicated, with a nastier domain and more complicated phase function.

A generalization of the smooth argument works

- We can follow the same reduction steps as in the smooth case. Step 1 (localization) is the same.
- Step 2 (residue): there are n poles in the annulus, and we need to express the residue sum somehow (the individual residues are not integrable). A trick allows us to do this via an integral over a simplex.
- Step 3 (Fourier-Laplace integral): the resulting integral is more complicated, with a nastier domain and more complicated phase function.
- However in the generic (transverse) case we automatically obtain a nondegenerate stationary point in dimension $n + d - 2$, and can use a modification of the Laplace approximation (which deals with boundary terms).

Generic double point in dimension 2

- Suppose that $F = G/H$ has a strictly minimal pole at $\mathbf{p} = (z_*, w_*)$, which is a double point of $\mathcal{V}$ such that $G(\mathbf{p}) \neq 0$. Then as $s \to \infty$ for r/s in $\mathrm{K}(\mathbf{p})$,

$$a_{rs} \sim (z_*)^{-r}(w_*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{(z_* w_*)^2\,\mathrm{Q}(\mathbf{p})}} + O(e^{-c(r+s)})\right]$$

where Q is the Hessian of H.

Generic double point in dimension 2

- Suppose that $F = G/H$ has a strictly minimal pole at $\mathbf{p} = (z_*, w_*)$, which is a double point of $\mathcal{V}$ such that $G(\mathbf{p}) \neq 0$. Then as $s \to \infty$ for r/s in $\mathrm{K}(\mathbf{p})$,

$$a_{rs} \sim (z_*)^{-r}(w_*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{(z_* w_*)^2\,\mathrm{Q}(\mathbf{p})}} + O(e^{-c(r+s)})\right]$$

  where Q is the Hessian of H.
- Note that

Generic double point in dimension 2

- Suppose that $F = G/H$ has a strictly minimal pole at $\mathbf{p} = (z_*, w_*)$, which is a double point of $\mathcal{V}$ such that $G(\mathbf{p}) \neq 0$. Then as $s \to \infty$ for r/s in $\mathrm{K}(\mathbf{p})$,

$$a_{rs} \sim (z_*)^{-r}(w_*)^{-s} \left[\frac{G(\mathbf{p})}{\sqrt{(z_* w_*)^2\, \mathrm{Q}(\mathbf{p})}} + O(e^{-c(r+s)}) \right]$$

  where Q is the Hessian of H.
- Note that
  - the expansion holds uniformly over compact subcones of K;

Generic double point in dimension 2

- Suppose that $F = G/H$ has a strictly minimal pole at $\mathbf{p} = (z_*, w_*)$, which is a double point of $\mathcal{V}$ such that $G(\mathbf{p}) \neq 0$. Then as $s \to \infty$ for r/s in $\mathrm{K}(\mathbf{p})$,

$$a_{rs} \sim (z_*)^{-r}(w_*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{(z_* w_*)^2\,\mathrm{Q}(\mathbf{p})}} + O(e^{-c(r+s)})\right]$$

  where Q is the Hessian of H.
- Note that
  - the expansion holds uniformly over compact subcones of K;
  - the hypothesis $G(\mathbf{p}) \neq 0$ is necessary; when $d > 1$, can have $G(\mathbf{p}) = H(\mathbf{p}) = 0$ even if G, H are relatively prime.

Example (Queueing network)

- Consider

$$F(x,y) = \frac{\exp(x+y)}{(1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{2y}{3} - \frac{x}{3})}$$

which is the “grand partition function” for a very simple queueing network.

Example (Queueing network)

- Consider

$$F(x,y) = \frac{\exp(x+y)}{(1-\frac{2x}{3}-\frac{y}{3})(1-\frac{2y}{3}-\frac{x}{3})}$$

  which is the "grand partition function" for a very simple queueing network.
- Most of the points of $\mathcal{V}$ are smooth, and we can apply the smooth point results to derive asymptotics in directions outside the cone $1/2 \leq r/s \leq 2$.

Example (Queueing network)

- Consider

$$F(x,y) = \frac{\exp(x+y)}{(1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{2y}{3} - \frac{x}{3})}$$

  which is the "grand partition function" for a very simple queueing network.
- Most of the points of $\mathcal{V}$ are smooth, and we can apply the smooth point results to derive asymptotics in directions outside the cone $1/2 \leq r/s \leq 2$.
- The point $(1,1)$ is a double point satisfying the above. In the cone $1/2 < r/s < 2$, we have $a_{rs} \sim 3e^2$.

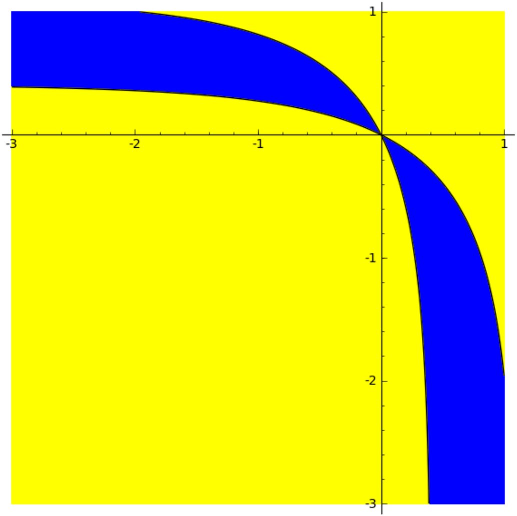
Example (Queueing network)

- Consider

$$F(x,y) = \frac{\exp(x+y)}{(1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{2y}{3} - \frac{x}{3})}$$

  which is the "grand partition function" for a very simple queueing network.
- Most of the points of $\mathcal{V}$ are smooth, and we can apply the smooth point results to derive asymptotics in directions outside the cone $1/2 \le r/s \le 2$.
- The point $(1,1)$ is a double point satisfying the above. In the cone $1/2 < r/s < 2$, we have $a_{rs} \sim 3e^2$.
- Note we say nothing here about the boundary of the cone.

$\log U$ for queueing example



Example (lemniscate)

- Consider $F = 1/H$ where

  $$H(x,y) = x^2y^2 - 2xy(x+y) + 5\,(x^2+y^2) + 14\,xy - 20(x+y) + 19.$$

  This is combinatorial, and H is an irreducible polynomial.

Example (lemniscate)

- Consider $F = 1/H$ where

  $$H(x,y) = x^2y^2 - 2xy(x+y) + 5\,(x^2+y^2) + 14\,xy - 20(x+y) + 19.$$

  This is combinatorial, and H is an irreducible polynomial.
- All points except $(1,1)$ are smooth, and $(1,1)$ is a transverse double point. Showing it is strictly minimal takes a little work.

Example (lemniscate)

- Consider $F = 1/H$ where

  $$H(x, y) = x^2y^2 - 2xy(x+y) + 5\,(x^2+y^2) + 14\,xy - 20(x+y) + 19.$$

  This is combinatorial, and H is an irreducible polynomial.
- All points except $(1, 1)$ are smooth, and $(1, 1)$ is a transverse double point. Showing it is strictly minimal takes a little work.
- In the cone $1/2 < r/s < 2$ we have $a_{rs} \sim 6$, outside we use the smooth point formula.

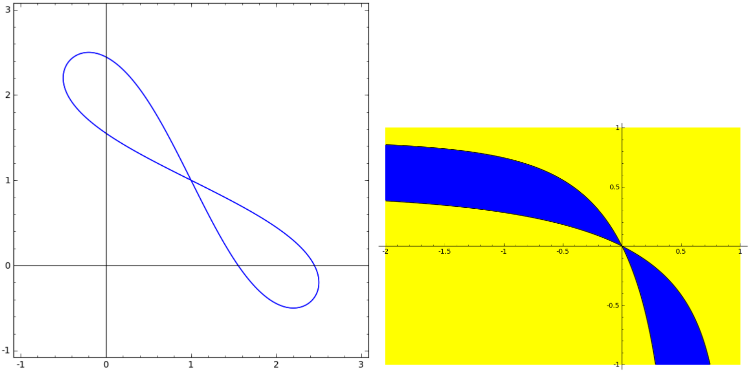
Example (lemniscate)

- Consider $F = 1/H$ where

  $$H(x, y) = x^2y^2 - 2xy(x+y) + 5\,(x^2+y^2) + 14\,xy - 20(x+y) + 19.$$

  This is combinatorial, and H is an irreducible polynomial.
- All points except $(1, 1)$ are smooth, and $(1, 1)$ is a transverse double point. Showing it is strictly minimal takes a little work.
- In the cone $1/2 < r/s < 2$ we have $a_{rs} \sim 6$, outside we use the smooth point formula.
- Note that H factors locally at $(1, 1)$ but not globally.

$\mathcal{V}$ and $\log \mathrm{U}$ for lemniscate



Multiple points: generic shape of formula($\mathbf{z}_*$)

- (smooth point, or multiple point with $n \le d$)

$$\mathbf{z}_*{}^{-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-n)/2-k}.$$

Multiple points: generic shape of $\mathrm{formula}(\mathbf{z}_*)$

- (smooth point, or multiple point with $n \leq d$)

$$\mathbf{z}_*{}^{-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-n)/2-k}.$$

  - (smooth/multiple point $n < d$)

$$a_0 = G(\mathbf{z}_*) C(\mathbf{z}_*)$$

    where C depends on the derivatives to order 2 of H;

Multiple points: generic shape of $\mathrm{formula}(\mathbf{z}_*)$

- (smooth point, or multiple point with $n \leq d$)

$$\mathbf{z}_*{}^{-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-n)/2-k}.$$

  - (smooth/multiple point $n < d$)

    $$a_0 = G(\mathbf{z}_*)C(\mathbf{z}_*)$$

    where C depends on the derivatives to order 2 of H;
  - (multiple point, $n = d$)

    $$a_0 = G(\mathbf{z}_*)(\det J)^{-1}$$

    where J is the Jacobian matrix $(\partial H_i/\partial z_j)$, other a_k are zero;

Multiple points: generic shape of $\text{formula}(\mathbf{z}_*)$

- (smooth point, or multiple point with $n \le d$)

$$\mathbf{z}_*{}^{-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-n)/2-k}.$$

  - (smooth/multiple point $n < d$)

  $$a_0 = G(\mathbf{z}_*) C(\mathbf{z}_*)$$

  where C depends on the derivatives to order 2 of H;
  - (multiple point, $n = d$)

  $$a_0 = G(\mathbf{z}_*)(\det J)^{-1}$$

  where J is the Jacobian matrix $(\partial H_i / \partial z_j)$, other a_k are zero;
- (multiple point, $n \ge d$)

$$\mathbf{z}_*{}^{-\mathbf{r}} G(\mathbf{z}_*) P\left(\frac{r_1}{z_1^*}, \ldots, \frac{r_d}{z_d^*}\right),$$

P a piecewise polynomial of degree $n - d$.

Aside: residue forms

- Instead of computing a residue and then integrating it directly, we can often repeat this process.

Aside: residue forms

- Instead of computing a residue and then integrating it directly, we can often repeat this process.
- The best way to understand this is via differential forms, in a coordinate-free way.

Aside: residue forms

- Instead of computing a residue and then integrating it directly, we can often repeat this process.
- The best way to understand this is via differential forms, in a coordinate-free way.
- This reduces the computation from d dimensions to $d - n$ where n is the number of sheets.

Aside: residue forms

- Instead of computing a residue and then integrating it directly, we can often repeat this process.
- The best way to understand this is via differential forms, in a coordinate-free way.
- This reduces the computation from d dimensions to $d - n$ where n is the number of sheets.
- When $n = d$, this is the only way we know to get the exponential decay beyond the leading term.

Aside: residue forms

- Instead of computing a residue and then integrating it directly, we can often repeat this process.
- The best way to understand this is via differential forms, in a coordinate-free way.
- This reduces the computation from d dimensions to $d - n$ where n is the number of sheets.
- When $n = d$, this is the only way we know to get the exponential decay beyond the leading term.
- When $n > d$, we first preprocess (see Lecture 4) to reduce to the case $n \leq d$.

Example (2 planes in 3-space)

- The GF is

$$F(x,y,z) = \frac{1}{(4-2x-y-z)(4-x-2y-z)}.$$

Example (2 planes in 3-space)

- The GF is

$$F(x, y, z) = \frac{1}{(4 - 2x - y - z)(4 - x - 2y - z)}.$$

- The critical points for some directions lie on one of the two sheets where a single factor vanishes, and smooth point analysis works. These occur when $\min\{r, s\} < (r + s)/3$.

Example (2 planes in 3-space)

- The GF is

$$F(x,y,z) = \frac{1}{(4-2x-y-z)(4-x-2y-z)}.$$

- The critical points for some directions lie on one of the two sheets where a single factor vanishes, and smooth point analysis works. These occur when $\min\{r,s\} < (r+s)/3$.
- The curve of intersection of the two sheets supplies the other directions. Each point on the line $\{(1,1,1) + \lambda(-1,-1,-3) \mid -1/3 < \lambda < 1\}$ gives asymptotics in a 2-D cone.

Example (2 planes in 3-space)

- The GF is

$$F(x, y, z) = \frac{1}{(4 - 2x - y - z)(4 - x - 2y - z)}.$$

- The critical points for some directions lie on one of the two sheets where a single factor vanishes, and smooth point analysis works. These occur when $\min\{r, s\} < (r + s)/3$.
- The curve of intersection of the two sheets supplies the other directions. Each point on the line $\{(1, 1, 1) + \lambda(-1, -1, -3) \mid -1/3 < \lambda < 1\}$ gives asymptotics in a 2-D cone.
- For example, $a_{3t,3t,2t} \sim (48\pi t)^{-1/2}$ with relative error less than 0.3% when $n = 30$.

Exercise: binomial coefficient power sums

- The dth Franel number is $f_n^{(d)} := \sum_k \binom{n}{k}^d$.

Exercise: binomial coefficient power sums

- The dth Franel number is $f_n^{(d)} := \sum_k \binom{n}{k}^d$.
- For odd $d \geq 3$, the GF is not algebraic (and probably for even d?)

Exercise: binomial coefficient power sums

- The dth Franel number is $f_n^{(d)} := \sum_k \binom{n}{k}^d$.
- For odd $d \geq 3$, the GF is not algebraic (and probably for even d?)
- The supercritical Riordan case holds as above.

Exercise: binomial coefficient power sums

- The dth Franel number is $f_n^{(d)} := \sum_k \binom{n}{k}^d$.
- For odd $d \geq 3$, the GF is not algebraic (and probably for even d?)
- The supercritical Riordan case holds as above.
- Derive the formula mentioned in Lecture 2 for the GF of $f_n^{(d)}$, for arbitrary d.

Exercise: binomial coefficient power sums

- The dth Franel number is $f_n^{(d)} := \sum_k \binom{n}{k}^d$.
- For odd $d \geq 3$, the GF is not algebraic (and probably for even d?)
- The supercritical Riordan case holds as above.
- Derive the formula mentioned in Lecture 2 for the GF of $f_n^{(d)}$, for arbitrary d.
- Compare with the exact result when $d = 6, n = 10$.

Exercise: double point asymptotics

- For the queueing example, compute the asymptotics in the cone $1/2 < r/s < 2$ by an iterated residue computation, rather than using the formula given above.

Exercise: double point asymptotics

- For the queueing example, compute the asymptotics in the cone $1/2 < r/s < 2$ by an iterated residue computation, rather than using the formula given above.
- Compute asymptotics for the queueing example in the cone $1/2 < r/s < 2$ by reducing to Fourier-Laplace integral as mentioned above.

Exercise: double point asymptotics

- For the queueing example, compute the asymptotics in the cone $1/2 < r/s < 2$ by an iterated residue computation, rather than using the formula given above.
- Compute asymptotics for the queueing example in the cone $1/2 < r/s < 2$ by reducing to Fourier-Laplace integral as mentioned above.
- Which method do you prefer?

Exercise: double point asymptotics

- For the queueing example, compute the asymptotics in the cone $1/2 < r/s < 2$ by an iterated residue computation, rather than using the formula given above.
- Compute asymptotics for the queueing example in the cone $1/2 < r/s < 2$ by reducing to Fourier-Laplace integral as mentioned above.
- Which method do you prefer?
- Which method can say something about asymptotics on the boundary of the cone?

Exercise: biased coin flips

- A coin has probability of heads p, which can be changed. The coin will be biased so that $p = 2/3$ for the first n flips, and $p = 1/3$ thereafter. A player desires to get r heads and s tails and is allowed to choose n. On average, how many choices of $n \leq r + s$ will be winning choices?

Exercise: biased coin flips

- A coin has probability of heads p, which can be changed. The coin will be biased so that $p = 2/3$ for the first n flips, and $p = 1/3$ thereafter. A player desires to get r heads and s tails and is allowed to choose n. On average, how many choices of $n \leq r + s$ will be winning choices?
- The answer is given by the convolution

$$a_{rs} = \sum_{a+b=n} \binom{n}{a} (2/3)^a (1/3)^b \binom{r+s-n}{r-a} (1/3)^{r-a} (2/3)^{s-b}$$

Exercise: biased coin flips

- A coin has probability of heads p, which can be changed. The coin will be biased so that $p = 2/3$ for the first n flips, and $p = 1/3$ thereafter. A player desires to get r heads and s tails and is allowed to choose n. On average, how many choices of $n \leq r + s$ will be winning choices?
- The answer is given by the convolution

$$a_{rs} = \sum_{a+b=n} \binom{n}{a} (2/3)^a (1/3)^b \binom{r+s-n}{r-a} (1/3)^{r-a} (2/3)^{s-b}$$

- Derive asymptotics for a_{rs} when $1/2 < r/s < 2$.

Lecture IV

Computational aspects

Lecture 4: Overview

- All our asymptotics are ultimately computed via Fourier-Laplace integrals. All standard references make simplifying assumptions that do not always hold in GF applications. In some cases, we needed to extend what is known.

Lecture 4: Overview

- All our asymptotics are ultimately computed via Fourier-Laplace integrals. All standard references make simplifying assumptions that do not always hold in GF applications. In some cases, we needed to extend what is known.
- Once the asymptotics have been derived, in order to apply them in terms of original data we require substantial algebraic computation. We have implemented some of this in Sage. Higher order terms in the expansions are particularly tricky.

Lecture 4: Overview

- All our asymptotics are ultimately computed via Fourier-Laplace integrals. All standard references make simplifying assumptions that do not always hold in GF applications. In some cases, we needed to extend what is known.
- Once the asymptotics have been derived, in order to apply them in terms of original data we require substantial algebraic computation. We have implemented some of this in Sage. Higher order terms in the expansions are particularly tricky.
- The algebraic computations are usually best carried out using defining ideals, rather than explicit formulae.

Table of Contents

Low-dimensional examples of F-L integrals

- Typical smooth point example looks like

$$\int_{-1}^{1} e^{-\lambda(1+i)x^2}\, dx.$$

Isolated nondegenerate critical point, exponential decay

Low-dimensional examples of F-L integrals

- Typical smooth point example looks like

$$\int_{-1}^{1} e^{-\lambda(1+i)x^2}\, dx.$$

  Isolated nondegenerate critical point, exponential decay
- Simplest double point example looks roughly like

$$\int_{-1}^{1}\int_{0}^{1} e^{-\lambda(x^2+2ixy)}\, dy\, dx.$$

  Note $\operatorname{Re} f = 0$ on $x = 0$, so rely on oscillation for smallness.

Low-dimensional examples of F-L integrals

- Typical smooth point example looks like

$$\int_{-1}^{1} e^{-\lambda(1+i)x^2}\, dx.$$

  Isolated nondegenerate critical point, exponential decay
- Simplest double point example looks roughly like

$$\int_{-1}^{1}\int_{0}^{1} e^{-\lambda(x^2+2ixy)}\, dy\, dx.$$

  Note $\operatorname{Re} f = 0$ on $x = 0$, so rely on oscillation for smallness.
- Multiple point with $n = 2, d = 1$ gives integral like

$$\int_{-1}^{1}\int_{0}^{1}\int_{-x}^{x} e^{-\lambda(z^2+2izy)}\, dy\, dx\, dz.$$

  Simplex corners now intrude, continuum of critical points.

Difficulties with F-L asymptotics

- All authors assume at least one of the following:

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;
  - purely imaginary phase;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;
  - purely imaginary phase;
  - isolated stationary point of phase, usually quadratically nondegenerate.

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;
  - purely imaginary phase;
  - isolated stationary point of phase, usually quadratically nondegenerate.
- Many of our applications to generating function asymptotics do not fit into this framework. In some cases, we needed to extend what is known.

Example

- Consider
$$I(\lambda) = \int_{-\varepsilon}^{\varepsilon} \int_0^1 e^{-\lambda\phi(p,t)}\, dp\, dt$$
where $\phi(p,t) = i\lambda t + \log\left[(1-p)v_1(e^{it}) + pv_2(e^{it})\right]$.

Example

- Consider
$$I(\lambda) = \int_{-\varepsilon}^{\varepsilon} \int_0^1 e^{-\lambda\phi(p,t)}\, dp\, dt$$
where $\phi(p,t) = i\lambda t + \log\left[(1-p)v_1(e^{it}) + pv_2(e^{it})\right]$.
- This arises in the simplest strictly minimal double point situation. Recall the v_i are the inverse poles near the double point.

Example

- Consider
$$I(\lambda) = \int_{-\varepsilon}^{\varepsilon} \int_0^1 e^{-\lambda\phi(p,t)}\, dp\, dt$$
where $\phi(p,t) = i\lambda t + \log\left[(1-p)v_1(e^{it}) + pv_2(e^{it})\right]$.
- This arises in the simplest strictly minimal double point situation. Recall the v_i are the inverse poles near the double point.
- The answer is
$$I(\lambda) \sim \frac{2\pi}{|v_1'(1) - v_2'(1)|\lambda}$$

Example

- Consider

$$I(\lambda) = \int_{-\varepsilon}^{\varepsilon} \int_0^1 e^{-\lambda\phi(p,t)}\, dp\, dt$$

where $\phi(p,t) = i\lambda t + \log\left[(1-p)v_1(e^{it}) + pv_2(e^{it})\right]$.

- This arises in the simplest strictly minimal double point situation. Recall the v_i are the inverse poles near the double point.
- The answer is

$$I(\lambda) \sim \frac{2\pi}{|v_1'(1) - v_2'(1)|\lambda}$$

- This doesn't satisfy the hypotheses of the last slide, and so we needed to derive the analogue of the Laplace approximation.

Table of Contents

Higher order terms

- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.

Higher order terms

- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.

Higher order terms

- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.
- Applications of higher order terms:

Higher order terms

- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.
- Applications of higher order terms:
  - When leading term cancels in deriving other formulae.

Higher order terms

- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.
- Applications of higher order terms:
  - When leading term cancels in deriving other formulae.
  - When leading term is zero because of numerator.

Higher order terms

- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.
- Applications of higher order terms:
  - When leading term cancels in deriving other formulae.
  - When leading term is zero because of numerator.
  - Better numerical approximations for smaller indices.

Hörmander's explicit formula

For an isolated nondegenerate stationary point in dimension d,

$$I(\lambda) \sim \left(\det\left(\frac{\lambda f''(\mathbf{0})}{2\pi}\right)\right)^{-1/2} \sum_{k\geq 0} \lambda^{-k} L_k(A, f)$$

where

$$\underline{f}(t) = f(t) - (1/2)tf''(0)t^T$$

$$\mathcal{D} = \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b}(-\mathrm{i}\partial_a)(-\mathrm{i}\partial_b)$$

$$\tilde{L}_k(A, f) = \sum_{l\leq 2k} \frac{\mathcal{D}^{l+k}(A\underline{f}^l)(0)}{(-1)^k 2^{l+k} l!(l+k)!}.$$

$\tilde{L}_k$ is a differential operator of order $2k$ acting on A at 0 (considering the order $3m$ zero of $\underline{f}^m$), whose coefficients are rational functions of $f''(0), \ldots, f^{(2k+2)}(0)$.

Example (nonoverlapping patterns)

- Given a word over alphabet $\{a_1, \ldots, a_d\}$, players alternate reading letters. If the last two letters are the same, we erase the letters seen so far, and continue.

Example (nonoverlapping patterns)

- Given a word over alphabet $\{a_1, \ldots, a_d\}$, players alternate reading letters. If the last two letters are the same, we erase the letters seen so far, and continue.
- For example, in $abaabbba$, there are two occurrences.

Example (nonoverlapping patterns)

- Given a word over alphabet $\{a_1, \ldots, a_d\}$, players alternate reading letters. If the last two letters are the same, we erase the letters seen so far, and continue.
- For example, in $abaabbba$, there are two occurrences.
- How many such snaps are there, for random words?

Example (nonoverlapping patterns)

- Given a word over alphabet $\{a_1, \dots, a_d\}$, players alternate reading letters. If the last two letters are the same, we erase the letters seen so far, and continue.
- For example, in $abaabbba$, there are two occurrences.
- How many such snaps are there, for random words?
- Answer: let ψ_n be the random variable counting snaps in words of length n. Then as $n \to \infty$,

$$\mathbb{E}(\psi_n) = (3/4)n - 15/32 + O(n^{-1})$$
$$\sigma^2(\psi_n) = (9/32)n + O(1).$$

Example (snaps continued)

- The details are as follows. Consider W given by

$$
\begin{aligned}
W(x_1, \ldots, x_d, y) &= \frac{A(x)}{1 - yB(x)} \\
A(x) &= 1/[1 - \sum_{j=1}^{d} x_j/(x_j + 1)] \\
B(x) &= 1 - (1 - e_1(x))A(x) \\
e_1(x) &= \sum_{i=j}^{d} x_j.
\end{aligned}
$$

Example (snaps continued)

- The details are as follows. Consider W given by

$$
\begin{aligned}
W(x_1, \ldots, x_d, y) &= \frac{A(x)}{1 - yB(x)} \\
A(x) &= 1/[1 - \sum_{j=1}^{d} x_j/(x_j + 1)] \\
B(x) &= 1 - (1 - e_1(x))A(x) \\
e_1(x) &= \sum_{i=j}^{d} x_j.
\end{aligned}
$$

- The symbolic method shows that $[x_1^n \ldots x_d^n, y^s]W(\mathbf{x}, y)$ counts words with n occurrences of each letter and s snaps.

Example (snaps continued)

We extract as usual. Note the first order cancellation in the variance computation. For $d = 3$,

$$\begin{aligned}
\mathbb{E}(\psi_n) &= \frac{[x^{n\mathbf{1}}]\frac{\partial W}{\partial y}(x,1)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (3/4)n - 15/32 + O(n^{-1}) \\
\mathbb{E}(\psi_n^2) &= \frac{[x^{n\mathbf{1}}]\left(\frac{\partial^2 W}{\partial y^2}(x,1) + \frac{\partial W}{\partial y}(x,1)\right)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (9/16)n^2 - (27/64)n + O(1) \\
\sigma^2(\psi_n) &= \mathbb{E}(\psi_n^2) - \mathbb{E}(\psi_n)^2 = (9/32)n + O(1).
\end{aligned}$$

Example (vanishing numerator)

- Let
$$F(x,y) = \sum_{rs} a_{rs} x^r y^s = \frac{y(1-2y)}{1-x-y}.$$

Example (vanishing numerator)

- Let
$$F(x,y) = \sum_{rs} a_{rs} x^r y^s = \frac{y(1-2y)}{1-x-y}.$$
- Here
$$a_{rs} = 2\binom{r+s-2}{r-1} - \binom{r+s-1}{r}.$$

Example (vanishing numerator)

- Let
$$F(x,y) = \sum_{rs} a_{rs} x^r y^s = \frac{y(1-2y)}{1-x-y}.$$
- Here
$$a_{rs} = 2\binom{r+s-2}{r-1} - \binom{r+s-1}{r}.$$
- When $r = s$, this simplifies to $\frac{1}{r}\binom{2r-2}{r-1}$, a shifted Catalan number. The dominant point is $(1/2, 1/2)$ by symmetry.

Example (vanishing numerator)

- Let
$$F(x, y) = \sum_{rs} a_{rs} x^r y^s = \frac{y(1-2y)}{1-x-y}.$$
- Here
$$a_{rs} = 2\binom{r+s-2}{r-1} - \binom{r+s-1}{r}.$$
- When $r = s$, this simplifies to $\frac{1}{r}\binom{2r-2}{r-1}$, a shifted Catalan number. The dominant point is $(1/2, 1/2)$ by symmetry.
- We know the asymptotics of these are of order $n^{-3/2}$. This is consistent, because the numerator of F vanishes at $(1/2, 1/2)$.

Example (vanishing numerator)

- Let
$$F(x,y) = \sum_{rs} a_{rs} x^r y^s = \frac{y(1-2y)}{1-x-y}.$$
- Here
$$a_{rs} = 2\binom{r+s-2}{r-1} - \binom{r+s-1}{r}.$$
- When $r = s$, this simplifies to $\frac{1}{r}\binom{2r-2}{r-1}$, a shifted Catalan number. The dominant point is $(1/2, 1/2)$ by symmetry.
- We know the asymptotics of these are of order $n^{-3/2}$. This is consistent, because the numerator of F vanishes at $(1/2, 1/2)$.
- Our general formula yields
$$a_{nn} \sim 4^n \left(\frac{1}{4\sqrt{\pi}} n^{-3/2} + \frac{3}{32\sqrt{\pi}} n^{-5/2} \right).$$

Computing numerical approximations

- Alex Raichev's Sage implementation computes higher order expansions for smooth and multiple points.

Computing numerical approximations

- Alex Raichev's Sage implementation computes higher order expansions for smooth and multiple points.
- The error from truncating at the kth term is of order $1/n^{1+k}$.

Computing numerical approximations

- Alex Raichev's Sage implementation computes higher order expansions for smooth and multiple points.
- The error from truncating at the kth term is of order $1/n^{1+k}$.
- The current implementation is not very sophisticated, and when $k \geq 3$ and $d \geq 4$, for example, usually fails to halt in reasonable time.

Computing numerical approximations

- Alex Raichev's Sage implementation computes higher order expansions for smooth and multiple points.
- The error from truncating at the kth term is of order $1/n^{1+k}$.
- The current implementation is not very sophisticated, and when $k \geq 3$ and $d \geq 4$, for example, usually fails to halt in reasonable time.
- To compute the kth term naively using Hörmander requires at least d^{3k} $d \times d$ matrix computations.

Computing numerical approximations

- Alex Raichev's Sage implementation computes higher order expansions for smooth and multiple points.
- The error from truncating at the kth term is of order $1/n^{1+k}$.
- The current implementation is not very sophisticated, and when $k \geq 3$ and $d \geq 4$, for example, usually fails to halt in reasonable time.
- To compute the kth term naively using Hörmander requires at least d^{3k} $d \times d$ matrix computations.
- There is surely a lot of room for improvement here.

Example (Snaps with $d = 3$)

n	1	2	4	8
$\mathbb{E}(\psi)$	0	1.000	2.509	5.521
$(3/4)n$	0.7500	1.500	3	6
$(3/4)n - 15/32$	0.2813	1.031	2.531	5.531
one-term relative error	undefined	0.5000	0.1957	0.08685
two-term relative error	undefined	0.03125	0.008832	0.001936
$\mathbb{E}(\psi^2)$	0	1.8000	7.496	32.80
$(9/16)n^2$	0.5625	2.250	9	36
$(9/16)n^2 - (27/64)n$	0.1406	1.406	7.312	32.63
one-term relative error	undefined	0.2500	0.2006	0.09768
two-term relative error	undefined	0.2188	0.02449	0.005220
$\sigma^2(\psi)$	0	0.8000	1.201	2.320
$(9/32)n$	0.2813	0.5625	1.125	2.250
relative error	undefined	0.2969	0.06294	0.03001

Example (2 planes in 3-space)

Using the formula we obtain

$$a_{3t,3t,2t} = \frac{1}{\sqrt{3\pi}} \left(\frac{1}{4} t^{-1/2} - \frac{25}{1152} t^{-3/2} + \frac{1633}{663552} t^{-5/2} \right) + O(t^{-7/2}).$$

The relative errors are:

rel. err. vs t	1	2	4	8	16	32
$k = 1$	-0.660	-0.315	-0.114	-0.0270	-0.00612	-0.00271
$k = 2$	-0.516	-0.258	-0.0899	-0.0158	-0.000664	0.00000780
$k = 3$	-0.532	-0.261	-0.0906	-0.0160	-0.000703	-0.00000184

Table of Contents

Computations in polynomial rings

- In order to apply our formulae, we need to, at least:

Computations in polynomial rings

- In order to apply our formulae, we need to, at least:
  - find the critical point $\mathbf{z}_*(\mathbf{r})$;

Computations in polynomial rings

- In order to apply our formulae, we need to, at least:
  - find the critical point $\mathbf{z}_*(\mathbf{r})$;
  - compute a rational function of derivatives of H, evaluated at $\mathbf{z}_*$.

Computations in polynomial rings

- In order to apply our formulae, we need to, at least:
  - find the critical point $\mathbf{z}_*(\mathbf{r})$;
  - compute a rational function of derivatives of H, evaluated at $\mathbf{z}_*$.
- The first can be solved by, for example, Gröbner basis methods.

Computations in polynomial rings

- In order to apply our formulae, we need to, at least:
  - find the critical point $\mathbf{z}_*(\mathbf{r})$;
  - compute a rational function of derivatives of H, evaluated at $\mathbf{z}_*$.
- The first can be solved by, for example, Gröbner basis methods.
- The second can cause big problems if done naively, leading to a symbolic mess, and loss of numerical precision. It is best to deal with annihilating ideals.

Example (Why ideals are better)

- Suppose x is the positive root of $p(x) := x^3 - x^2 + 11x - 2$, and we want to compute $g(x) := x^5/(867x^4 - 1)$.

Example (Why ideals are better)

- Suppose x is the positive root of $p(x) := x^3 - x^2 + 11x - 2$, and we want to compute $g(x) := x^5/(867x^4 - 1)$.
- If we compute x symbolically and then substitute into g, we obtain a huge mess involving radicals, which evaluates numerically to 0.193543073868354.

Example (Why ideals are better)

- Suppose x is the positive root of $p(x) := x^3 - x^2 + 11x - 2$, and we want to compute $g(x) := x^5/(867x^4 - 1)$.
- If we compute x symbolically and then substitute into g, we obtain a huge mess involving radicals, which evaluates numerically to 0.193543073868354.
- If we compute x numerically and then substitute, we obtain 0.193543073867096.

Example (Why ideals are better)

- Suppose x is the positive root of $p(x) := x^3 - x^2 + 11x - 2$, and we want to compute $g(x) := x^5/(867x^4 - 1)$.
- If we compute x symbolically and then substitute into g, we obtain a huge mess involving radicals, which evaluates numerically to 0.193543073868354.
- If we compute x numerically and then substitute, we obtain 0.193543073867096.
- Instead we can compute the minimal polynomial of $y := g(x)$ by Gröbner methods. This gives

$$11454803y^3 - 2227774y^2 + 2251y - 32 = 0$$

and evaluating numerically yields 0.193543073868734.

Example (Polyomino computation)

- Recall the GF for horizontally convex polyominoes is

$$F(x,y) = \frac{xy(1-x)^3}{(1-x^4) - xy(1-x-x^2+x^3+x^2y)}.$$

Example (Polyomino computation)

- Recall the GF for horizontally convex polyominoes is

$$F(x,y) = \frac{xy(1-x)^3}{(1-x^4) - xy(1-x-x^2+x^3+x^2y)}.$$

- Solving $\{H = 0, \nabla H = 0\}$ yields only the point $(1,0)$. Thus dominant points in direction $\lambda := s/r, 0 < \lambda < 1$, are all smooth.

Example (Polyomino computation)

- Recall the GF for horizontally convex polyominoes is
$$F(x,y) = \frac{xy(1-x)^3}{(1-x^4) - xy(1-x-x^2+x^3+x^2y)}.$$
- Solving $\{H = 0, \nabla H = 0\}$ yields only the point $(1,0)$. Thus dominant points in direction $\lambda := s/r, 0 < \lambda < 1$, are all smooth.
- The ideal in $\mathbb{C}[x,y]$ defined by $\{sxH_x - ryH_y, H\}$ has a Gröbner basis giving a quartic minimal polynomial for $x_*(\lambda)$, and $y_*(\lambda)$ is a linear function of $x_*(\lambda)$ (also satisfies a quartic).

Example (Polyomino computation)

- Recall the GF for horizontally convex polyominoes is

$$F(x, y) = \frac{xy(1-x)^3}{(1-x^4) - xy(1-x-x^2+x^3+x^2y)}.$$

- Solving $\{H = 0, \nabla H = 0\}$ yields only the point $(1, 0)$. Thus dominant points in direction $\lambda := s/r, 0 < \lambda < 1$, are all smooth.
- The ideal in $\mathbb{C}[x, y]$ defined by $\{sxH_x - ryH_y, H\}$ has a Gröbner basis giving a quartic minimal polynomial for $x_*(\lambda)$, and $y_*(\lambda)$ is a linear function of $x_*(\lambda)$ (also satisfies a quartic).
- Specifically, the elimination polynomial for x is

$$(1+\lambda)x^4+4(1+\lambda)^2x^3+10(\lambda^2+\lambda-1)x^2+4(2\lambda-1)^2x+(1-\lambda)(1-2\lambda).$$

Example (Polyomino computation continued)

- The leading coefficient in the asymptotic expansion has the form $(2\pi)^{-1/2}C$ where C is algebraic.

Example (Polyomino computation continued)

- The leading coefficient in the asymptotic expansion has the form $(2\pi)^{-1/2}C$ where C is algebraic.
- For *generic* λ, the minimal polynomial of C has degree 8.

Example (Polyomino computation continued)

- The leading coefficient in the asymptotic expansion has the form $(2\pi)^{-1/2}C$ where C is algebraic.
- For *generic* λ, the minimal polynomial of C has degree 8.
- However, for example when $r = 2s$ there is major simplification: the minimal polynomials for x and y respectively are $3x^2 + 18x - 5$ and $75y^2 - 288y + 256$, etc.

Example (Polyomino computation continued)

- The leading coefficient in the asymptotic expansion has the form $(2\pi)^{-1/2}C$ where C is algebraic.
- For *generic* λ, the minimal polynomial of C has degree 8.
- However, for example when $r = 2s$ there is major simplification: the minimal polynomials for x and y respectively are $3x^2 + 18x - 5$ and $75y^2 - 288y + 256$, etc.
- Now given (r, s), solving numerically for C as a root gives a more accurate answer than if we had solved for x_*, y_* above and substituted.

Computations in local rings

- In order to apply our smooth/multiple point formulae, we need to, at least:

Computations in local rings

- In order to apply our smooth/multiple point formulae, we need to, at least:
  - classify the local geometry at point $\mathbf{z}_*$;

Computations in local rings

- In order to apply our smooth/multiple point formulae, we need to, at least:
  - classify the local geometry at point $\mathbf{z}_*$;
  - compute (derivatives of) the factors H_i near $\mathbf{z}_*$.

Computations in local rings

- In order to apply our smooth/multiple point formulae, we need to, at least:
  - classify the local geometry at point $\mathbf{z}_*$;
  - compute (derivatives of) the factors H_i near $\mathbf{z}_*$.
- Unfortunately, computations in the local ring are not effective (as far as we know). If a polynomial factors as an analytic function, but the factors are not polynomial, we can't deal with it algorithmically (yet).

Computations in local rings

- In order to apply our smooth/multiple point formulae, we need to, at least:
  - classify the local geometry at point $\mathbf{z}_*$;
  - compute (derivatives of) the factors H_i near $\mathbf{z}_*$.
- Unfortunately, computations in the local ring are not effective (as far as we know). If a polynomial factors as an analytic function, but the factors are not polynomial, we can't deal with it algorithmically (yet).
- Smooth points are easily detected. There are some sufficient conditions, and some necessary conditions, for $\mathbf{z}_*$ to be a multiple point. But in general we don't know how to classify singularities algorithmically.

Example (local factorization of lemniscate)

- Let $H(x,y) =$
  $19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$,
  and analyse $1/H$.

Example (local factorization of lemniscate)

- Let $H(x, y) = 19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$, and analyse $1/H$.
- Here $\mathcal{V}$ is smooth at every point except $(1, 1)$, which we see by solving the system $\{H = 0, \nabla H = 0\}$.

Example (local factorization of lemniscate)

- Let $H(x,y) = 19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$, and analyse $1/H$.
- Here $\mathcal{V}$ is smooth at every point except $(1,1)$, which we see by solving the system $\{H = 0, \nabla H = 0\}$.
- At $(1,1)$, changing variables to $h(u,v) := H(1+u, 1+v)$, we see that $h(u,v) = 4u^2 + 10uv + 4v^2 + C(u,v)$ where C has no terms of degree less than 3.

Example (local factorization of lemniscate)

- Let $H(x,y) =$ $19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$, and analyse $1/H$.
- Here $\mathcal{V}$ is smooth at every point except $(1,1)$, which we see by solving the system $\{H = 0, \nabla H = 0\}$.
- At $(1,1)$, changing variables to $h(u,v) := H(1+u, 1+v)$, we see that $h(u,v) = 4u^2 + 10uv + 4v^2 + C(u,v)$ where C has no terms of degree less than 3.
- The quadratic part factors into distinct factors, showing that $(1,1)$ is a transverse multiple point.

Example (local factorization of lemniscate)

- Let $H(x,y) = 19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$, and analyse $1/H$.
- Here $\mathcal{V}$ is smooth at every point except $(1,1)$, which we see by solving the system $\{H = 0, \nabla H = 0\}$.
- At $(1,1)$, changing variables to $h(u,v) := H(1+u, 1+v)$, we see that $h(u,v) = 4u^2 + 10uv + 4v^2 + C(u,v)$ where C has no terms of degree less than 3.
- The quadratic part factors into distinct factors, showing that $(1,1)$ is a transverse multiple point.
- Note that our double point formula does not require details of the individual factors. However this is not the case for general multiple points.

Reduction of multiple points

- If we have $n > d$ transverse smooth factors meeting at a point $\mathbf{p}$, we can reduce to the case $n \leq d$ at the cost of increasing the number of summands.

Reduction of multiple points

- If we have $n > d$ transverse smooth factors meeting at a point $\mathbf{p}$, we can reduce to the case $n \leq d$ at the cost of increasing the number of summands.
- If we have repeated factors, we can reduce to the case of distinct factors using exactly the same idea.

Reduction of multiple points

- If we have $n > d$ transverse smooth factors meeting at a point $\mathbf{p}$, we can reduce to the case $n \leq d$ at the cost of increasing the number of summands.
- If we have repeated factors, we can reduce to the case of distinct factors using exactly the same idea.
- If this is not done, we arrive at Fourier-Laplace integrals with non-isolated stationary points, which are hard to analyse.

Reduction of multiple points

- If we have $n > d$ transverse smooth factors meeting at a point $\mathbf{p}$, we can reduce to the case $n \leq d$ at the cost of increasing the number of summands.
- If we have repeated factors, we can reduce to the case of distinct factors using exactly the same idea.
- If this is not done, we arrive at Fourier-Laplace integrals with non-isolated stationary points, which are hard to analyse.
- However after doing the above we always reduce to the case of an isolated point, which we can handle.

Example (Algebraic reduction, sketch)

- Let $H = H_1 H_2 H_3 := (1-x)(1-y)(1-xy)$.

Example (Algebraic reduction, sketch)

- Let $H = H_1 H_2 H_3 := (1-x)(1-y)(1-xy)$.
- In the local ring at $(1,1)$, each factor should be in the ideal generated by the other two (Nullstellensatz).

Example (Algebraic reduction, sketch)

- Let $H = H_1 H_2 H_3 := (1-x)(1-y)(1-xy)$.
- In the local ring at $(1,1)$, each factor should be in the ideal generated by the other two (Nullstellensatz).
- In fact it is true globally, since $H_3 = H_1 + H_2 - H_1 H_2$. (Nullstellensatz certificate).

Example (Algebraic reduction, sketch)

- Let $H = H_1 H_2 H_3 := (1-x)(1-y)(1-xy)$.
- In the local ring at $(1,1)$, each factor should be in the ideal generated by the other two (Nullstellensatz).
- In fact it is true globally, since $H_3 = H_1 + H_2 - H_1 H_2$. (Nullstellensatz certificate).
- Thus eventually we obtain

$$F = \frac{1}{H_1 H_2 H_3} = \cdots = \frac{2-y}{(1-y)(1-xy)^2)} + \frac{1}{(1-x)(1-xy)^2}.$$

Example (Algebraic reduction, sketch)

- Let $H = H_1 H_2 H_3 := (1-x)(1-y)(1-xy)$.
- In the local ring at $(1,1)$, each factor should be in the ideal generated by the other two (Nullstellensatz).
- In fact it is true globally, since $H_3 = H_1 + H_2 - H_1 H_2$. (Nullstellensatz certificate).
- Thus eventually we obtain

$$F = \frac{1}{H_1 H_2 H_3} = \cdots = \frac{2-y}{(1-y)(1-xy)^2)} + \frac{1}{(1-x)(1-xy)^2}.$$

- The next step, reducing the multiplicity of factors can be done at the residue stage (residue for higher order pole) or by other methods, and is both easy and algorithmic.

Example (Algebraic reduction, sketch)

- Let $H = H_1 H_2 H_3 := (1-x)(1-y)(1-xy)$.
- In the local ring at $(1,1)$, each factor should be in the ideal generated by the other two (Nullstellensatz).
- In fact it is true globally, since $H_3 = H_1 + H_2 - H_1 H_2$. (Nullstellensatz certificate).
- Thus eventually we obtain

$$F = \frac{1}{H_1 H_2 H_3} = \cdots = \frac{2-y}{(1-y)(1-xy)^2)} + \frac{1}{(1-x)(1-xy)^2}.$$

- The next step, reducing the multiplicity of factors can be done at the residue stage (residue for higher order pole) or by other methods, and is both easy and algorithmic.
- Thus we can reduce to a (possibly large) sum of (polynomial multiples of) transverse double point asymptotic series.

Exercises

A computer algebra system will help for some of these.

- Use Hörmander's formula to compute L_0, L_1, L_2 for $F(x, y) = (1 - x - y)^{-1}$, at the minimal point $(1/2, 1/2)$. This gives asymptotics for the main diagonal coefficients $\binom{2n}{n}$.

Exercises

A computer algebra system will help for some of these.

- Use Hörmander's formula to compute L_0, L_1, L_2 for $F(x,y) = (1-x-y)^{-1}$, at the minimal point $(1/2, 1/2)$. This gives asymptotics for the main diagonal coefficients $\binom{2n}{n}$.
- The small change from $y(1-2y)/(1-x-y)$ to $(1-2y)/(1-x-y)$ should make no difference to our basic computational procedure. Show that, nevertheless, the results are very different. Explain.

Exercises

A computer algebra system will help for some of these.

- Use Hörmander's formula to compute L_0, L_1, L_2 for $F(x,y) = (1-x-y)^{-1}$, at the minimal point $(1/2, 1/2)$. This gives asymptotics for the main diagonal coefficients $\binom{2n}{n}$.
- The small change from $y(1-2y)/(1-x-y)$ to $(1-2y)/(1-x-y)$ should make no difference to our basic computational procedure. Show that, nevertheless, the results are very different. Explain.
- Compute the expectation and variance of the number of snaps in a standard deck of cards (no asymptotics required).

Exercises

A computer algebra system will help for some of these.

- Use Hörmander's formula to compute L_0, L_1, L_2 for $F(x, y) = (1 - x - y)^{-1}$, at the minimal point $(1/2, 1/2)$. This gives asymptotics for the main diagonal coefficients $\binom{2n}{n}$.
- The small change from $y(1 - 2y)/(1 - x - y)$ to $(1 - 2y)/(1 - x - y)$ should make no difference to our basic computational procedure. Show that, nevertheless, the results are very different. Explain.
- Compute the expectation and variance of the number of snaps in a standard deck of cards (no asymptotics required).
- Carry out the polyomino computation in detail.

Lecture V

Extensions

Overview

- We first look in turn at some of our standard assumptions in force over the last few lectures, and discuss what happens when each is weakened.

Overview

- We first look in turn at some of our standard assumptions in force over the last few lectures, and discuss what happens when each is weakened.
- Removing the combinatorial assumption leads to topological issues which we address in the framework of stratified Morse theory.

Overview

- We first look in turn at some of our standard assumptions in force over the last few lectures, and discuss what happens when each is weakened.
- Removing the combinatorial assumption leads to topological issues which we address in the framework of stratified Morse theory.
- The Fourier-Laplace integrals arising from the reductions can be more complicated that those previously studied.

Overview

- We first look in turn at some of our standard assumptions in force over the last few lectures, and discuss what happens when each is weakened.
- Removing the combinatorial assumption leads to topological issues which we address in the framework of stratified Morse theory.
- The Fourier-Laplace integrals arising from the reductions can be more complicated that those previously studied.
- We then look at going beyond the class of rational (meromorphic) singularities.

Table of Contents

Assumption: unique smooth dominant simple pole

- If there is periodicity, we typically obtain a finite number of contributing points whose contributions must be summed. This leads to the appropriate cancellation. A routine modification.

Assumption: unique smooth dominant simple pole

- If there is periodicity, we typically obtain a finite number of contributing points whose contributions must be summed. This leads to the appropriate cancellation. A routine modification.
- A toral point is one for which every point on its torus is a minimal singularity (such as $1/(1-x^2y^3)$. These occur in quantum random walks. A routine modification.

Assumption: unique smooth dominant simple pole

- If there is periodicity, we typically obtain a finite number of contributing points whose contributions must be summed. This leads to the appropriate cancellation. A routine modification.
- A toral point is one for which every point on its torus is a minimal singularity (such as $1/(1 - x^2y^3)$. These occur in quantum random walks. A routine modification.
- If the dominant point is smooth but H is not locally squarefree, then we obtain polynomial corrections that are easily computed. A routine modification.

Example (Periodicity)

- Let $F(z, w) = 1/(1 - 2zw + w^2)$ be the generating function for Chebyshev polynomials of the second kind.

Example (Periodicity)

- Let $F(z, w) = 1/(1 - 2zw + w^2)$ be the generating function for Chebyshev polynomials of the second kind.
- For directions (r, s) with $0 < s/r < 1$, there is a dominant point at

$$\mathbf{p} = \left(\frac{r}{\sqrt{r^2 - s^2}}, \sqrt{\frac{r - s}{r + s}} \right)$$

.

Example (Periodicity)

- Let $F(z,w) = 1/(1-2zw+w^2)$ be the generating function for Chebyshev polynomials of the second kind.
- For directions (r,s) with $0 < s/r < 1$, there is a dominant point at
$$\mathbf{p} = \left(\frac{r}{\sqrt{r^2-s^2}}, \sqrt{\frac{r-s}{r+s}}\right)$$
.
- There is also a dominant point at $-\mathbf{p}$. Adding the contributions yields
$$a_{rs} \sim \sqrt{\frac{2}{\pi}}(-1)^{(s-r)/2}\left(\frac{2r}{\sqrt{s^2-r^2}}\right)^{-r}\left(\sqrt{\frac{s-r}{s+r}}\right)^{-s}\sqrt{\frac{s+r}{r(s-r)}}$$
when $r+s$ is even and zero otherwise.

Assumption: transversality

- If sheets at a multiple point are not transversal, the phase of the Fourier-Laplace integral vanishes on a set of positive dimension.

Assumption: transversality

- If sheets at a multiple point are not transversal, the phase of the Fourier-Laplace integral vanishes on a set of positive dimension.
- If this occurs because there are too many sheets, the reduction from Lecture 4 works.

Assumption: transversality

- If sheets at a multiple point are not transversal, the phase of the Fourier-Laplace integral vanishes on a set of positive dimension.
- If this occurs because there are too many sheets, the reduction from Lecture 4 works.
- If it occurs because the dimension of the space spanned by normals is just too small, then it is a little harder to deal with.

Assumption: transversality

- If sheets at a multiple point are not transversal, the phase of the Fourier-Laplace integral vanishes on a set of positive dimension.
- If this occurs because there are too many sheets, the reduction from Lecture 4 works.
- If it occurs because the dimension of the space spanned by normals is just too small, then it is a little harder to deal with.
- Each term in our expansions depends on finitely many derivatives of G and H, so if sheets have contact to sufficiently high order, the results are the same as if they coincided. Thus if we can reduce in the local ring, all is well. Otherwise we may need to attack the F-L integral directly.

Example (tangential curves)

- Suppose that $\mathcal{V}$ looks like two curves intersecting at a strictly minimal point $(1, 1)$, with branches $y = g_j(x)$.

Example (tangential curves)

- Suppose that $\mathcal{V}$ looks like two curves intersecting at a strictly minimal point $(1,1)$, with branches $y = g_j(x)$.
- Suppose further that the first derivatives are equal and $f_j''(\theta) = -d_j\theta^2 + \dots$.

Example (tangential curves)

- Suppose that $\mathcal{V}$ looks like two curves intersecting at a strictly minimal point $(1, 1)$, with branches $y = g_j(x)$.
- Suppose further that the first derivatives are equal and $f_j''(\theta) = -d_j \theta^2 + \dots$.
- Then the cone K of directions is a single ray and

$$a_{rs} \sim \frac{2G(1,1)\sqrt{s}}{\sqrt{2\pi}\left(\sqrt{d_1} + \sqrt{d_2}\right)}.$$

Example (tangential curves)

- Suppose that $\mathcal{V}$ looks like two curves intersecting at a strictly minimal point $(1,1)$, with branches $y = g_j(x)$.
- Suppose further that the first derivatives are equal and $f_j''(\theta) = -d_j\theta^2 + \dots$.
- Then the cone K of directions is a single ray and

$$a_{rs} \sim \frac{2G(1,1)\sqrt{s}}{\sqrt{2\pi}\left(\sqrt{d_1} + \sqrt{d_2}\right)}.$$

- When $d_0 = d_1$ this gives the same result as a single repeated smooth factor.

Assumption: no change in local geometry

- If the phase of the Fourier-Laplace integral vanishes to order more than 2, more complicated behaviour ensues.

Assumption: no change in local geometry

- If the phase of the Fourier-Laplace integral vanishes to order more than 2, more complicated behaviour ensues.
- If the order of vanishing is 2 everywhere except for 3 at a certain direction, for example, we obtain a phase transition and Airy phenomena.

Example (Airy phenomena)

- The core of a rooted planar map is the largest 2-connected subgraph containing the root edge.

Example (Airy phenomena)

- The core of a rooted planar map is the largest 2-connected subgraph containing the root edge.
- The probability distribution of the size k of the core in a random planar map with size n is described by

$$p(n,k) = \frac{k}{n}[x^k y^n z^n]\frac{xz\psi'(z)}{(1 - x\psi(z))(1 - y\phi(z))}\,.$$

where $\psi(z) = (z/3)(1 - z/3)^2$ and $\phi(z) = 3(1 + z)^2$.

Example (Airy phenomena)

- The core of a rooted planar map is the largest 2-connected subgraph containing the root edge.
- The probability distribution of the size k of the core in a random planar map with size n is described by

$$p(n,k) = \frac{k}{n}[x^k y^n z^n] \frac{xz\psi'(z)}{(1 - x\psi(z))(1 - y\phi(z))} .$$

  where $\psi(z) = (z/3)(1 - z/3)^2$ and $\phi(z) = 3(1 + z)^2$.
- In directions away from $n = 3k$, our ordinary smooth point analysis holds. When $n = 3k$ we can redo the F-L integral easily and obtain asymptotics of order $n^{-1/3}$.

Example (Airy phenomena)

- The core of a rooted planar map is the largest 2-connected subgraph containing the root edge.
- The probability distribution of the size k of the core in a random planar map with size n is described by

$$p(n,k) = \frac{k}{n}[x^k y^n z^n] \frac{xz\psi'(z)}{(1 - x\psi(z))(1 - y\phi(z))} \, .$$

  where $\psi(z) = (z/3)(1 - z/3)^2$ and $\phi(z) = 3(1 + z)^2$.
- In directions away from $n = 3k$, our ordinary smooth point analysis holds. When $n = 3k$ we can redo the F-L integral easily and obtain asymptotics of order $n^{-1/3}$.
- Determining the behaviour as we approach this diagonal at a moderate rate is harder (Manuel Lladser PhD thesis), and recovers the results of Banderier-Flajolet-Schaeffer-Soria 2001.

Table of Contents

Non-combinatorial case: Overview

- Some applications require us to consider more general GFs, with coefficients that may not be nonnegative. Finding dominant points is now much harder.

Non-combinatorial case: Overview

- Some applications require us to consider more general GFs, with coefficients that may not be nonnegative. Finding dominant points is now much harder.
- Going back to Cauchy's integral, we use homology rather than homotopy to compute its asymptotics. Using the method of steepest descent as formalized by Morse theory, we can do this almost algorithmically in the smooth case. The integral is determined by critical points which are the same as the critical points we saw previously.

Non-combinatorial case: Overview

- Some applications require us to consider more general GFs, with coefficients that may not be nonnegative. Finding dominant points is now much harder.
- Going back to Cauchy's integral, we use homology rather than homotopy to compute its asymptotics. Using the method of steepest descent as formalized by Morse theory, we can do this almost algorithmically in the smooth case. The integral is determined by critical points which are the same as the critical points we saw previously.
- When $d = 2$, this has been implemented algorithmically, but not for higher d.

Non-combinatorial case: Overview

- Some applications require us to consider more general GFs, with coefficients that may not be nonnegative. Finding dominant points is now much harder.
- Going back to Cauchy's integral, we use homology rather than homotopy to compute its asymptotics. Using the method of steepest descent as formalized by Morse theory, we can do this almost algorithmically in the smooth case. The integral is determined by critical points which are the same as the critical points we saw previously.
- When $d = 2$, this has been implemented algorithmically, but not for higher d.
- There is a lesser known version of Morse theory due to Whitney, called stratified Morse theory, which deals with singularities. There is substantial discussion of this in the book.

Cauchy integral formula is homological

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.

Cauchy integral formula is homological

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to replace T by a contour that is more suitable for explicit computation. This may involve additional residue terms.

Cauchy integral formula is homological

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to replace T by a contour that is more suitable for explicit computation. This may involve additional residue terms.
- The homology of $\mathbb{C}^d \setminus \mathcal{V}$ is the key to decomposing the integral.

Cauchy integral formula is homological

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to replace T by a contour that is more suitable for explicit computation. This may involve additional residue terms.
- The homology of $\mathbb{C}^d \setminus \mathcal{V}$ is the key to decomposing the integral.
- It is natural to try a saddle point/steepest descent approach.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \Re log(\mathbf{z})$ as a height function; try to choose contour to minimize $\max h$.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \Re log(\mathbf{z})$ as a height function; try to choose contour to minimize $\max h$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \leq d-1$. The top dimensional stratum is the set of smooth points.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \Re log(\mathbf{z})$ as a height function; try to choose contour to minimize $\max h$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \leq d - 1$. The top dimensional stratum is the set of smooth points.
- The critical points are those where the restriction of h to a stratum has derivative zero. Generically, there are finite many.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \Re log(\mathbf{z})$ as a height function; try to choose contour to minimize $\max h$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \leq d-1$. The top dimensional stratum is the set of smooth points.
- The critical points are those where the restriction of h to a stratum has derivative zero. Generically, there are finite many.
- The Cauchy integral decomposes into a sum

$$\sum n_i \int_{C_i} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \mathbf{F}(\mathbf{z})\, \mathbf{dz}$$

where C_i is a quasi-local cycle for ${\mathbf{z}_*}^{(i)} \in \mathrm{crit}(\mathbf{r})$.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \Re log(\mathbf{z})$ as a height function; try to choose contour to minimize $\max h$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \le d-1$. The top dimensional stratum is the set of smooth points.
- The critical points are those where the restriction of h to a stratum has derivative zero. Generically, there are finite many.
- The Cauchy integral decomposes into a sum

$$\sum n_i \int_{C_i} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \mathbf{F}(\mathbf{z}) \, \mathbf{dz}$$

  where C_i is a quasi-local cycle for ${\mathbf{z}_*}^{(i)} \in \mathrm{crit}(\mathbf{r})$.
- Key problem: find the highest critical points with nonzero n_i. These are the dominant ones.

Bicolored supertrees

Example

- Consider

$$F(x,y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$

for which we want asymptotics on the main diagonal.

Bicolored supertrees

Example

- Consider
$$F(x,y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$
for which we want asymptotics on the main diagonal.
- The critical points are, listed in increasing height, $(1+\sqrt{5}, (3-\sqrt{5})/16), (2, \frac{1}{8}), (1-\sqrt{5}, (3+\sqrt{5})/16)$.

Bicolored supertrees

Example

- Consider
$$F(x,y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$
for which we want asymptotics on the main diagonal.
- The critical points are, listed in increasing height, $(1+\sqrt{5}, (3-\sqrt{5})/16), (2, \frac{1}{8}), (1-\sqrt{5}, (3+\sqrt{5})/16)$.
- In fact $(2, 1/8)$ dominates. The analysis is a substantial part of the PhD thesis of Tim DeVries (U. Pennsylvania).

Bicolored supertrees

Example

- Consider

$$F(x,y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$

  for which we want asymptotics on the main diagonal.
- The critical points are, listed in increasing height, $(1+\sqrt{5}, (3-\sqrt{5})/16), (2, \frac{1}{8}), (1-\sqrt{5}, (3+\sqrt{5})/16)$.
- In fact $(2, 1/8)$ dominates. The analysis is a substantial part of the PhD thesis of Tim DeVries (U. Pennsylvania).
- The answer:

$$a_{nn} \sim \frac{4^n\sqrt{2}\Gamma(5/4)}{4\pi} n^{-5/4}.$$

Table of Contents

Inverting diagonalization

- Recall the diagonal method shows that the diagonal of a rational bivariate GF is algebraic.

Inverting diagonalization

- Recall the diagonal method shows that the diagonal of a rational bivariate GF is algebraic.
- Conversely, every univariate algebraic GF is the diagonal of some rational bivariate GF (next slide).

Inverting diagonalization

- Recall the diagonal method shows that the diagonal of a rational bivariate GF is algebraic.
- Conversely, every univariate algebraic GF is the diagonal of some rational bivariate GF (next slide).
- The latter result does not generalize strictly to higher dimensions, but something close to it is true. Our multivariate framework means that increasing dimension causes no difficulties in principle, so we can reduce to the rational case.

Inverting diagonalization

- Recall the diagonal method shows that the diagonal of a rational bivariate GF is algebraic.
- Conversely, every univariate algebraic GF is the diagonal of some rational bivariate GF (next slide).
- The latter result does not generalize strictly to higher dimensions, but something close to it is true. Our multivariate framework means that increasing dimension causes no difficulties in principle, so we can reduce to the rational case.
- The elementary diagonal of $F(z_0, \ldots, z_d) = \sum_{r_0,\ldots,r_d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ is

$$\operatorname{diag} F := f(z_1, \ldots, z_d) = \sum_{r_1,\ldots,r_d} a_{r_1,r_1,\ldots,r_d} z_1^{r_1} \ldots z_d^{r_d}.$$

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
where $u(0, \underline{0}) \neq 0$ and $1 \leq k \in \mathbb{N}$.

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
where $u(0, \underline{0}) \neq 0$ and $1 \leq k \in \mathbb{N}$.
- Define
$$R(z_0, \mathbf{z}) = \frac{z_0^2 P_1(z_0, z_0 z_1, z_2, \dots)}{k P(z_0, z_0 z_1, z_2, \dots)}$$
$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \dots z_d).$$

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
where $u(0, \underline{0}) \neq 0$ and $1 \leq k \in \mathbb{N}$.
- Define
$$R(z_0, \mathbf{z}) = \frac{z_0^2 P_1(z_0, z_0 z_1, z_2, \dots)}{k P(z_0, z_0 z_1, z_2, \dots)}$$
$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \dots z_d).$$
- The Argument Principle shows that $F = \operatorname{diag} R$:
$$\frac{1}{2\pi i} \int_C \tilde{R}(w, \mathbf{z}) \frac{dw}{w} = \sum \operatorname{Res} \tilde{R}(w, \mathbf{z}) = F(\mathbf{z}).$$

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
where $u(0, \underline{0}) \neq 0$ and $1 \leq k \in \mathbb{N}$.
- Define
$$R(z_0, \mathbf{z}) = \frac{z_0^2 P_1(z_0, z_0 z_1, z_2, \dots)}{k P(z_0, z_0 z_1, z_2, \dots)}$$
$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \dots z_d).$$
- The Argument Principle shows that $F = \operatorname{diag} R$:
$$\frac{1}{2\pi i} \int_C \tilde{R}(w, \mathbf{z}) \frac{dw}{w} = \sum \operatorname{Res} \tilde{R}(w, \mathbf{z}) = F(\mathbf{z}).$$
- Higher order terms are essential: the numerator of $\tilde{R}$ always vanishes at the dominant point. The Catalan example from Lecture 4 was created using this method.

Safonov's general construction

- In general, apply a sequence of blowups (monomial substitutions) to reduce to the case above.

Safonov's general construction

- In general, apply a sequence of blowups (monomial substitutions) to reduce to the case above.
- This is a standard idea from algebraic geometry: resolution of singularities.

Safonov's general construction

- In general, apply a sequence of blowups (monomial substitutions) to reduce to the case above.
- This is a standard idea from algebraic geometry: resolution of singularities.
- Definition: Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have $d+1$ variables and let M be a $d \times d$ matrix with nonnegative entries. The M-diagonal of F is the formal power series in d variables whose coefficients are given by $b_{r_2,\ldots r_d} = a_{s_1,s_1,s_2,\ldots s_d}$ and $(s_1, \ldots, s_d) = (r_1, \ldots, r_d)M$.

Safonov's general construction

- In general, apply a sequence of blowups (monomial substitutions) to reduce to the case above.
- This is a standard idea from algebraic geometry: resolution of singularities.
- Definition: Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have $d+1$ variables and let M be a $d \times d$ matrix with nonnegative entries. The M-diagonal of F is the formal power series in d variables whose coefficients are given by $b_{r_2,\dots r_d} = a_{s_1,s_1,s_2,\dots s_d}$ and $(s_1, \dots, s_d) = (r_1, \dots, r_d)M$.
- Theorem: Let f be an algebraic function of d variables. Then there is a unimodular integer matrix M with positive entries and a rational function F in $d+1$ variables such that f is the M-diagonal of F.

Safonov's general construction

- In general, apply a sequence of blowups (monomial substitutions) to reduce to the case above.
- This is a standard idea from algebraic geometry: resolution of singularities.
- Definition: Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have $d+1$ variables and let M be a $d \times d$ matrix with nonnegative entries. The M-diagonal of F is the formal power series in d variables whose coefficients are given by $b_{r_2,\dots r_d} = a_{s_1,s_1,s_2,\dots s_d}$ and $(s_1, \dots, s_d) = (r_1, \dots, r_d)M$.
- Theorem: Let f be an algebraic function of d variables. Then there is a unimodular integer matrix M with positive entries and a rational function F in $d+1$ variables such that f is the M-diagonal of F.
- The example $x\sqrt{1-x-y}$ shows that the elementary diagonal cannot always be used.

Example (Narayana numbers)

- The bivariate GF $F(x, y)$ for the Narayana numbers

$$a_{rs} = \frac{1}{r}\binom{r}{s}\binom{r-1}{s-1}$$

satisfies $P(F(x, y), x, y) = 0$, where

$$\begin{aligned} P(w, x, y) &= w^2 - w\left[1 + x(y-1)\right] + xy \\ &= \left[w - F(x, y)\right]\left[w - \overline{F}(x, y)\right]. \end{aligned}$$

where $\overline{F}$ is the algebraic conjugate.

Example (Narayana numbers)

- The bivariate GF $F(x,y)$ for the Narayana numbers

$$a_{rs} = \frac{1}{r}\binom{r}{s}\binom{r-1}{s-1}$$

satisfies $P(F(x,y),x,y) = 0$, where

$$\begin{aligned} P(w,x,y) &= w^2 - w\left[1 + x(y-1)\right] + xy \\ &= \left[w - F(x,y)\right]\left[w - \overline{F}(x,y)\right]. \end{aligned}$$

where $\overline{F}$ is the algebraic conjugate.

- Using the above construction we obtain the lifting

$$G(u,x,y) = \frac{u(1-2u-ux(1-y))}{1-u-xy-ux(1-y)}.$$

Example (Narayana numbers continued)

- The above lifting yields asymptotics by smooth point analysis in the usual way. The critical point equations yield

$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$

and we obtain asymptotics starting with s^{-2}. For example

$$a_{2s,s} \sim \frac{16^s}{8\pi s^2}.$$

Example (Narayana numbers continued)

- The above lifting yields asymptotics by smooth point analysis in the usual way. The critical point equations yield

$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$

  and we obtain asymptotics starting with s^{-2}. For example

$$a_{2s,s} \sim \frac{16^s}{8\pi s^2}.$$

- Interestingly, specializing $y = 1$ commutes with lifting (and yields the shifted Catalan numbers as in Lecture 4). Is this always true?

Technical issues

- Safonov's lifting often takes us away from the combinatorial case. Therefore the Morse theory approach will probably be needed.

Technical issues

- Safonov's lifting often takes us away from the combinatorial case. Therefore the Morse theory approach will probably be needed.
- Dominant singularities can be at infinity.

Technical issues

- Safonov's lifting often takes us away from the combinatorial case. Therefore the Morse theory approach will probably be needed.
- Dominant singularities can be at infinity.
- There are other lifting procedures, some of which go from dimension d to $2d$. They seem complicated, and we have not yet tried them in detail.

Technical issues

- Safonov's lifting often takes us away from the combinatorial case. Therefore the Morse theory approach will probably be needed.
- Dominant singularities can be at infinity.
- There are other lifting procedures, some of which go from dimension d to $2d$. They seem complicated, and we have not yet tried them in detail.
- However in some cases they work better - for example $2xy/(2+x+y)$ is a lifting of $x\sqrt{1-x}$, whereas Safonov's method appears not to work easily.

Possible research projects

- Systematically compare the diagonal method and our methods.

Possible research projects

- Systematically compare the diagonal method and our methods.
- Systematically generate sums of squares identities and include them in OEIS.

Possible research projects

- Systematically compare the diagonal method and our methods.
- Systematically generate sums of squares identities and include them in OEIS.
- Develop a good theory for algebraic singularities (using resolution of singularities somehow).

Possible research projects

- Systematically compare the diagonal method and our methods.
- Systematically generate sums of squares identities and include them in OEIS.
- Develop a good theory for algebraic singularities (using resolution of singularities somehow).
- Improve efficiency of algorithms for computing higher order terms in expansions. Implement them in Sage.

Possible research projects

- Systematically compare the diagonal method and our methods.
- Systematically generate sums of squares identities and include them in OEIS.
- Develop a good theory for algebraic singularities (using resolution of singularities somehow).
- Improve efficiency of algorithms for computing higher order terms in expansions. Implement them in Sage.
- Develop better computational methods for computing symbolically with symmetric functions.

Possible research projects

- Systematically compare the diagonal method and our methods.
- Systematically generate sums of squares identities and include them in OEIS.
- Develop a good theory for algebraic singularities (using resolution of singularities somehow).
- Improve efficiency of algorithms for computing higher order terms in expansions. Implement them in Sage.
- Develop better computational methods for computing symbolically with symmetric functions.
- Make the computation of dominant points algorithmic in the noncombinatorial case.

Exercises

- Prove that the numerator of Safonov's lifting must vanish at the dominant point, as claimed above.

Exercises

- Prove that the numerator of Safonov's lifting must vanish at the dominant point, as claimed above.
- Show that $x\sqrt{1-x-y}$ cannot occur as the elementary diagonal of a rational function in 3 variables, as claimed above.

Exercises

- Prove that the numerator of Safonov's lifting must vanish at the dominant point, as claimed above.
- Show that $x\sqrt{1-x-y}$ cannot occur as the elementary diagonal of a rational function in 3 variables, as claimed above.
- Derive asymptotics for the following GF (Vince and Bóna 2012)
$$F(x,y) = 1 - \sqrt{(1-x)^2 + (1-y)^2 - 1}$$

Exercises

- Prove that the numerator of Safonov's lifting must vanish at the dominant point, as claimed above.
- Show that $x\sqrt{1-x-y}$ cannot occur as the elementary diagonal of a rational function in 3 variables, as claimed above.
- Derive asymptotics for the following GF (Vince and Bóna 2012)
$$F(x,y) = 1 - \sqrt{(1-x)^2 + (1-y)^2 - 1}$$
- In the Cauchy integral for $\sqrt{1-x}$, make a substitution to convert to an integral of a rational function. How general is this procedure?