Lecture I

Motivation, review, overview

1 Preliminaries

Recommended books for univariate asymptotics

- H. Wilf, *generatingfunctionology*, `http://www.math.upenn.edu/~wilf/DownldGF.html`
- M. Kauers and P. Paule, *The Concrete Tetrahedron*, `www.risc.jku.at/people/mkauers/publications/kauers11h.pdf`.
- P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, `http://algo.inria.fr/flajolet/Publications/AnaCombi/anacombi.html`
- A. Odlyzko, *Asymptotic Enumeration Methods*, `www.dtc.umn.edu/~odlyzko/doc/enumeration.html`..

Main references for all lectures

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. `https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/`
- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.
- Sage implementations by Alex Raichev: `https://github.com/araichev/amgf`.

Lecture plan

- These lectures discuss results obtained over more than 10 years of work with Robin Pemantle and others, explained in detail in our book.
- Outline of lectures:
  - (i) Motivation, review of univariate case, overview of results.
  - (ii) Smooth points in dimension 2.
  - (iii) Higher dimensions, multiple points.
  - (iv) Computational issues.
  - (v) Extensions and open problems.
- Exercises are of varying levels of difficulty. We can discuss some in the problem sessions. Those marked (C) involve probably publishable research, for which I am seeking collaborators, should be accessible to PhD students.

2 Introduction and motivation

Lecture 1: Overview

- In one variable, starting with a sequence $a_{\mathbf{r}}$ of interest, we form its generating function $F(\mathbf{z})$. Cauchy's integral theorem allows us to express $a_{\mathbf{r}}$ as an integral. The exponential growth rate of $a_{\mathbf{r}}$ is determined by the location of a dominant singularity $\mathbf{z}_*$ of F. More precise estimates depend on the local geometry of the singular set $\mathcal{V}$ of F near $\mathbf{z}_*$.
- In the multivariate case, all the above is still true. However, we need to specify the direction in which we want asymptotics; we then need to worry about uniformity; the definition of "dominant" is a little different; the local geometry of $\mathcal{V}$ can be much nastier; the local analysis is more complicated.

Standing assumptions

- Unless otherwise specified, the following hold throughout.
- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The *generating function* (GF) is the formal power series $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$.
- The *combinatorial case*: all $a_{\mathbf{r}} \geq 0$.
- The *aperiodic case*: $a_{\mathbf{r}}$ is not supported on a proper sublattice of $\mathbb{N}^d$.

2.1 Univariate case

From sequence to generating function and back

- Most sequences of interest satisfy recurrences. We analyse them using the GF. Sequence operations correspond to algebraic operations on power series (e.g. $a_n \leftrightarrow F(z)$ implies $na_n \leftrightarrow zF'(z)$).
- The GF can often be determined by translating the recurrence into a functional equation for F, then solving it.
- Example: (Fibonacci)

$$\begin{aligned} a_r &= a_{r-1} + a_{r-2} \quad \text{if } r \geq 2 \\ a_0 &= 0, a_1 = 1 \end{aligned}$$

automatically yields $F(z) = z/(1 - z - z^2)$.

- Our focus this week is on the next step: deriving a formula (usually asymptotic approximation) for a_r, given a nice representation of F. This is *coefficient extraction*.

Univariate case: exponential growth rate

- Let U be the open disc of convergence of F, having radius ρ, $\partial\,\mathrm{U}$ its boundary.
- Let C_R denote the circle of radius R centred at 0. If $R < \rho$ then by Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i}\int_C z^{-n-1}F(z)\,dz.$$

- This directly yields $|a_n| \le (2\pi i R)KR^{-n-1}/(2\pi i) = KR^{-n}$.
- Letting $R \to \rho_-$ shows that the exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{n}\log|a_n| = -\log\rho.$$

- Suppose that $\rho < \infty$. Then (Vivanti-Pringsheim) $z = \rho$ is a singularity of F, and is the only singularity of F on $\partial\,\mathrm{U}$.
- Further analysis depends on the type of singularity.

From singularities to asymptotic expansions

There are standard methods for dealing with each type of singularity, all relying on choosing appropriate contours of integration. The most common:

- if ρ is a pole, use the residue theorem;
- if F is rational, can also use partial fraction decomposition;
- if ρ is algebraic/logarithmic, use *singularity analysis* (Flajolet-Odlyzko 1990);
- if ρ is essential, use the *saddle point method*.

Example 1 (Univariate pole example: derangements).

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$, so $a_r = O(1)$.
- Using a circle of radius $1+\varepsilon$ we obtain, by Cauchy's residue theorem,

$$a_r = \frac{1}{2\pi i}\int_{C_{1+\varepsilon}} z^{-r-1}F(z)\,dz - \mathrm{Res}(z^{-r-1}F(z); z = 1).$$

- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.
- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.
- Since there are no more poles, we can push the contour of integration to ∞ in this case, so the error in the approximation decays faster than any exponential function of r.

Univariate rational functions: general solution

- Given a rational function $p(z)/q(z)$ with $q(0) = 1$, factor it as $q(z) = \prod_i (1 - \phi_i z)^{n_i}$ with all ϕ_i distinct.
- Use partial fractions to expand

$$F(z) = \sum_i \sum_{j=1}^{n_i} \frac{c_{ij}}{(1 - \phi_i z)^j}.$$

- This allows for an exact formula, and restricting to the largest ϕ_i (corresponding to the minimal zeros of q) gives the asymptotic expansion.
- For example, Fibonacci yields $a_r \sim 5^{-1/2}[(1 + \sqrt{5})/2]^r$.
- Repeated roots provide *polynomial correction* to the exponential factor. For example, $1/(1 - 2z)^3 = \sum_r \binom{r+2}{2} 2^r z^r$.

Example 2 (Essential singularity: saddle point method).
- Here $F(z) = \exp(z)$. The Cauchy integral formula on a circle C_R of radius R gives $a_n \leq F(R)/R^n$.
- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.
- The integral over C_n has most mass near $z = n$, so that

$$\begin{aligned} a_n &= \frac{F(n)}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) \frac{F(ne^{i\theta})}{F(n)} \, d\theta \\ &\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp\left(-in\theta + \log F(ne^{i\theta}) - \log F(n)\right) \, d\theta. \end{aligned}$$

Example 3 (Saddle point example continued).
- The Maclaurin expansion yields

$$-in\theta + \log F(ne^{i\theta}) - \log F(n) = -n\theta^2/2 + O(n\theta^3).$$

- This gives, with $b_n = 2\pi n^n e^{-n} a_n$,

$$b_n \approx \int_{-\varepsilon}^{\varepsilon} \exp(-n\theta^2/2) \, d\theta \approx \int_{-\infty}^{\infty} \exp(-n\theta^2/2) \, d\theta = \sqrt{2\pi/n}.$$

- This recaptures *Stirling's approximation*, since $n! = 1/a_n$:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

2.2 Multivariate case

Multivariate asymptotics — some quotations

- (Bender 1974) "Practically nothing is known about asymptotics for recursions in two variables even when a GF is available. Techniques for obtaining asymptotics from bivariate GFs would be quite useful."
- (Odlyzko 1995) "A major difficulty in estimating the coefficients of mvGFs is that the geometry of the problem is far more difficult. ... Even rational multivariate functions are not easy to deal with."
- (Flajolet/Sedgewick 2009) "Roughly, we regard here a bivariate GF as a collection of univariate GFs ...."

Finding multivariate GFs

- Unlike the univariate case, a constant coefficient linear recursion need not yield a rational function. This occurs, for example, in lattice walks where steps go forward in some dimensions and backward in others.
- The *kernel method* (see Chapter 2.3) is often useful for dealing with these cases.
- Linear recursions with polynomial coefficients yield linear PDEs, which can be hard to solve, certainly harder than the ODEs in the univariate case.
- We will not deal with this issue in these lectures - we assume that the GF is given in explicit form (say rational or algebraic) and concentrate on extraction of Maclaurin coefficients.

Diagonal method

- Suppose that $d = 2$ and we want asymptotics from $F(z, w)$ on the diagonal $r = s$.
- The *diagonal GF* is $F_{1,1}(x) = \sum_n a_{nn} x^n$.
- We can compute, for some circle γ_x around $t = 0$,

$$\begin{aligned} F_{1,1}(x) &= [t^0]F(x/t, t) \\ &= \frac{1}{2\pi i}\int_{\gamma_x} \frac{F(x/t,t)}{t}\,dt \\ &= \sum_k \operatorname{Res}(F(x/t,t)/t; t = s_k(x)) \end{aligned}$$

  where the $s_k(x)$ are the singularities satisfying $\lim_{x\to 0} s_k(x) = 0$.

- If F is rational, then $F_{1,1}$ is algebraic.

Example 4 (Delannoy lattice walks). • Consider walks in $\mathbb{Z}^2$, starting from $(0,0)$, with steps in $\{(1,0),(0,1),(1,1)\}$ (*Delannoy walks*).

- Here $F(x,y) = (1-x-y-xy)^{-1}$.
- This corresponds to the recurrence $a_{rs} = a_{r,s-1} + a_{r-1,s} + a_{r-1,s-1}$.
- How to compute a_{rs} for large r, s?
- For example, what does $a_{7n,5n}$ look like as $n \to \infty$?

Use the diagonal method?

- We could try to compute the diagonal GF $F_{pq}(z) := \sum_{n\geq 0} a_{pn,qn} z^n$ as above.
- This would work fairly well for $p = q = 1$, but is generally a bad idea (see Chapter 13.1):
  - The computational complexity increases rapidly with $p + q$.
  - We can't handle irrational diagonals, or derive uniform asymptotics (if p/q changes slightly, what do we do?).
  - If $d > 2$, diagonals will not be algebraic in general, even if F is rational.
  - Fancier methods exist (based on *holonomic* or *D-finite* theory), but again computational complexity is a major obstacle.

Our plan

- Thoroughly investigate asymptotic coefficient extraction, starting with meromorphic $F(\mathbf{z}) := F(z_1, \ldots, z_d)$ (pole singularities).
- Directly generalize the $d = 1$ analysis for poles.
- Use the Cauchy Integral Formula in dimension d.
- Use residue analysis to derive asymptotics.
- Amazingly little was known *even about rational F in 2 variables.* We aimed to create a general theory.

Some difficulties when $d > 1$

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the *singular variety* $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points, and may self-intersect;
  - real dimension of contour is d, that of $\mathcal{V}$ is $2d - 2$, so less room to avoid each other;
  - topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated;
- Analysis: the (Leray) residue formula is much harder to use.

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of *critical points.*
- For computing asymptotics in direction $\overline{\mathbf{r}}$, we may restrict to a dominant point $\mathbf{z}_*(\overline{\mathbf{r}})$ lying in the positive orthant. (*)
- There is an asymptotic expansion formula($\mathbf{z}_*$) for $a_{\mathbf{r}}$, where formula($\mathbf{z}_*$) is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.
- This yields
$$a_{\mathbf{r}} \sim \mathrm{formula}(\mathbf{z}_*)$$
where the expansion is uniform on compact subsets of directions, provided the geometry does not change.
- The set $\mathrm{crit}(\overline{\mathbf{r}})$ is computable via symbolic algebra.
- To determine the dominant point requires a little more work, but usually not much. (*)

Obvious questions

- Can we always find asymptotics in a given direction in this way?
- How do we find the dominant point?
- How easy is it to carry out all the computations?
- What about higher order terms in the expansions?
- How does our method compare with others?
- How does it all work? (I want to see the details)

Exercises

Exercises: finding GFs

- Find (a defining equation for) the GF for the sequence (a_n) defined by $a_0 = 0; a_n = n + (2/n)\sum_{0\le k<n} a_k$ for $n \ge 1$.
- (C) Find an explicit form for the GF of the sequence given by

$$p(n,j) = \frac{2n-1-j}{2n-1}p(n-1,j) + \frac{j-1}{2n-1}p(n-1,j-1)$$

  with initial condition $p(1,2) = 1$.
- Express the GF for the sequence given by the recursion

$$f(r,s) = f(r-1,s) + f(r,s-1) - \frac{(r+s-1)}{(r+s)}f(r-1,s-1)$$
$$f(0,s) = 1, f(r,0) = 1$$

  as explicitly as you can.

Exercises: diagonal method

- Find (by hand) a closed form for the GF for the leading diagonal in the Delannoy case (that is, compute $F_{1,1}$).
- Repeat this for $F_{2,1}$.
- Challenge for D-finiteness experts: for Delannoy walks, what is the largest $p + q$ (where $\gcd\{p, q\} = 1$) for which you can compute an asymptotic approximation of $a_{pn,qn}$, with an error of less than 0.01% when $n = 10$?

Lecture II

Smooth points in dimension 2

Lecture 2: Overview

- If the dominant singularity is a *smooth point* of $\mathcal{V}$, the local geometry is simple. In the generic case, the local analysis is also straightforward. We can derive explicit results that apply to a huge number of applications. In dimension 2, these are even more explicit.
- We first consider the case where the dominant singularity is *strictly minimal*, meaning that F is analytic on the open polydisc D defined by z_*, which is the only singularity on $\overline{D}$. In this case we can use univariate residue theory accompanied by elementary deformations of the contour of integration.

3 Basic smooth point formula in dimension 2

Proof of generic smooth point formula — overview

- A point $\mathbf{z}$ of $\mathcal{V}$ is *smooth* if $\nabla H(\mathbf{z}) \neq 0$ ($\mathbf{z}$ is a *simple pole*).
- At a smooth point, we can reduce to an iterated integral where the inner integral is 1-dimensional.
- We can use univariate residue theory to approximate the inner integral.
- It remains then to integrate over the remaining $d-1$ variables.
- The first and last steps are unnecessary in the univariate case.
- We focus here on the $d-1=1$ case but everything works in general dimension.

Reduction step 1: localization

- Suppose that (z_*, w_*) is a smooth strictly minimal pole with nonzero coordinates, and let $\rho = |z_*|, \sigma = |w_*|$. Let C_a denote the circle of radius a centred at 0.
- By Cauchy, for small $\delta > 0$,

$$a_{rs} = (2\pi i)^{-2} \int_{C_\rho} z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

- The inner integral is small away from z_*, so that for some small neighbourhood N of z_* in C_ρ,

$$a_{rs} \approx I := (2\pi i)^{-2} \int_N z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

- Note that this is because of strict minimality: off N, the function $F(z,\cdot)$ has radius of convergence greater than σ, and compactness allows us to do everything uniformly.

Reduction step 2: residue

- By smoothness, there is a local parametrization $w = g(z) := 1/v(z)$ near z_*.

- If δ is small enough, the function $w \mapsto F(z,w)/w$ has a unique pole in the annulus $\sigma - \partial \le |w| \le \sigma + \delta$. Let $\Psi(z)$ be the *residue* there.

- By Cauchy,

$$I = I' + (2\pi i)^{-1} v(z)^s \Psi(z),$$

  where

$$I' := (2\pi i)^{-2} \int_N z^{-r} \int_{C_{\sigma+\delta}} w^{-s} F(z,w) \, \frac{dw}{w} \, \frac{dz}{z}.$$

- Clearly $|z_*^r I'| \to 0$, and hence

$$a_{rs} \approx (2\pi i)^{-1} \int_N z^{-r} v(z)^s \Psi(z) \, dz.$$

Reduction step 3: Fourier-Laplace integral

- We make the substitution

$$\begin{aligned} f(\theta) &= -\log \frac{v(z_* e^{i\theta})}{v(z_*)} + i\frac{r\theta}{s} \\ A(\theta) &= \Psi(z_* \exp(i\theta)). \end{aligned}$$

- This yields

$$a_{rs} \sim \frac{1}{2\pi} z_*^{-r} w_*^{-s} \int_D \exp(-sf(\theta)) A(\theta) \, d\theta$$

  where D is a small neighbourhood of $0 \in \mathbb{R}$.

Fourier-Laplace integrals

- We have been led to asymptotic ($\lambda >> 0$) analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\theta)} A(\theta) \, d\theta$$

where:

  - $\mathbf{0} \in D, f(\mathbf{0}) = 0$.
  - $\operatorname{Re} f \geq 0$; the *phase* f and *amplitude* A are analytic.
  - D is a neighbourhood of 0.

- Such integrals are well known in many areas including mathematical physics. Potential difficulties in analysis: interplay between exponential and oscillatory decay of f, degeneracy of f, boundary issues.

Laplace approximation to Fourier-Laplace integrals

- Integration by parts shows that unless $f'(0) = 0$, $I(\lambda)$ is rapidly decreasing (except for boundary terms).

- If 0 is an isolated stationary point and the boundary terms can be neglected, then we have a good chance of computing an asymptotic expansion for the integral.

- If furthermore $f''(0) \neq 0$ (the *nondegeneracy* condition), we have the nicest formula: the standard *Laplace approximation* for the leading term is

$$I(\lambda) \sim A(0)\sqrt{\frac{2\pi}{\lambda f''(0)}}.$$

Our specific F-L integral

- Note that

$$f'(0) = -i\left(\frac{z_* v'(z_*)}{v(z_*)} - \frac{r}{s}\right).$$

- Thus if $\alpha := r/s$ and $\alpha \neq zv'(z_*)/v(z_*)$, our "reduction" is of no use, whereas when $\alpha = zv'(z_*)/v(z_*)$ (*critical point equation)*, we definitely get a result of order $|z_*|^{-r}|w_*|^{-s}$ as $r \to \infty$ with $r/s = \alpha$.

- Furthermore

$$f''(0) = \frac{z_*^2 v''(z_*)}{v(z_*)} + \frac{z_* v'(z_*)}{v(z_*)} - \left(\frac{z_* v'(z_*)}{v(z_*)}\right)^2.$$

- So given (z_*, w_*), for this value of α we can derive asymptotics using the Laplace approximation as above.

Converting back to the original data

- We have made several reductions and obtained an asymptotic approximation for a_{rs}, in terms of derived data.
- The derivatives of f can be be expressed in terms of derivatives of H by using the chain rule and solving equations.
- Substituting at the point $\theta = 0$ and solving yields

$$
\begin{aligned}
f'(0) &= i\frac{r}{s} - i\frac{zH_z}{wH_w} \\
f''(0) &= Q := -(wH_w)^2 zH_z - wH_w(zH_z)^2 - (wH_w)^2 z^2 H_{zz} \\
&\quad - (zH_z)^2 w^2 H_{ww} + zwH_zH_wH_{zw}.
\end{aligned}
$$

  where these are evaluated at (z_*, w_*).
- The residue can also be computed in terms of H. We can now put everything together to give an explicit formula in terms of original data.

Generic smooth point asymptotics in dimension 2

- Suppose that $F = G/H$ has a strictly minimal simple pole at $\mathbf{p} = (z^*, w^*)$. If $Q(\mathbf{p}) \neq 0$, then when $s \to \infty$ with $(rwH_w - szH_z)_{|\mathbf{p}} = 0$,

$$
a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{2\pi}}\sqrt{\frac{-wH_w(\mathbf{p})}{sQ(\mathbf{p})}} + O(s^{-3/2})\right].
$$

  The apparent lack of symmetry is illusory, since $wH_w/s = zH_z/r$ at $\mathbf{p}$.
- This, the simplest multivariate case, already covers hugely many applications.
- Here $\mathbf{p}$ is given, which specifies the only direction in which we can say anything useful. But we can vary $\mathbf{p}$ and obtain asymptotics that are uniform in the direction.

4 Illustrative examples

Important special case: Riordan arrays

- A *Riordan array* is a bivariate sequence with GF of the form

$$
F(x, y) = \frac{\phi(x)}{1 - yv(x)}.
$$

- Examples include: Pascal, Catalan, Motzkin, Schröder, etc, triangles; sums of IID random variables; many plane lattice walk models.

- In this case, if we define

$$\mu(x) := xv'(x)/v(x)$$
$$\sigma^2(x) := x^2 v''(x)/v(x) + \mu(x) - \mu(x)^2$$

the previous formula boils down (under extra assumptions) to

$$a_{rs} \sim (x_*)^{-r} v(x_*)^s \frac{\phi(x_*)}{\sqrt{2\pi s \sigma^2(x_*)}}$$

where x_* satisfies $\mu(x_*) = r/s$.

Example 5 (Delannoy walks).

- Recall that $F(x,y) = (1 - x - y - xy)^{-1}$. This is Riordan with $\phi(x) = (1-x)^{-1}$ and $v(x) = (1+x)/(1-x)$. Here $\mathcal{V}$ is globally smooth and for each (r,s) there is a unique solution to $\mu(x) = r/s$.

- Solving, and using the formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{r}{\Delta - s}\right]^r \left[\frac{s}{\Delta - r}\right]^s \sqrt{\frac{rs}{2\pi\Delta(r + s - \Delta)^2}}.$$

where $\Delta = \sqrt{r^2 + s^2}$.

- Extracting the diagonal is now easy: $a_{7n,5n} \sim AC^n n^{-1/2}$ where $A \approx 0.236839621050264$, $C \approx 30952.9770838817$.

- Compare Panholzer-Prodinger, Bull. Aust. Math. Soc. 2012.

Example 6 (Equal numbers of parts).

- What is the probability that two independently and uniformly chosen elements of a combinatorial class have the same number of parts, k, given that they have the same total size n?

- Compare Banderier-Hitczenko, Discrete Mathematics 2012.

- If (a_{nk}) is Riordan defined by ϕ, v, then the numerator is

$$b_n := \sum_k a_{nk}^2 = [t^n u^n] \frac{\phi(t)\phi(u)}{1 - v(t)v(u)}$$

- Aside: this formula gives interesting sum of squares identities..

Example 7 (Equal numbers of parts continued).

- The smooth point formula applies, provided $\lim_{x\to\rho_-} v(x) > 1$, where ρ is the radius of convergence of v. This is the *supercritical* case.

- In the supercritical case, let c be the positive root of $v(x) = 1$. Then

$$b_n \sim c^{-2n} \frac{\phi(c)^2}{\sqrt{4\pi\mu_v(c)\sigma_v^2(c)}} n^{-1/2}.$$

- Aside: we can proceed analogously for arbitrary $d \geq 2$.
- See M.C. Wilson, *Diagonal asymptotics for products of combinatorial classes*, Combinatorics, Probability and Computing (Flajolet memorial issue).

Example 8 (Polyominoes).

- A *horizontally convex polyomino* (HCP) is a union of cells $[a, a+1] \times [b, b+1]$ in the two-dimensional integer lattice such that the interior of the figure is connected and every row is connected.
- The GF for horizontally convex polyominoes (k = rows, n = squares) is

$$\begin{aligned} F(x, y) &= \sum_{n,k} a_{nk} x^n y^k \\ &= \frac{xy(1-x)^3}{(1-x)^4 - xy(1 - x - x^2 + x^3 + x^2 y)}. \end{aligned}$$

Example 9 (Polyominoes continued).

- Here $\mathcal{V}$ is smooth everywhere except $(1, 0)$, which cannot contribute to asymptotics except when $s = 0$, so we ignore that.
- For each direction with $0 < \lambda := k/n \leq 1$, there are 4 critical points. Finding the dominant one symbolically is a little tricky. It lies in the first quadrant and there is a unique such point.
- The x and y-coordinates of the dominant point are each given by a quartic (with coefficients that are polynomial in λ). Thus they are algebraic, but complicated to express.
- For each λ we can solve numerically if desired. The general asymptotic shape is clear from the smooth point formula.
- More on this example in Lecture 4.

Example 10 (Symmetric Eulerian numbers).

- Let $a_{rs}/(r!s!)$ be the number of permutations of the set $[r + s + 1] := \{1, 2, \ldots, r + s + 1\}$ with precisely r descents.
- The exponential GF is

$$F(x, y) = \frac{e^x - e^y}{xe^y - ye^x} = \frac{(e^x - e^y)/(x - y)}{(xe^y - ye^x)/(x - y)}$$

- Here $\mathcal{V}$ is globally smooth. The dominant point for $r = s$ is $(1, 1)$ and for other directions it is given by

$$\begin{aligned} (1-x)s &= (y-1)r \\ xe^y &= ye^x. \end{aligned}$$

- The smooth point formula gives the asymptotic form, and for a fixed direction we can solve numerically.

Exercises

Exercises: 2D smooth points

- Write down explicitly the Fourier-Laplace integral for the Delannoy example. Is it obvious that $f'(0) = 0$ from this representation?
- What extra assumptions on ϕ and v are required in order for the smooth point analysis to apply to a Riordan array, and for which directions does our method yield asymptotics?
- Given an equation of the form $f(z) = z\phi(f(z))$ where $f(x) = \sum_n a_n z^n$, use the Lagrange Inversion Formula to show that

  $$na_n = [x^n y^n] \frac{y}{1 - x\phi(y)}.$$

  and hence derive first order asymptotics for a_n. When is the approximation valid?
- (C) Use the formula for b_n above to systematically derive identities involving sums of squares that are not in OEIS.

Lecture III

Higher dimensions, other geometries

Lecture 3: Overview

- We can generalize the smooth point analysis to the case of multiple points. In higher dimensions, there is a nice geometric interpretation in terms of convex geometry of the logarithmic domain of convergence.
- We derive explicit formulae for multiple points. The residue computations can be done in terms of residue forms, which enables us to derive stronger results.

5 Higher dimensional smooth points

Higher dimensions

- The smooth point argument from the previous lecture generalizes directly to dimension d.
- The difference is that the ensuing Fourier-Laplace integral is in dimension $d - 1$.

- There is a generalization of the Laplace approximation, namely

$$I(\lambda) \sim A(\mathbf{0})\sqrt{\frac{1}{\lambda \det \frac{\mathrm{Q}}{2\pi}}}.$$

- There are technical issues involved in proving this, because the phase f is neither purely real nor purely imaginary. See Chapter 5.

Smooth formulae for general d

- $\mathbf{z}$ is a critical point for $\mathbf{r}$ iff

$$\nabla_{\log} H(\mathbf{z}) := (z_1 H_1, \ldots, z_d H_d) \text{ is parallel to } \mathbf{r}.$$

- When $\mathbf{z}_*$ is a critical point for $\mathbf{r}$, then, with Q denoting the Hessian of the derived function f in the Fourier-Laplace integral, k any coordinate where $H_k := \partial H/\partial z_k \neq 0$:

$$a_{\mathbf{r}} \sim \mathbf{z_*}^{-\mathbf{r}} \frac{1}{\sqrt{\det 2\pi\, \mathrm{Q}(\mathbf{r})}} \frac{G(\mathbf{z})}{z_k H_k(\mathbf{z}_*)} r_k^{(1-d)/2}.$$

- This specializes when $d = 2$ to the previous formula.

Example 11 (Alignments).

- A $(d, r_1, \ldots, r_d)$-*alignment* is a d-row binary matrix with jth row sum r_j and no zero columns.
- These have applications to bioinformatics.
- The generating function for the number of $(d, \cdot)$-alignments is

$$F(\mathbf{z}) = \sum a(r_1, \ldots, r_d)\mathbf{z}^{\mathbf{r}} = \frac{1}{2 - \prod_{i=1}^{d}(1 + z_i)}.$$

- Our hypotheses are satisfied: smooth, combinatorial, aperiodic. For each $\bar{\mathbf{r}}$, there is a dominant point in the positive orthant.

Example 12 (Alignments continued).

- For the diagonal direction we have $\mathbf{z}_*(\bar{\mathbf{1}}) = (2^{1/d} - 1)\mathbf{1}$ (by symmetry), so the number of "square" alignments satisfies

$$a(n, n \ldots, n) \sim (2^{1/d} - 1)^{-dn} \frac{1}{(2^{1/d} - 1)2^{(d^2-1)/2d}\sqrt{d(\pi n)^{d-1}}}$$

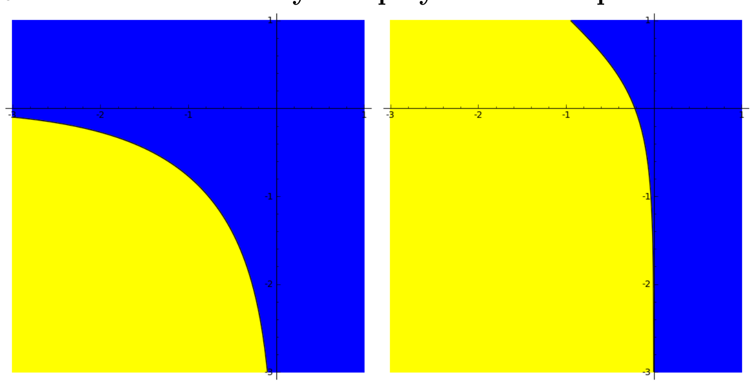
- Confirms a result of Griggs, Hanlon, Odlyzko & Waterman, *Graphs and Combinatorics* 1990, with less work, and extends to generalized alignments.

6 Geometric interpretation

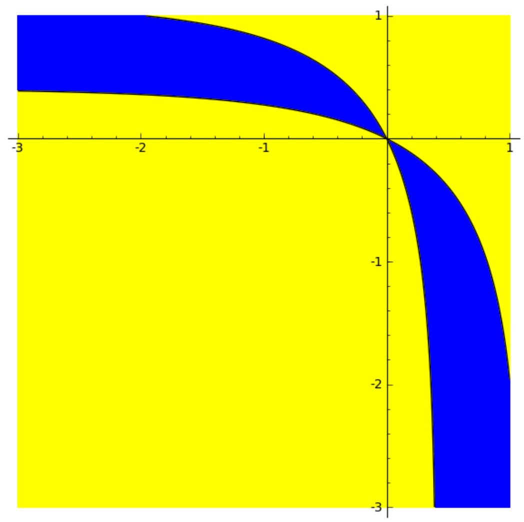
Logarithmic domain

- Recall U is the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the *logarithmic domain of convergence*.
- This is convex with boundary $\log \mathcal{V} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in \mathcal{V}\}$.
- Each point $\mathbf{x}_*$ of $\log \mathcal{V}$ yields a minimal point $\mathbf{z}_* := \exp(\mathbf{x}_*)$ of $\mathcal{V}$, lying in the positive orthant.
- The cone spanned by normals to supporting hyperplanes at $\mathbf{x}^* \in \log \mathcal{V}$ we denote by $\mathrm{K}(\mathbf{z}_*)$.
- If $\mathbf{z}_*$ is smooth, this is a single ray determined by the image of $\mathbf{z}_*$ under the *logarithmic Gauss map* $\nabla_{\log} H$.

$\log \mathrm{U}$ for smooth Delannoy and polyomino examples



$\log \mathrm{U}$ for nonsmooth example



Interpretation of smooth asymptotic formula

- The stationary point of the F-L integral for direction $\mathbf{r}$ corresponds to a *critical point* of $\mathcal{V}$ that lies on $\partial\,\mathrm{U}$.
- The dominant point $\mathbf{z}$ in the first orthant is $\exp(\mathbf{x})$, where the outward normal to $\log \mathrm{U}$ at $\mathbf{x}$ is parallel to $\mathbf{r}$.
- If $\mathcal{V}$ is smooth everywhere, then asymptotics in all directions are supplied by such points.
- The quantity Q is essentially the *Gaussian curvature* of $\log \mathcal{V}$.

Alternative smooth point formula

- $$a_{\mathbf{r}} \sim {\mathbf{z}_*}^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2}\kappa(\mathbf{z}_*)}}\,\frac{G(\mathbf{z}_*)}{|\,\nabla_{\log} H(\mathbf{z}_*)|}$$

  where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.

Nonsmooth points

- Arbitrarily complicated singularities are possible. We should be satisfied with a general procedure rather than a formula. Today we discuss *multiple points*.
- The point $\mathbf{z} \in \mathcal{V}$ is a multiple point if every small neighbourhood of $\mathbf{z}$ in $\mathcal{V}$ is the union of finitely many smooth hypersurfaces.

- We have good results when the intersection of these sheets is *transverse*.
- For multiple points that are not transverse, we also have results.
- We also have some results for cone points (Chapter 11, very difficult, not presented this week).

7 Multiple points

A generalization of the smooth argument works

- We can follow the same reduction steps as in the smooth case. Step 1 (localization) is the same.
- Step 2 (residue): there are n poles in the annulus, and we need to express the residue sum somehow (the individual residues are not integrable). A trick allows us to do this via an integral over a simplex.
- Step 3 (Fourier-Laplace integral): the resulting integral is more complicated, with a nastier domain and more complicated phase function.
- However in the generic (transverse) case we automatically obtain a nondegenerate stationary point in dimension $n+d-2$, and can use a modification of the Laplace approximation (which deals with boundary terms).

Generic double point in dimension 2

- Suppose that $F = G/H$ has a strictly minimal pole at $\mathbf{p} = (z_*, w_*)$, which is a double point of $\mathcal{V}$ such that $G(\mathbf{p}) \neq 0$. Then as $s \to \infty$ for r/s in $\mathrm{K}(\mathbf{p})$,

$$a_{rs} \sim (z_*)^{-r}(w_*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{(z_* w_*)^2\,\mathrm{Q}(\mathbf{p})}} + O(e^{-c(r+s)})\right]$$

  where Q is the Hessian of H.
- Note that
  - the expansion holds uniformly over compact subcones of K;
  - the hypothesis $G(\mathbf{p}) \neq 0$ is necessary; when $d > 1$, can have $G(\mathbf{p}) = H(\mathbf{p}) = 0$ even if G, H are relatively prime.

Example 13 (Queueing network).

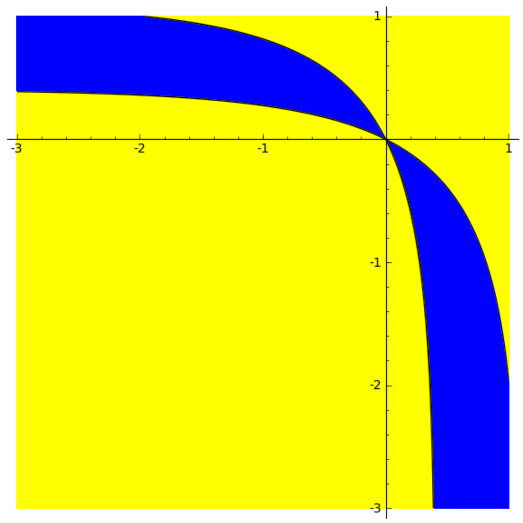
- Consider

$$F(x, y) = \frac{\exp(x + y)}{(1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{2y}{3} - \frac{x}{3})}$$

  which is the "grand partition function" for a very simple queueing network.

- Most of the points of $\mathcal{V}$ are smooth, and we can apply the smooth point results to derive asymptotics in directions outside the cone $1/2 \leq r/s \leq 2$.
- The point $(1,1)$ is a double point satisfying the above. In the cone $1/2 < r/s < 2$, we have $a_{rs} \sim 3e^2$.
- Note we say nothing here about the boundary of the cone.

$\log \mathrm{U}$ **for queueing example**



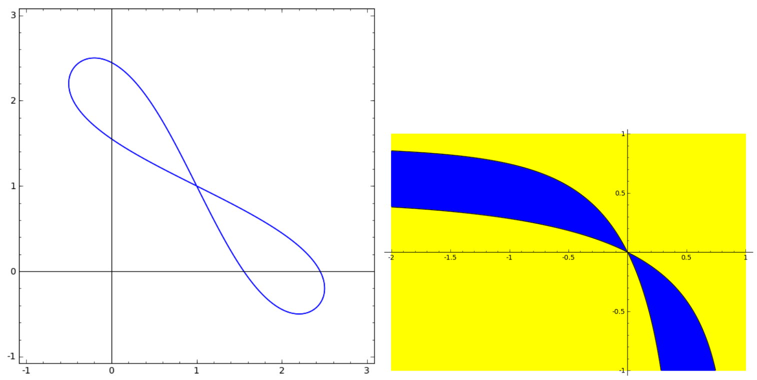
Example 14 (lemniscate).

- Consider $F = 1/H$ where

$$H(x,y) = x^2y^2 - 2xy(x+y) + 5\,(x^2+y^2) + 14\,xy - 20(x+y) + 19.$$

  This is combinatorial, and H is an irreducible polynomial.
- All points except $(1,1)$ are smooth, and $(1,1)$ is a transverse double point. Showing it is strictly minimal takes a little work.
- In the cone $1/2 < r/s < 2$ we have $a_{rs} \sim 6$, outside we use the smooth point formula.
- Note that H factors locally at $(1,1)$ but not globally.

$\mathcal{V}$ **and** $\log \mathrm{U}$ **for lemniscate**



Multiple points: generic shape of formula($\mathbf{z}_*$)

- (smooth point, or multiple point with $n \le d$)

$$\mathbf{z}_*{}^{-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-n)/2-k}.$$

  - (smooth/multiple point $n < d$)

$$a_0 = G(\mathbf{z}_*) C(\mathbf{z}_*)$$

    where C depends on the derivatives to order 2 of H;
  - (multiple point, $n = d$)

$$a_0 = G(\mathbf{z}_*)(\det J)^{-1}$$

    where J is the Jacobian matrix $(\partial H_i / \partial z_j)$, other a_k are zero;

- (multiple point, $n \ge d$)

$$\mathbf{z}_*{}^{-\mathbf{r}} G(\mathbf{z}_*) P\left(\frac{r_1}{z_1^*}, \dots, \frac{r_d}{z_d^*}\right),$$

  P a piecewise polynomial of degree $n - d$.

Aside: residue forms

- Instead of computing a residue and then integrating it directly, we can often repeat this process.
- The best way to understand this is via differential forms, in a coordinate-free way.
- This reduces the computation from d dimensions to $d - n$ where n is the number of sheets.

- When $n = d$, this is the only way we know to get the exponential decay beyond the leading term.
- When $n > d$, we first preprocess (see Lecture 4) to reduce to the case $n \leq d$.

Example 15 (2 planes in 3-space).

- The GF is

$$F(x, y, z) = \frac{1}{(4 - 2x - y - z)(4 - x - 2y - z)}.$$

- The critical points for some directions lie on one of the two sheets where a single factor vanishes, and smooth point analysis works. These occur when $\min\{r, s\} < (r + s)/3$.
- The curve of intersection of the two sheets supplies the other directions. Each point on the line $\{(1, 1, 1) + \lambda(-1, -1, -3) \mid -1/3 < \lambda < 1\}$ gives asymptotics in a 2-D cone.
- For example, $a_{3t,3t,2t} \sim (48\pi t)^{-1/2}$ with relative error less than 0.3% when $n = 30$.

Exercises

Exercise: binomial coefficient power sums

- The dth *Franel number* is $f_n^{(d)} := \sum_k \binom{n}{k}^d$.
- For odd $d \geq 3$, the GF is not algebraic (and probably for even d?)
- The supercritical Riordan case holds as above.
- Derive the formula mentioned in Lecture 2 for the GF of $f_n^{(d)}$, for arbitrary d.
- Compare with the exact result when $d = 6, n = 10$.

Exercise: double point asymptotics

- For the queueing example, compute the asymptotics in the cone $1/2 < r/s < 2$ by an iterated residue computation, rather than using the formula given above.
- Compute asymptotics for the queueing example in the cone $1/2 < r/s < 2$ by reducing to Fourier-Laplace integral as mentioned above.
- Which method do you prefer?
- Which method can say something about asymptotics on the boundary of the cone?

Exercise: biased coin flips

- A coin has probability of heads p, which can be changed. The coin will be biased so that $p = 2/3$ for the first n flips, and $p = 1/3$ thereafter. A player desires to get r heads and s tails and is allowed to choose n. On average, how many choices of $n \leq r + s$ will be winning choices?

- The answer is given by the convolution

$$a_{rs} = \sum_{a+b=n} \binom{n}{a} (2/3)^a (1/3)^b \binom{r+s-n}{r-a} (1/3)^{r-a} (2/3)^{s-b}$$

- Derive asymptotics for a_{rs} when $1/2 < r/s < 2$.

Lecture IV

Computational aspects

Lecture 4: Overview

- All our asymptotics are ultimately computed via Fourier-Laplace integrals. All standard references make simplifying assumptions that do not always hold in GF applications. In some cases, we needed to extend what is known.

- Once the asymptotics have been derived, in order to apply them in terms of original data we require substantial algebraic computation. We have implemented some of this in Sage. Higher order terms in the expansions are particularly tricky.

- The algebraic computations are usually best carried out using defining ideals, rather than explicit formulae.

8 Asymptotics of Fourier-Laplace integrals

Low-dimensional examples of F-L integrals

- Typical smooth point example looks like

$$\int_{-1}^{1} e^{-\lambda(1+i)x^2} \, dx.$$

  Isolated nondegenerate critical point, exponential decay

- Simplest double point example looks roughly like

$$\int_{-1}^{1} \int_{0}^{1} e^{-\lambda(x^2+2ixy)}\, dy\, dx.$$

Note $\mathrm{Re}\, f = 0$ on $x = 0$, so rely on oscillation for smallness.

- Multiple point with $n = 2, d = 1$ gives integral like

$$\int_{-1}^{1} \int_{0}^{1} \int_{-x}^{x} e^{-\lambda(z^2+2izy)}\, dy\, dx\, dz.$$

Simplex corners now intrude, continuum of critical points.

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;
  - purely imaginary phase;
  - isolated stationary point of phase, usually quadratically nondegenerate.
- Many of our applications to generating function asymptotics do not fit into this framework. In some cases, we needed to extend what is known.

Example 16.
- Consider

$$I(\lambda) = \int_{-\varepsilon}^{\varepsilon} \int_{0}^{1} e^{-\lambda\phi(p,t)}\, dp\, dt$$

where $\phi(p,t) = i\lambda t + \log\left[(1-p)v_1(e^{it}) + pv_2(e^{it})\right]$.

- This arises in the simplest strictly minimal double point situation. Recall the v_i are the inverse poles near the double point.
- The answer is

$$I(\lambda) \sim \frac{2\pi}{|v_1'(1) - v_2'(1)|\lambda}$$

- This doesn't satisfy the hypotheses of the last slide, and so we needed to derive the analogue of the Laplace approximation.

9 Higher order terms

Higher order terms

- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.
- Applications of higher order terms:
  - When leading term cancels in deriving other formulae.
  - When leading term is zero because of numerator.
  - Better numerical approximations for smaller indices.

Hörmander's explicit formula

For an isolated nondegenerate stationary point in dimension d,

$$I(\lambda) \sim \left(\det\left(\frac{\lambda f''(\mathbf{0})}{2\pi}\right)\right)^{-1/2} \sum_{k\geq 0} \lambda^{-k} L_k(A, f)$$

where

$$\begin{aligned}
\underline{f}(t) &= f(t) - (1/2)tf''(0)t^T \\
\mathcal{D} &= \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b}(-\mathrm{i}\partial_a)(-\mathrm{i}\partial_b) \\
\tilde{L}_k(A, f) &= \sum_{l\leq 2k} \frac{\mathcal{D}^{l+k}(A\underline{f}^l)(0)}{(-1)^k 2^{l+k} l!(l+k)!}.
\end{aligned}$$

$\tilde{L}_k$ is a differential operator of order $2k$ acting on A at 0 (considering the order $3m$ zero of $\underline{f}^m$), whose coefficients are rational functions of $f''(0), \ldots, f^{(2k+2)}(0)$.

Example 17 (nonoverlapping patterns).
- Given a word over alphabet $\{a_1, \ldots, a_d\}$, players alternate reading letters. If the last two letters are the same, we erase the letters seen so far, and continue.
- For example, in *abaabbba*, there are two occurrences.
- How many such *snaps* are there, for random words?
- Answer: let ψ_n be the random variable counting snaps in words of length n. Then as $n \to \infty$,

$$\begin{aligned}
\mathbb{E}(\psi_n) &= (3/4)n - 15/32 + O(n^{-1}) \\
\sigma^2(\psi_n) &= (9/32)n + O(1).
\end{aligned}$$

Example 18 (snaps continued). • The details are as follows. Consider W given by

$$W(x_1, \dots, x_d, y) = \frac{A(x)}{1 - yB(x)}$$
$$A(x) = 1/[1 - \sum_{j=1}^{d} x_j/(x_j+1)]$$
$$B(x) = 1 - (1 - e_1(x))A(x)$$
$$e_1(x) = \sum_{i=j}^{d} x_j.$$

- The symbolic method shows that $[x_1^n \dots x_d^n, y^s]W(\mathbf{x}, y)$ counts words with n occurrences of each letter and s snaps.

Example 19 (snaps continued). We extract as usual. Note the first order cancellation in the variance computation. For $d = 3$,

$$\begin{aligned}
\mathbb{E}(\psi_n) &= \frac{[x^{n\mathbf{1}}]\frac{\partial W}{\partial y}(x,1)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (3/4)n - 15/32 + O(n^{-1}) \\
\mathbb{E}(\psi_n^2) &= \frac{[x^{n\mathbf{1}}]\left(\frac{\partial^2 W}{\partial y^2}(x,1) + \frac{\partial W}{\partial y}(x,1)\right)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (9/16)n^2 - (27/64)n + O(1) \\
\sigma^2(\psi_n) &= \mathbb{E}(\psi_n^2) - \mathbb{E}(\psi_n)^2 = (9/32)n + O(1).
\end{aligned}$$

Example 20 (vanishing numerator). • Let

$$F(x, y) = \sum_{rs} a_{rs} x^r y^s = \frac{y(1-2y)}{1-x-y}.$$

- Here
$$a_{rs} = 2\binom{r+s-2}{r-1} - \binom{r+s-1}{r}.$$
- When $r = s$, this simplifies to $\frac{1}{r}\binom{2r-2}{r-1}$, a shifted *Catalan number*. The dominant point is $(1/2, 1/2)$ by symmetry.
- We know the asymptotics of these are of order $n^{-3/2}$. This is consistent, because the numerator of F vanishes at $(1/2, 1/2)$.
- Our general formula yields
$$a_{nn} \sim 4^n \left(\frac{1}{4\sqrt{\pi}} n^{-3/2} + \frac{3}{32\sqrt{\pi}} n^{-5/2}\right).$$

Computing numerical approximations

- Alex Raichev's Sage implementation computes higher order expansions for smooth and multiple points.
- The error from truncating at the kth term is of order $1/n^{1+k}$.
- The current implementation is not very sophisticated, and when $k \geq 3$ and $d \geq 4$, for example, usually fails to halt in reasonable time.
- To compute the kth term naively using Hörmander requires at least d^{3k} $d \times d$ matrix computations.
- There is surely a lot of room for improvement here.

Example 21 (Snaps with $d = 3$).

n	1	2	4	8
$\mathbb{E}(\psi)$	0	1.000	2.509	5.521
$(3/4)n$	0.7500	1.500	3	6
$(3/4)n - 15/32$	0.2813	1.031	2.531	5.531
one-term relative error	undefined	0.5000	0.1957	0.08685
two-term relative error	undefined	0.03125	0.008832	0.001936
$\mathbb{E}(\psi^2)$	0	1.8000	7.496	32.80
$(9/16)n^2$	0.5625	2.250	9	36
$(9/16)n^2 - (27/64)n$	0.1406	1.406	7.312	32.63
one-term relative error	undefined	0.2500	0.2006	0.09768
two-term relative error	undefined	0.2188	0.02449	0.005220
$\sigma^2(\psi)$	0	0.8000	1.201	2.320
$(9/32)n$	0.2813	0.5625	1.125	2.250
relative error	undefined	0.2969	0.06294	0.03001

Example 22 (2 planes in 3-space). Using the formula we obtain

$$a_{3t,3t,2t} = \frac{1}{\sqrt{3\pi}} \left(\frac{1}{4} t^{-1/2} - \frac{25}{1152} t^{-3/2} + \frac{1633}{663552} t^{-5/2} \right) + O(t^{-7/2}).$$

The relative errors are:

rel. err. vs t	1	2	4	8	16	32
$k = 1$	-0.660	-0.315	-0.114	-0.0270	-0.00612	-0.00271
$k = 2$	-0.516	-0.258	-0.0899	-0.0158	-0.000664	0.00000780
$k = 3$	-0.532	-0.261	-0.0906	-0.0160	-0.000703	-0.00000184

10 Computations in rings

Computations in polynomial rings

- In order to apply our formulae, we need to, at least:
  - find the critical point $\mathbf{z}_*(\mathbf{r})$;
  - compute a rational function of derivatives of H, evaluated at $\mathbf{z}_*$.
- The first can be solved by, for example, Gröbner basis methods.

- The second can cause big problems if done naively, leading to a symbolic mess, and loss of numerical precision. It is best to deal with annihilating ideals.

Example 23 (Why ideals are better).

- Suppose x is the positive root of $p(x) := x^3 - x^2 + 11x - 2$, and we want to compute $g(x) := x^5/(867x^4 - 1)$.
- If we compute x symbolically and then substitute into g, we obtain a huge mess involving radicals, which evaluates numerically to 0.193543073868354.
- If we compute x numerically and then substitute, we obtain 0.193543073867096.
- Instead we can compute the minimal polynomial of $y := g(x)$ by Gröbner methods. This gives

$$11454803y^3 - 2227774y^2 + 2251y - 32 = 0$$

and evaluating numerically yields 0.193543073868734.

Example 24 (Polyomino computation).

- Recall the GF for horizontally convex polyominoes is

$$F(x,y) = \frac{xy(1-x)^3}{(1-x^4) - xy(1-x-x^2+x^3+x^2y)}.$$

- Solving $\{H = 0, \nabla H = 0\}$ yields only the point $(1,0)$. Thus dominant points in direction $\lambda := s/r, 0 < \lambda < 1$, are all smooth.
- The ideal in $\mathbb{C}[x,y]$ defined by $\{sxH_x - ryH_y, H\}$ has a Gröbner basis giving a quartic minimal polynomial for $x_*(\lambda)$, and $y_*(\lambda)$ is a linear function of $x_*(\lambda)$ (also satisfies a quartic).
- Specifically, the elimination polynomial for x is

$$(1+\lambda)x^4 + 4(1+\lambda)^2x^3 + 10(\lambda^2+\lambda-1)x^2 + 4(2\lambda-1)^2x + (1-\lambda)(1-2\lambda).$$

Example 25 (Polyomino computation continued).

- The leading coefficient in the asymptotic expansion has the form $(2\pi)^{-1/2}C$ where C is algebraic.
- For *generic* λ, the minimal polynomial of C has degree 8.
- However, for example when $r = 2s$ there is major simplification: the minimal polynomials for x and y respectively are $3x^2+18x-5$ and $75y^2 - 288y + 256$, etc.
- Now given (r,s), solving numerically for C as a root gives a more accurate answer than if we had solved for x_*, y_* above and substituted.

10.1 Local factorizations

Computations in local rings

- In order to apply our smooth/multiple point formulae, we need to, at least:
  - classify the local geometry at point $\mathbf{z}_*$;
  - compute (derivatives of) the factors H_i near $\mathbf{z}_*$.
- Unfortunately, computations in the local ring are not effective (as far as we know). If a polynomial factors as an analytic function, but the factors are not polynomial, we can't deal with it algorithmically (yet).
- Smooth points are easily detected. There are some sufficient conditions, and some necessary conditions, for $\mathbf{z}_*$ to be a multiple point. But in general we don't know how to classify singularities algorithmically.

Example 26 (local factorization of lemniscate). • Let $H(x, y) = 19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$, and analyse $1/H$.

- Here $\mathcal{V}$ is smooth at every point except $(1, 1)$, which we see by solving the system $\{H = 0, \nabla H = 0\}$.
- At $(1, 1)$, changing variables to $h(u, v) := H(1 + u, 1 + v)$, we see that $h(u, v) = 4u^2 + 10uv + 4v^2 + C(u, v)$ where C has no terms of degree less than 3.
- The quadratic part factors into distinct factors, showing that $(1, 1)$ is a transverse multiple point.
- Note that our double point formula does not require details of the individual factors. However this is not the case for general multiple points.

Reduction of multiple points

- If we have $n > d$ transverse smooth factors meeting at a point $\mathbf{p}$, we can reduce to the case $n \leq d$ at the cost of increasing the number of summands.
- If we have repeated factors, we can reduce to the case of distinct factors using exactly the same idea.
- If this is not done, we arrive at Fourier-Laplace integrals with non-isolated stationary points, which are hard to analyse.
- However after doing the above we always reduce to the case of an isolated point, which we can handle.

Example 27 (Algebraic reduction, sketch). • Let $H = H_1H_2H_3 := (1 - x)(1 - y)(1 - xy)$.

- In the local ring at $(1,1)$, each factor should be in the ideal generated by the other two (Nullstellensatz).
- In fact it is true globally, since $H_3 = H_1 + H_2 - H_1 H_2$. (Nullstellensatz certificate).
- Thus eventually we obtain

$$F = \frac{1}{H_1 H_2 H_3} = \cdots = \frac{2-y}{(1-y)(1-xy)^2)} + \frac{1}{(1-x)(1-xy)^2}.$$

- The next step, reducing the multiplicity of factors can be done at the residue stage (residue for higher order pole) or by other methods, and is both easy and algorithmic.
- Thus we can reduce to a (possibly large) sum of (polynomial multiples of) transverse double point asymptotic series.

Exercises

Exercises

A computer algebra system will help for some of these.

- Use Hörmander's formula to compute L_0, L_1, L_2 for $F(x,y) = (1-x-y)^{-1}$, at the minimal point $(1/2, 1/2)$. This gives asymptotics for the main diagonal coefficients $\binom{2n}{n}$.
- The small change from $y(1-2y)/(1-x-y)$ to $(1-2y)/(1-x-y)$ should make no difference to our basic computational procedure. Show that, nevertheless, the results are very different. Explain.
- Compute the expectation and variance of the number of snaps in a standard deck of cards (no asymptotics required).
- Carry out the polyomino computation in detail.

Lecture V

Extensions

Overview

- We first look in turn at some of our standard assumptions in force over the last few lectures, and discuss what happens when each is weakened.
- Removing the combinatorial assumption leads to topological issues which we address in the framework of stratified Morse theory.

- The Fourier-Laplace integrals arising from the reductions can be more complicated that those previously studied.
- We then look at going beyond the class of rational (meromorphic) singularities.

11 Easy generalizations

Assumption: unique smooth dominant simple pole

- If there is periodicity, we typically obtain a finite number of contributing points whose contributions must be summed. This leads to the appropriate cancellation. A routine modification.
- A *toral point* is one for which every point on its torus is a minimal singularity (such as $1/(1 - x^2 y^3)$. These occur in quantum random walks. A routine modification.
- If the dominant point is smooth but H is not locally squarefree, then we obtain polynomial corrections that are easily computed. A routine modification.

Example 28 (Periodicity).

- Let $F(z, w) = 1/(1 - 2zw + w^2)$ be the generating function for Chebyshev polynomials of the second kind.
- For directions (r, s) with $0 < s/r < 1$, there is a dominant point at

$$\mathbf{p} = \left(\frac{r}{\sqrt{r^2 - s^2}}, \sqrt{\frac{r - s}{r + s}} \right)$$

.

- There is also a dominant point at $-\mathbf{p}$. Adding the contributions yields

$$a_{rs} \sim \sqrt{\frac{2}{\pi}} (-1)^{(s-r)/2} \left(\frac{2r}{\sqrt{s^2 - r^2}} \right)^{-r} \left(\sqrt{\frac{s - r}{s + r}} \right)^{-s} \sqrt{\frac{s + r}{r(s - r)}}$$

  when $r + s$ is even and zero otherwise.

Assumption: transversality

- If sheets at a multiple point are not transversal, the phase of the Fourier-Laplace integral vanishes on a set of positive dimension.
- If this occurs because there are too many sheets, the reduction from Lecture 4 works.
- If it occurs because the dimension of the space spanned by normals is just too small, then it is a little harder to deal with.

- Each term in our expansions depends on finitely many derivatives of G and H, so if sheets have contact to sufficiently high order, the results are the same as if they coincided. Thus if we can reduce in the local ring, all is well. Otherwise we may need to attack the F-L integral directly.

Example 29 (tangential curves).

- Suppose that $\mathcal{V}$ looks like two curves intersecting at a strictly minimal point $(1,1)$, with branches $y = g_j(x)$.
- Suppose further that the first derivatives are equal and $f_j''(\theta) = -d_j\theta^2 + \dots$.
- Then the cone K of directions is a single ray and

$$a_{rs} \sim \frac{2G(1,1)\sqrt{s}}{\sqrt{2\pi}\left(\sqrt{d_1} + \sqrt{d_2}\right)}.$$

- When $d_0 = d_1$ this gives the same result as a single repeated smooth factor.

Assumption: no change in local geometry

- If the phase of the Fourier-Laplace integral vanishes to order more than 2, more complicated behaviour ensues.
- If the order of vanishing is 2 everywhere except for 3 at a certain direction, for example, we obtain a *phase transition* and *Airy phenomena*.

Example 30 (Airy phenomena).

- The *core* of a rooted planar map is the largest 2-connected subgraph containing the root edge.
- The probability distribution of the size k of the core in a random planar map with size n is described by

$$p(n,k) = \frac{k}{n}[x^k y^n z^n]\frac{xz\psi'(z)}{(1 - x\psi(z))(1 - y\phi(z))}.$$

where $\psi(z) = (z/3)(1 - z/3)^2$ and $\phi(z) = 3(1 + z)^2$.

- In directions away from $n = 3k$, our ordinary smooth point analysis holds. When $n = 3k$ we can redo the F-L integral easily and obtain asymptotics of order $n^{-1/3}$.
- Determining the behaviour as we approach this diagonal at a moderate rate is harder (Manuel Lladser PhD thesis), and recovers the results of Banderier-Flajolet-Schaeffer-Soria 2001.

12 Removing the combinatorial assumption

Non-combinatorial case: Overview

- Some applications require us to consider more general GFs, with coefficients that may not be nonnegative. Finding dominant points is now much harder.
- Going back to Cauchy's integral, we use homology rather than homotopy to compute its asymptotics. Using the *method of steepest descent* as formalized by *Morse theory*, we can do this almost algorithmically in the smooth case. The integral is determined by *critical points* which are the same as the critical points we saw previously.
- When $d = 2$, this has been implemented algorithmically, but not for higher d.
- There is a lesser known version of Morse theory due to Whitney, called *stratified Morse theory*, which deals with singularities. There is substantial discussion of this in the book.

Cauchy integral formula is homological

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to replace T by a contour that is more suitable for explicit computation. This may involve additional residue terms.
- The homology of $\mathbb{C}^d \setminus \mathcal{V}$ is the key to decomposing the integral.
- It is natural to try a saddle point/steepest descent approach.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \mathrm{Log}(\mathbf{z})$ as a height function; try to choose contour to minimize $\max h$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \leq d - 1$. The top dimensional stratum is the set of smooth points.
- The critical points are those where the restriction of h to a stratum has derivative zero. Generically, there are finite many.

- The Cauchy integral decomposes into a sum

$$\sum n_i \int_{C_i} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \mathbf{F}(\mathbf{z})\, \mathbf{dz}$$

  where C_i is a *quasi-local cycle* for $\mathbf{z}_*{}^{(i)} \in \mathrm{crit}(\mathbf{r})$.

- Key problem: find the highest critical points with nonzero n_i. These are the dominant ones.

Bicolored supertrees

Example 31.

- Consider

$$F(x, y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$

  for which we want asymptotics on the main diagonal.

- The critical points are, listed in increasing height, $(1+\sqrt{5}, (3-\sqrt{5})/16), (2, \frac{1}{8}), (1-\sqrt{5}, (3+\sqrt{5})/16)$.

- In fact $(2, 1/8)$ dominates. The analysis is a substantial part of the PhD thesis of Tim DeVries (U. Pennsylvania).

- The answer:

$$a_{nn} \sim \frac{4^n\sqrt{2}\Gamma(5/4)}{4\pi} n^{-5/4}.$$

13 Algebraic singularities

Inverting diagonalization

- Recall the diagonal method shows that the diagonal of a rational bivariate GF is algebraic.

- Conversely, every univariate algebraic GF is the diagonal of some rational bivariate GF.

- The latter result does not generalize strictly to higher dimensions, but something close to it is true.

- Our multivariate framework means that increasing dimension causes no difficulties in principle, so we can reduce to the rational case.

- The mathematical idea behind this is *resolution of singularities*.

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
where $u(0, \underline{0}) \neq 0$ and $1 \leq k \in \mathbb{N}$.

- Define
$$R(z_0, \mathbf{z}) = \frac{z_0^2 P_1(z_0, z_0 z_1, z_2, \dots)}{kP(z_0, z_0 z_1, z_2, \dots)}$$
$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \dots z_d).$$

- By the Argument Principle
$$\frac{1}{2\pi i} \int_C \tilde{R}(w, \mathbf{z}) \frac{dw}{w} = \sum \operatorname{Res} \tilde{R}(w, \mathbf{z}) = F(\mathbf{z}).$$

- Higher order terms are essential: the numerator of $\tilde{R}$ always vanishes at the dominant point. The Catalan example from Lecture 4 was created using this method.

Safonov's general construction

- In general, apply a sequence of *blowups* (monomial substitutions) to reduce to the case above.

- This is a standard idea from algebraic geometry: resolution of singularities.

- Definition: Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have $d+1$ variables and let M be a $d \times d$ matrix with nonnegative entries. The *M-diagonal* of F is the formal power series in d variables whose coefficients are given by $b_{r_2, \dots r_d} = a_{s_1, s_1, s_2, \dots s_d}$ and $(s_1, \dots, s_d) = (r_1, \dots, r_d)M$.

- Theorem: Let f be an algebraic function of d variables. Then there is a unimodular integer matrix M with positive entries and a rational function F in $d+1$ variables such that f is the M-diagonal of F.

- The example $x\sqrt{1-x-y}$ shows that the main diagonal cannot always be used.

Example 32 (Narayana numbers).

- The bivariate GF $F(x, y)$ for the *Narayana numbers*
$$a_{rs} = \frac{1}{r} \binom{r}{s} \binom{r-1}{s-1}$$
satisfies $P(F(x, y), x, y) = 0$, where
$$\begin{aligned} P(w, x, y) &= w^2 - w\,[1 + x(y-1)] + xy \\ &= [w - F(x, y)] \left[w - \overline{F}(x, y)\right]. \end{aligned}$$
where $\overline{F}$ is the algebraic conjugate.

- Using the above construction we obtain the lifting
$$G(u,x,y) = \frac{u(1-2u-ux(1-y))}{1-u-xy-ux(1-y)}.$$
with $b_{rrs} = a_{rs}$.

Example 33 (Narayana numbers continued).

- The above lifting yields asymptotics by smooth point analysis in the usual way. The critical point equations yield
$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$
and we obtain asymptotics starting with s^{-2}. For example
$$a_{2s,s} \sim \frac{16^s}{8\pi s^2}.$$
- Interestingly, specializing $y = 1$ commutes with lifting (and yields the shifted Catalan numbers as in Lecture 4). Is this always true?

Technical issues

- Safonov's lifting often takes us away from the combinatorial case. Therefore the Morse theory approach will probably be needed.
- Dominant singularities can be at infinity.
- There are other lifting procedures, some of which go from dimension d to $2d$. They seem complicated, and we have not yet tried them in detail.
- However in some cases they work better - for example $2xy/(2+x+y)$ is a lifting of $x\sqrt{1-x}$, whereas Safonov's method appears not to work easily.

13.1 Further work

Possible research projects

- Systematically compare the diagonal method and our methods.
- Systematically generate sums of squares identities and include them in OEIS.
- Develop a good theory for algebraic singularities (using resolution of singularities somehow).
- Improve efficiency of algorithms for computing higher order terms in expansions. Implement them in Sage.
- Develop better computational methods for computing symbolically with symmetric functions.
- Make the computation of dominant points algorithmic in the noncombinatorial case.

Exercises

Exercises

- Prove that the numerator of Safonov's lifting must vanish at the dominant point, as claimed above.
- Show that $x\sqrt{1-x-y}$ cannot occur as the leading diagonal of a rational function in 3 variables, as claimed above.
- Derive asymptotics for the following GF (Vince and Bóna 2012)

$$F(x,y) = 1 - \sqrt{(1-x)^2 + (1-y)^2 - 1}$$

- In the Cauchy integral for $\sqrt{1-x}$, make a substitution to convert to an integral of a rational function. How general is this procedure?