Analytic Combinatorics in Several Variables

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Main references

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/

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- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.
- Sage implementations by Alex Raichev: `https://github.com/araichev/amgf`.

Lecture plan

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Example (Some test problems)

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- (lemniscate — a second order linear recurrence)

$$\left(x^2y^2 - 2xy(x+y) + 5\left(x^2+y^2\right) + 14\,xy - 20(x+y) + 19\right)^{-1}.$$

(no asymptotics today — see Chapter 10)

Overview

- In one variable, starting with a sequence $a_{\mathbf{r}}$ of interest, we form its generating function $F(\mathbf{z})$. Cauchy's integral theorem allows us to express $a_{\mathbf{r}}$ as an integral. The exponential growth rate of $a_{\mathbf{r}}$ is determined by the location of a dominant singularity $\mathbf{z}_*$ of F. More precise estimates depend on the local geometry of the singular set $\mathcal{V}$ of F near $\mathbf{z}_*$.

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- In the multivariate case, all the above is still true. However, we need to specify the direction in which we want asymptotics; we then need to worry about uniformity; the definition of "dominant" is a little different; the local geometry of $\mathcal{V}$ can be much nastier; the local analysis is more complicated.

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- Assume $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ where G, H are analytic (e.g. polynomials).
- The combinatorial case: all $a_{\mathbf{r}} \geq 0$. The aperiodic case: $a_{\mathbf{r}}$ is not supported on a proper sublattice of $\mathbb{N}^d$.

Example (Univariate pole: derangements)

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$. Using a circle of radius $1-\varepsilon$ yields, by Cauchy's theorem

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- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.

Example (Essential singularity: saddle point method)

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- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.
- The integral over C_n has most mass near $z = n$, so that

$$\begin{aligned} a_n &= \frac{F(n)}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) \frac{F(ne^{i\theta})}{F(n)} \, d\theta \\ &\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp\left(-in\theta + \log F(ne^{i\theta}) - \log F(n)\right) \, d\theta. \end{aligned}$$

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- This recaptures Stirling's approximation, since $n! = 1/a_n$:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

Multivariate asymptotics — some quotations

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- We aimed to improve the multivariate situation.

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F_{1,1}(x) &= [t^0] F(x/t, t) \\
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  where $s_k(x)$ is a singularity satisfying $\lim_{x \to 0} s_k(x) = 0$.
- If F is rational, then $F_{1,1}$ is algebraic.

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- Instead we use a direct approach based on Cauchy's Integral Formula in dimension d.

Cauchy integral formula

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
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- To derive asymptotics, it is natural to try a saddle point/steepest descent approach.

Topological overview - stratified Morse theory

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- Key problem: find the highest critical points with nonzero n_i. These are the dominant ones.

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- We can (with some effort) convert quantities in our formula back to the original data.

Difficulties with F-L asymptotics

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$$I(\lambda) = \int_D \exp(-\lambda f(\mathbf{x})) A(\mathbf{x}) \, d\mathbf{x}.$$

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- Many of our applications to generating function asymptotics do not fit into this framework. We needed to extend what is known (see Chapter 5).

Low-dimensional examples of F-L integrals

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- Multiple point with $n = 2, d = 1$ gives integral like
$$\int_{-1}^{1}\int_{0}^{1}\int_{-x}^{x} e^{-\lambda(z^2+2izy)}\, dy\, dx\, dz.$$
Simplex corners now intrude, continuum of critical points.

Logarithmic domain

- Let U be the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence. This is known to be convex.

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- In the combinatorial case, for each $\overline{\mathbf{r}}$ there is a dominant point $\mathbf{z}_*(\overline{\mathbf{r}}) := \exp(\mathbf{x}_*)$ where $\mathbf{x}_* \in \partial \log U$. In the aperiodic case, there are no more.

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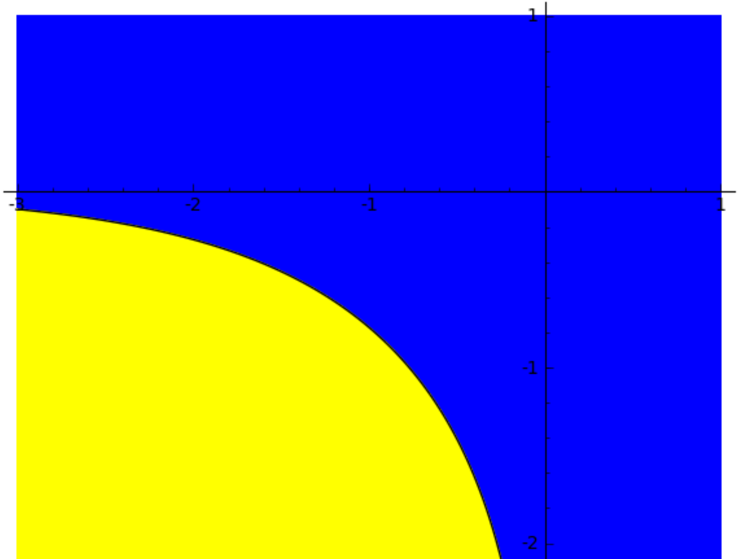
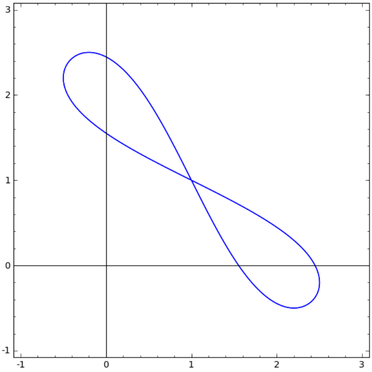
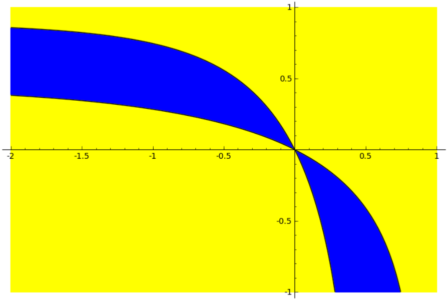
- Thus for each $\overline{\mathbf{r}}$ we can find $\mathbf{z}_*(\overline{\mathbf{r}})$, on the boundary of $\mathcal{V}$ and in the positive orthant of $\mathbb{R}^d$, that controls asymptotics in direction $\overline{\mathbf{r}}$.

$\log \mathrm{U}$ for Delannoy example



$\mathcal{V}$ and $\log \mathrm{U}$ for leminiscate example





Smooth formulae for general d

- $\mathbf{z}_*$ turns out to be a critical point for $\overline{\mathbf{r}}$ iff the outward normal to $\log \mathcal{V}$ is parallel to $\mathbf{r}$. In other words, for some $\lambda \in \mathbb{C}$, $\mathbf{z}_*$ solves

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- $$a_{\mathbf{r}} \sim \mathbf{z}_*(\overline{\mathbf{r}})^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2} \kappa(\mathbf{z}_*)}} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|}$$

  where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.

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  where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.
- The Gaussian curvature can be computed explicitly in terms of derivatives of H to second order.

Example (Alignments)

- Recall $F(\mathbf{z}) = \sum a(r_1, \ldots, r_d)\mathbf{z}^{\mathbf{r}} = \frac{1}{2-\prod_{i=1}^{d}(1+z_i)}$. Here $\mathcal{V}$ is globally smooth, and GF is combinatorial and aperiodic.

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- For example, for the main diagonal we have $\mathbf{z}_*(\bar{\mathbf{1}}) = (2^{1/d} - 1)\mathbf{1}$ (by symmetry), so the number of "square" alignments satisfies

$$a(n, n \ldots, n) \sim (2^{1/d} - 1)^{-dn} \frac{1}{(2^{1/d} - 1)2^{(d^2-1)/2d}\sqrt{d(\pi n)^{d-1}}}$$

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- Confirms a result of Griggs, Hanlon, Odlyzko & Waterman, *Graphs and Combinatorics* 1990, with less work, and extends to generalized alignments.

Important special case: Riordan arrays

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$$F(x,y) = \frac{\phi(x)}{1 - yv(x)}.$$

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- Examples: many plane lattice walk models (Pascal, Catalan, Motzkin, Schröder, etc); sums of IID random variables.
- In this case, if we define

$$\mu(x) := xv'(x)/v(x)$$
$$\sigma^2(x) := x^2 v''(x)/v(x) + \mu(x) - \mu(x)^2$$

the previous formula boils down (under minor extra assumptions) to

$$a_{rs} \sim (x_*)^{-r} v(x_*)^s \frac{\phi(x_*)}{\sqrt{2\pi s \sigma^2(x_*)}}$$

where x_* satisfies $\mu(x_*) = r/s$.

Example (Delannoy walks)

- Recall that $F(x, y) = (1 - x - y - xy)^{-1}$. This is Riordan with $\phi(x) = (1 - x)^{-1}$ and $v(x) = (1 + x)/(1 - x)$. Here $\mathcal{V}$ is globally smooth.

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- Using the formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{r}{\Delta - s}\right]^r \left[\frac{s}{\Delta - r}\right]^s \sqrt{\frac{rs}{2\pi\Delta(r + s - \Delta)^2}}.$$

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- Compare Panholzer-Prodinger, Bull. Aust. Math. Soc. 2012.

Non-combinatorial case: bicolored supertrees

Example (highest critical point doesn't contribute)

- Consider

$$F(x, y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$

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- The critical points are, listed in increasing height, $(1+\sqrt{5}, (3-\sqrt{5})/16), (2, \frac{1}{8}), (1-\sqrt{5}, (3+\sqrt{5})/16)$.

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- In fact $(2, 1/8)$ dominates.
- The answer:

$$a_{nn} \sim \frac{4^n\sqrt{2}\Gamma(5/4)}{4\pi} n^{-5/4}.$$

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- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.

Hörmander's explicit formula

For an isolated nondegenerate stationary point in dimension d,

$$I(\lambda) \sim \left(\det\left(\frac{\lambda f''(\mathbf{0})}{2\pi}\right)\right)^{-1/2} \sum_{k\geq 0} \lambda^{-k} L_k(A, f)$$

where L_k is a differential operator of order $2k$ evaluated at $\mathbf{0}$. Specifically,

$$\begin{aligned} \underline{f}(t) &= f(t) - (1/2) t f''(0) t^T \\ \mathcal{D} &= \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b} (-\mathrm{i}\partial_a)(-\mathrm{i}\partial_b) \\ L_k(A, f) &= \sum_{l \leq 2k} \frac{\mathcal{D}^{l+k}(A \underline{f}^l)(0)}{(-1)^k 2^{l+k} l!(l+k)!}. \end{aligned}$$

Example (nonoverlapping patterns)

- Given a word over alphabet $\{a_1, \dots, a_d\}$, players alternate reading letters. If the last two letters are the same, we erase the letters seen so far, and continue.

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- How many such snaps are there, for random words?
- Answer: let ψ_n be the random variable counting snaps in words of length n. Then as $n \to \infty$,

$$\mathbb{E}(\psi_n) = (3/4)n - 15/32 + O(n^{-1})$$
$$\sigma^2(\psi_n) = (9/32)n + O(1).$$

Example (snaps continued)

- The details are as follows. Consider W given by

$$W(x_1, \ldots, x_d, y) = \frac{A(x)}{1 - yB(x)}$$

$$A(x) = 1/[1 - \sum_{j=1}^{d} x_j/(x_j + 1)]$$

$$B(x) = 1 - (1 - e_1(x))A(x)$$

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- The symbolic method shows that $[x_1^n \ldots x_d^n, y^s]W(\mathbf{x}, y)$ counts words with n occurrences of each letter and s snaps.

Example (snaps continued)

We extract as usual. Note the first order cancellation in the variance computation. For $d = 3$,

$$
\begin{aligned}
\mathbb{E}(\psi_n) &= \frac{[x^{n\mathbf{1}}]\frac{\partial W}{\partial y}(x,1)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (3/4)n - 15/32 + O(n^{-1}) \\
\mathbb{E}(\psi_n^2) &= \frac{[x^{n\mathbf{1}}]\left(\frac{\partial^2 W}{\partial y^2}(x,1) + \frac{\partial W}{\partial y}(x,1)\right)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (9/16)n^2 - (27/64)n + O(1) \\
\sigma^2(\psi_n) &= \mathbb{E}(\psi_n^2) - \mathbb{E}(\psi_n)^2 = (9/32)n + O(1).
\end{aligned}
$$

Example (Snaps with $d = 3$)

n	1	2	4	8
$\mathbb{E}(\psi)$	0	1.000	2.509	5.521
$(3/4)n$	0.7500	1.500	3	6
$(3/4)n - 15/32$	0.2813	1.031	2.531	5.531
one-term relative error	undefined	0.5000	0.1957	0.08685
two-term relative error	undefined	0.03125	0.008832	0.001936
$\mathbb{E}(\psi^2)$	0	1.8000	7.496	32.80
$(9/16)n^2$	0.5625	2.250	9	36
$(9/16)n^2 - (27/64)n$	0.1406	1.406	7.312	32.63
one-term relative error	undefined	0.2500	0.2006	0.09768
two-term relative error	undefined	0.2188	0.02449	0.005220
$\sigma^2(\psi)$	0	0.8000	1.201	2.320
$(9/32)n$	0.2813	0.5625	1.125	2.250
relative error	undefined	0.2969	0.06294	0.03001

Inverting diagonalization

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- The latter result does not generalize strictly to higher dimensions, but something close to it is true. Our multivariate framework means that increasing dimension causes no difficulties in principle, so we can reduce to the rational case.
- The elementary diagonal of $F(z_0, \dots, z_d) = \sum_{r_0,\dots,r_d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ is

$$\operatorname{diag} F := f(z_1, \dots, z_d) = \sum_{r_1,\dots,r_d} a_{r_1,r_1,\dots,r_d} z_1^{r_1} \dots z_d^{r_d}.$$

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
where $u(0, \underline{0}) \neq 0$ and $1 \leq k \in \mathbb{N}$.

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- Define
$$R(z_0, \mathbf{z}) = \frac{z_0^2 P_1(z_0, z_0 z_1, z_2, \dots)}{kP(z_0, z_0 z_1, z_2, \dots)}$$
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$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \ldots z_d).$$
- The Argument Principle shows that $F = \operatorname{diag} R$:
$$\frac{1}{2\pi i} \int_C \tilde{R}(w, \mathbf{z}) \, \frac{dw}{w} = \sum \operatorname{Res} \tilde{R}(w, \mathbf{z}) = F(\mathbf{z}).$$

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$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \dots z_d).$$
- The Argument Principle shows that $F = \operatorname{diag} R$:
$$\frac{1}{2\pi i} \int_C \tilde{R}(w, \mathbf{z}) \frac{dw}{w} = \sum \operatorname{Res} \tilde{R}(w, \mathbf{z}) = F(\mathbf{z}).$$
- Higher order terms are essential: the numerator of $\tilde{R}$ always vanishes at the dominant point.

Safonov's general construction

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- Definition: Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have $d+1$ variables and let M be a $d \times d$ matrix with nonnegative entries. The M-diagonal of F is the formal power series in d variables whose coefficients are given by $b_{r_2,\dots r_d} = a_{s_1,s_1,s_2,\dots s_d}$ and $(s_1,\dots,s_d) = (r_1,\dots,r_d)M$.

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- Theorem: Let f be an algebraic function of d variables. Then there is a unimodular integer matrix M with positive entries and a rational function F in $d+1$ variables such that f is the M-diagonal of F.
- The example $x\sqrt{1-x-y}$ shows that the elementary diagonal cannot always be used.

Example (Narayana numbers)

- The bivariate GF $F(x,y)$ for the Narayana numbers

$$a_{rs} = \frac{1}{r}\binom{r}{s}\binom{r-1}{s-1}$$

satisfies $P(F(x,y),x,y) = 0$, where

$$\begin{aligned} P(w,x,y) &= w^2 - w\left[1 + x(y-1)\right] + xy \\ &= \left[w - F(x,y)\right]\left[w - \overline{F}(x,y)\right]. \end{aligned}$$

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- Using the above construction we obtain the lifting

$$G(u,x,y) = \frac{u(1-2u-ux(1-y))}{1-u-xy-ux(1-y)}.$$

Example (Narayana numbers continued)

- The above lifting yields asymptotics by smooth point analysis in the usual way. The critical point equations yield

$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$

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- Interestingly, specializing $y = 1$ commutes with lifting. Is this always true?

Technical issues

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- Dominant singularities can be at infinity.
- There are other lifting procedures, some of which go from dimension d to $2d$. They seem complicated, and we have not yet tried them in detail.
- However in some cases they work better - for example $2xy/(2+x+y)$ is a lifting of $x\sqrt{1-x}$, whereas Safonov's method appears not to work easily.

Research projects

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- Make the computation of dominant points algorithmic in the noncombinatorial case.