ACSV: help wanted from computer algebra(ists)

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Standing assumptions

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- To avoid discussing complicated topology, assume all coefficients of F are nonnegative.

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- Lattice walks in quarter plane, negative drift:

$$G = (1 + x)(1 - 2t(1 + x^2))$$
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- Cubical tensors

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- This yields an asymptotic expansion

$$a_{\mathbf{r}} \sim \sum_{\mathbf{p} \in \mathrm{contrib}(\overline{\mathbf{r}})} \mathbf{p}^{-\mathbf{r}} \mathcal{A}(\mathbf{p})$$

  that is uniform on compact subsets of directions, provided the geometry at $\mathbf{p}$ does not change.

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- There is an element of $\mathrm{contrib}(\overline{\mathbf{r}})$ having all positive coordinates.
- We have full results for smooth and transverse multiple local geometry of critical points.
- We can check $\mathbf{p} \in \mathrm{contrib}(\overline{\mathbf{r}})$ by checking whether $\overline{\mathbf{r}}$ belongs to a certain real positive cone $\mathrm{K}(\mathbf{p})$.

Simplest asymptotic formulae

- Smooth point:

$$a_{\mathbf{r}} \sim \mathbf{z}_*(\overline{\mathbf{r}})^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{d-1}\kappa(\mathbf{z}_*(\overline{\mathbf{r}}))}} \frac{G(\mathbf{z}_*(\overline{\mathbf{r}}))}{|\nabla_{\log} H(\mathbf{z}_*(\overline{\mathbf{r}}))|}$$

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- The Gaussian curvature can be computed explicitly in terms of derivatives of H to second order.
- Simplest multiple point:

$$a_{\mathbf{r}} \sim \mathbf{z}_*(\overline{\mathbf{r}})^{-\mathbf{r}} G(\mathbf{z}_*(\overline{\mathbf{r}})) \det J(\mathbf{z}_*(\overline{\mathbf{r}}))^{-1}$$

  where J is the Jacobian matrix $(\partial H_i / \partial z_j)$ and $H = \prod_i H_i$.

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  - Auxiliary functions to convert G/H to various forms.

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- Exception handling.
- Putting everything together (non-interactive mode).

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- Determine whether a direction lies inside a cone not given by rational extreme rays (`Cone` package can't handle).
- Improvements to speed of higher order asymptotic computations.

Hörmander's explicit formula for integral asymptotics

For an isolated nondegenerate stationary point $\mathbf{0}$ in dimension d,

$$I(\lambda) \sim \left(\det\left(\frac{\lambda f''(\mathbf{0})}{2\pi}\right)\right)^{-1/2} \sum_{k \geq 0} \lambda^{-k} L_k(A, f)$$

where L_k is a differential operator of order $2k$ evaluated at $\mathbf{0}$:

$$\underline{f}(t) = f(t) - (1/2) t f''(\mathbf{0}) t^T$$

$$\mathcal{D} = \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b} (-\mathrm{i}\partial_a)(-\mathrm{i}\partial_b)$$

$$L_k(A, f) = \sum_{l \leq 2k} \frac{\mathcal{D}^{l+k}(A \underline{f}^l)(\mathbf{0})}{(-1)^k 2^{l+k} l! (l+k)!}.$$

For example $L_0(A, f) = A$,
$L_1(A, f) = -\mathcal{D}(A)/2 - \mathcal{D}^2(A\underline{f})/8 - \mathcal{D}^3(A\underline{f}^2)/48.$

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- Detecting whether we are in this case is easy (irreducible factors are everywhere smooth).
- However if we are not in this case, we currently have no way to proceed. Such problems do arise rather frequently in applications.

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- Computation in this ring is trickier than in polynomial rings. However there is a theory of computation in local rings and apparently SINGULAR implements some of it.
- Help wanted in finding the state of the art!

Example (local factorization of lemniscate)

- Given $F = 1/H$ where H is irreducible, given by $H(x,y) = 19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$.

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- At $(1, 1)$, changing variables to $h(u, v) := H(1 + u, 1 + v)$, we see that $h(u, v) = 4u^2 + 10uv + 4v^2 + C(u, v)$ where C has no terms of degree less than 3.

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- The quadratic part factors into distinct factors, showing that $(1, 1)$ is a transverse multiple point.
- The current implementation does not deal with this at all, but we can compute by hand in this case to see that $a_{rr} \sim 1/6$.

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- Thus it is not always even obvious whether a point is smooth, and vanishing numerator affects exponential rate.

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- Here $F = G/H$ where $G = x - y$, $H_1 = 1 - (1/6)x - (5/6)y^2$, $H_2 = 1 - (5/6)x^2 - (1/6)y^2$, $H = H_1 H_2$.

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- How to do this algorithmically?