Lattice path asymptotics via Analytic Combinatorics in Several Variables

Mark C. Wilson
Department of Computer Science
University of Auckland

CANADAM, Saskatoon, 2015-06-01

Main references

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. `https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/`

Main references

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/
- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.

Main references

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/
- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.
- Sage implementations by Alex Raichev: https://github.com/araichev/amgf.

Example (A test problem)

- How many n-step nearest neighbour walks are there, if walks start from the origin, are confined to the first quadrant, and take steps in $\{(0,-1),(-1,1),(1,1)\}$? Call this a_n.

Example (A test problem)

- How many n-step nearest neighbour walks are there, if walks start from the origin, are confined to the first quadrant, and take steps in $\{(0,-1),(-1,1),(1,1)\}$? Call this a_n.
- Conjectured by Bostan & Kauers:

$$a_n \sim 3^n \sqrt{\frac{3}{4\pi n}}.$$

Overview

- Consider nearest-neighbour walks in $\mathbb{Z}^d$, defined by a set $S \subseteq \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ of allowed steps. Define

$$S_j = \{i : (i, j) \in S\} \qquad \text{for each } j \in \{-1, 0, 1\}.$$

Overview

- Consider nearest-neighbour walks in $\mathbb{Z}^d$, defined by a set $S \subseteq \{-1,0,1\}^d \setminus \{\mathbf{0}\}$ of allowed steps. Define

$$S_j = \{i : (i,j) \in S\} \qquad \text{for each } j \in \{-1,0,1\}.$$

- We can consider unrestricted walks, walks restricted to a halfspace, and walks restricted to the positive orthant. The last is the most challenging, and we concentrate on it today.

Overview

- Consider nearest-neighbour walks in $\mathbb{Z}^d$, defined by a set $S \subseteq \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ of allowed steps. Define

$$S_j = \{i : (i, j) \in S\} \qquad \text{for each } j \in \{-1, 0, 1\}.$$

- We can consider unrestricted walks, walks restricted to a halfspace, and walks restricted to the positive orthant. The last is the most challenging, and we concentrate on it today.
- We can keep track of the endpoint, and also the length. This gives a $d+1$-variate sequence $a_{\mathbf{r},n}$ with generating function $\sum_{\mathbf{r},n} a_{\mathbf{r},n} \mathbf{x}^{\mathbf{r}} t^n$.

Overview

- Consider nearest-neighbour walks in $\mathbb{Z}^d$, defined by a set $S \subseteq \{-1,0,1\}^d \setminus \{\mathbf{0}\}$ of allowed steps. Define

$$S_j = \{i : (i,j) \in S\} \qquad \text{for each } j \in \{-1,0,1\}.$$

- We can consider unrestricted walks, walks restricted to a halfspace, and walks restricted to the positive orthant. The last is the most challenging, and we concentrate on it today.
- We can keep track of the endpoint, and also the length. This gives a $d+1$-variate sequence $a_{\mathbf{r},n}$ with generating function $\sum_{\mathbf{r},n} a_{\mathbf{r},n} \mathbf{x}^{\mathbf{r}} t^n$.
- Summing over $\mathbf{r}$ gives a univariate series $\sum_n f(n) t^n$.

Overview

- Consider nearest-neighbour walks in $\mathbb{Z}^d$, defined by a set $S \subseteq \{-1,0,1\}^d \setminus \{\mathbf{0}\}$ of allowed steps. Define

$$S_j = \{i : (i,j) \in S\} \qquad \text{for each } j \in \{-1,0,1\}.$$

- We can consider unrestricted walks, walks restricted to a halfspace, and walks restricted to the positive orthant. The last is the most challenging, and we concentrate on it today.
- We can keep track of the endpoint, and also the length. This gives a $d+1$-variate sequence $a_{\mathbf{r},n}$ with generating function $\sum_{\mathbf{r},n} a_{\mathbf{r},n}\mathbf{x}^{\mathbf{r}}t^n$.
- Summing over $\mathbf{r}$ gives a univariate series $\sum_n f(n)t^n$.
- We seek in particular the asymptotics of $f(n)$.

Previous work, I

- Bousquet-Mélou & Mishna (2010) showed that for $d = 2$ there are 79 inequivalent nontrivial cases.

Previous work, I

- Bousquet-Mélou & Mishna (2010) showed that for $d = 2$ there are 79 inequivalent nontrivial cases.
- They introduced the symmetry group $G(S)$ and showed that this is finite in exactly 23 cases.

Previous work, I

- Bousquet-Mélou & Mishna (2010) showed that for $d = 2$ there are 79 inequivalent nontrivial cases.
- They introduced the symmetry group $G(S)$ and showed that this is finite in exactly 23 cases.
- They used this to show for 22 cases that F is D-finite. For 19 of these, used the orbit sum method and for 3 more, the half orbit sum method.

Previous work, II

- Bostan & Kauers (2009): for $d = 2$, conjectured asymptotics for $f(n)$ in the 23 cases.

Previous work, II

- Bostan & Kauers (2009): for $d = 2$, conjectured asymptotics for $f(n)$ in the 23 cases.
- Bostan & Kauers (2010): for $d = 2$, explicitly showed the 23rd case (Gessel walks) has algebraic f.

Previous work, II

- Bostan & Kauers (2009): for $d = 2$, conjectured asymptotics for $f(n)$ in the 23 cases.
- Bostan & Kauers (2010): for $d = 2$, explicitly showed the 23rd case (Gessel walks) has algebraic f.
- Melczer & Mishna (2014): for arbitrary d, G maximal, derived asymptotics for $f(n)$.

Previous work, II

- Bostan & Kauers (2009): for $d = 2$, conjectured asymptotics for $f(n)$ in the 23 cases.
- Bostan & Kauers (2010): for $d = 2$, explicitly showed the 23rd case (Gessel walks) has algebraic f.
- Melczer & Mishna (2014): for arbitrary d, G maximal, derived asymptotics for $f(n)$.
- Bostan, Chyzak, van Hoeij, Kauers & Pech: for $d = 2$, expressed f in terms of hypergeometric integrals in the 23 cases. We use their numbering of the cases.

Previous work, II

- Bostan & Kauers (2009): for $d = 2$, conjectured asymptotics for $f(n)$ in the 23 cases.
- Bostan & Kauers (2010): for $d = 2$, explicitly showed the 23rd case (Gessel walks) has algebraic f.
- Melczer & Mishna (2014): for arbitrary d, G maximal, derived asymptotics for $f(n)$.
- Bostan, Chyzak, van Hoeij, Kauers & Pech: for $d = 2$, expressed f in terms of hypergeometric integrals in the 23 cases. We use their numbering of the cases.
- Open: proof of asymptotics of $f(n)$ for cases 5–16. We solve that here.

ACSV

- Robin Pemantle and I derived general formulae for asymptotics of coefficients of rational functions $F = G/H$ in dimension d (see the book).

ACSV

- Robin Pemantle and I derived general formulae for asymptotics of coefficients of rational functions $F = G/H$ in dimension d (see the book).
- Analysis is based on the geometry of the singular variety (zero-set of H) near contributing critical points $\mathbf{z}_*$ depending on the direction $\mathbf{r}$.

ACSV

- Robin Pemantle and I derived general formulae for asymptotics of coefficients of rational functions $F = G/H$ in dimension d (see the book).
- Analysis is based on the geometry of the singular variety (zero-set of H) near contributing critical points $\mathbf{z}_*$ depending on the direction $\mathbf{r}$.
- The ultimate justification involves Morse theory, but this can be mostly ignored in the aperiodic combinatorial case.

ACSV

- Robin Pemantle and I derived general formulae for asymptotics of coefficients of rational functions $F = G/H$ in dimension d (see the book).
- Analysis is based on the geometry of the singular variety (zero-set of H) near contributing critical points $\mathbf{z}_*$ depending on the direction $\mathbf{r}$.
- The ultimate justification involves Morse theory, but this can be mostly ignored in the aperiodic combinatorial case.
- We deal in particular with multiple points (locally a transverse intersection of k smooth factors). If $1 \leq k \leq d$, formulae are of the form

$$a_{\mathbf{r}} \sim \mathbf{z}_*{}^{-\mathbf{r}} \sum_l b_l ||\mathbf{r}||^{-(d-k)/2-l}.$$

Diagonals

- The orbit sum approach yields F as the positive part of a rational series.

Diagonals

- The orbit sum approach yields F as the positive part of a rational series.
- This is the leading diagonal of a closely related series F.

Diagonals

- The orbit sum approach yields F as the positive part of a rational series.
- This is the leading diagonal of a closely related series F.
- The GF for walks restricted to the quarter plane has the form

$$f = \operatorname{diag} \frac{xyP(x^{-1}, y^{-1})}{(1 - txyS(x^{-1}, y^{-1}))\,(1-x)(1-y)}$$

where

$$S(x,y) = \sum_{(i,j)\in S} x^i y^j$$

$$P(x,y) = \sum_{\sigma \in G} \operatorname{sign}(\sigma)\sigma(xy).$$

Singularities

- The factor $H_1 := 1 - txyS(x^{-1}, y^{-1})$ is a polynomial. Its gradient simplifies to $(-1 + ty\partial S/\partial x, -1 + tx\partial S/\partial y, -1)$ and thus this factor is everywhere smooth.

Singularities

- The factor $H_1 := 1 - txyS(x^{-1}, y^{-1})$ is a polynomial. Its gradient simplifies to $(-1 + ty\partial S/\partial x, -1 + tx\partial S/\partial y, -1)$ and thus this factor is everywhere smooth.
- Other singularities come from factors of $(1 - x), (1 - y)$ and possibly from clearing denominators of $xyP(x^{-1}, y^{-1})$.

Singularities

- The factor $H_1 := 1 - txyS(x^{-1}, y^{-1})$ is a polynomial. Its gradient simplifies to $(-1 + ty\partial S/\partial x, -1 + tx\partial S/\partial y, -1)$ and thus this factor is everywhere smooth.
- Other singularities come from factors of $(1 - x), (1 - y)$ and possibly from clearing denominators of $xyP(x^{-1}, y^{-1})$.
- When F is combinatorial, there is a dominant singularity for direction $\mathbf{1}$ lying in the positive orthant.

Critical points

- H_1 contains a smooth critical point for the direction $(1,1,1)$ if and only if $\nabla S(x^{-1}, y^{-1}) = 0$.

Critical points

- H_1 contains a smooth critical point for the direction $(1, 1, 1)$ if and only if $\nabla S(x^{-1}, y^{-1}) = 0$.
- This occurs if and only if

$$\sum_{i=-1,j} y^j - x^{-2} \sum_{i=1,j} y^j = 0$$
$$\sum_{i \in S_{-1}} x^i - y^{-2} \sum_{i \in S_1} x^i = 0.$$

Critical points

- H_1 contains a smooth critical point for the direction $(1,1,1)$ if and only if $\nabla S(x^{-1}, y^{-1}) = 0$.
- This occurs if and only if

$$\sum_{i=-1,j} y^j - x^{-2} \sum_{i=1,j} y^j = 0$$
$$\sum_{i \in S_{-1}} x^i - y^{-2} \sum_{i \in S_1} x^i = 0.$$

- If S has a vertical axis of symmetry, then $(x^2 - 1) \sum_j y^j = 0$.

Structure of G

- Write

$$\begin{aligned} S(x,y) &= y^{-1}A_{-1}(x) + A_0(x) + yA_1(x) \\ &= x^{-1}B_{-1}(y) + B_0(y) + xB_1(y). \end{aligned}$$

Structure of G

- Write

$$S(x,y) = y^{-1}A_{-1}(x) + A_0(x) + yA_1(x)$$
$$= x^{-1}B_{-1}(y) + B_0(y) + xB_1(y).$$

- G is generated by the involutions (considered as algebra homomorphisms)

$$(x,y) \mapsto \left(x^{-1}\frac{B_{-1}(y)}{B_1(y)}, y\right)$$
$$(x,y) \mapsto \left(x, y^{-1}\frac{A_{-1}(x)}{A_1(x)}\right)$$

Structure of G

- Write

$$S(x,y) = y^{-1}A_{-1}(x) + A_0(x) + yA_1(x)$$
$$= x^{-1}B_{-1}(y) + B_0(y) + xB_1(y).$$

- G is generated by the involutions (considered as algebra homomorphisms)

$$(x,y) \mapsto \left(x^{-1}\frac{B_{-1}(y)}{B_1(y)}, y\right)$$
$$(x,y) \mapsto \left(x, y^{-1}\frac{A_{-1}(x)}{A_1(x)}\right)$$

- If S has vertical symmetry then $B_1 = B_{-1}$, these maps commute, and G has order 4.

Vertical axis of symmetry, I

- This covers Cases 1–16. The possible denominators from P are $x^2 + 1, x^2 + x + 1$. Neither can contribute because the problem is combinatorial and aperiodic. The dominant point has $x = 1$.

Vertical axis of symmetry, I

- This covers Cases 1–16. The possible denominators from P are $x^2 + 1, x^2 + x + 1$. Neither can contribute because the problem is combinatorial and aperiodic. The dominant point has $x = 1$.
- The numerator vanishes iff $|S_1| = |S_{-1}|$. In that case cancellation occurs and $k = 1$. This solves Cases 1–4: leading term $C|S|^n n^{-1/2}$.

Vertical axis of symmetry, I

- This covers Cases 1–16. The possible denominators from P are x^2+1, x^2+x+1. Neither can contribute because the problem is combinatorial and aperiodic. The dominant point has $x=1$.
- The numerator vanishes iff $|S_1| = |S_{-1}|$. In that case cancellation occurs and $k=1$. This solves Cases 1–4: leading term $C|S|^n n^{-1/2}$.
- Otherwise, there is a double point ($k=2$) at $(1,1,|S|)$. Its contribution is nonzero if and only if the numerator does not vanish and the direction $(1,1,1)$ lies in a certain cone.

Vertical axis of symmetry, I

- This covers Cases 1–16. The possible denominators from P are x^2+1, x^2+x+1. Neither can contribute because the problem is combinatorial and aperiodic. The dominant point has $x=1$.
- The numerator vanishes iff $|S_1| = |S_{-1}|$. In that case cancellation occurs and $k=1$. This solves Cases 1–4: leading term $C|S|^n n^{-1/2}$.
- Otherwise, there is a double point ($k=2$) at $(1,1,|S|)$. Its contribution is nonzero if and only if the numerator does not vanish and the direction $(1,1,1)$ lies in a certain cone.
- The direction lies in the cone iff $\partial S/\partial x(1,1) \geq 0$, iff $|S_1| \geq |S_{-1}|$ (happens in Cases 1–10).

Vertical axis of symmetry, I

- This covers Cases 1–16. The possible denominators from P are $x^2 + 1, x^2 + x + 1$. Neither can contribute because the problem is combinatorial and aperiodic. The dominant point has $x = 1$.
- The numerator vanishes iff $|S_1| = |S_{-1}|$. In that case cancellation occurs and $k = 1$. This solves Cases 1–4: leading term $C|S|^n n^{-1/2}$.
- Otherwise, there is a double point ($k = 2$) at $(1, 1, |S|)$. Its contribution is nonzero if and only if the numerator does not vanish and the direction $(1, 1, 1)$ lies in a certain cone.
- The direction lies in the cone iff $\partial S/\partial x(1, 1) \geq 0$, iff $|S_1| \geq |S_{-1}|$ (happens in Cases 1–10).
- Thus for Cases 5–10 we have leading term $C|S|^n n^{-1}$.

Vertical axis of symmetry, II

- There is a smooth critical point where $y^2 = |S_1|/|S_{-1}|$, so y is a quadratic irrational at worst.

Vertical axis of symmetry, II

- There is a smooth critical point where $y^2 = |S_1|/|S_{-1}|$, so y is a quadratic irrational at worst.
- . The exponential rate is

$$S(1, y^{-1}) = |S_0| + y^{-1}|S_1| + y|S_{-1}| = |S_0| + 2\sqrt{|S_1||S_{-1}|}.$$

Vertical axis of symmetry, II

- There is a smooth critical point where $y^2 = |S_1|/|S_{-1}|$, so y is a quadratic irrational at worst.
- . The exponential rate is

$$S(1, y^{-1}) = |S_0| + y^{-1}|S_1| + y|S_{-1}| = |S_0| + 2\sqrt{|S_1||S_{-1}|}.$$

- The arithmetic-geometric mean inequality shows that this is smaller than $|S|$, with equality if and only if $|S_1| = |S_{-1}|$.

Vertical axis of symmetry, II

- There is a smooth critical point where $y^2 = |S_1|/|S_{-1}|$, so y is a quadratic irrational at worst.
- . The exponential rate is

$$S(1, y^{-1}) = |S_0| + y^{-1}|S_1| + y|S_{-1}| = |S_0| + 2\sqrt{|S_1||S_{-1}|}.$$

- The arithmetic-geometric mean inequality shows that this is smaller than $|S|$, with equality if and only if $|S_1| = |S_{-1}|$.
- This holds in Cases 11–16.

Interesting smooth point situation

- Normally the polynomial correction starts with n^{-1}, since $(3-1)/2 = 1$. The lth term is of order n^{-l}.

Interesting smooth point situation

- Normally the polynomial correction starts with n^{-1}, since $(3-1)/2 = 1$. The lth term is of order n^{-l}.
- If the numerator vanishes at the dominant point, the $l = 1$ term vanishes.

Interesting smooth point situation

- Normally the polynomial correction starts with n^{-1}, since $(3-1)/2 = 1$. The lth term is of order n^{-l}.
- If the numerator vanishes at the dominant point, the $l = 1$ term vanishes.
- This happens in all cases 11-16. The numerator simplifies at the smooth point to $(1+x)(1-y^2|S_{-1}|/|S_1|$, which is zero from the critical point equation for y.

Interesting smooth point situation

- Normally the polynomial correction starts with n^{-1}, since $(3-1)/2 = 1$. The lth term is of order n^{-l}.
- If the numerator vanishes at the dominant point, the $l = 1$ term vanishes.
- This happens in all cases 11-16. The numerator simplifies at the smooth point to $(1+x)(1 - y^2|S_{-1}|/|S_1|$, which is zero from the critical point equation for y.
- The leading term asymptotic is $C(|S_0| + 2\sqrt{|S_1||S_{-1}|})^n n^{-2}$.

Explanation

- The key quantity for walks with vertical symmetry is the difference between the upward and downward steps.

Explanation

- The key quantity for walks with vertical symmetry is the difference between the upward and downward steps.
- If this is positive, there are more possible walks that don't cross the boundary, so the quarter plane restriction is encountered less often. Asymptotics come from the point $(1, 1, 1/|S|)$.

Explanation

- The key quantity for walks with vertical symmetry is the difference between the upward and downward steps.
- If this is positive, there are more possible walks that don't cross the boundary, so the quarter plane restriction is encountered less often. Asymptotics come from the point $(1, 1, 1/|S|)$.
- If negative, asymptotics come from the highest smooth point.

Explanation

- The key quantity for walks with vertical symmetry is the difference between the upward and downward steps.
- If this is positive, there are more possible walks that don't cross the boundary, so the quarter plane restriction is encountered less often. Asymptotics come from the point $(1, 1, 1/|S|)$.
- If negative, asymptotics come from the highest smooth point.
- This explains Cases 1-16 in a unified way.

Other cases

- Cases 17–19 also follow as above, with slightly different formulae and more work.

Other cases

- Cases 17–19 also follow as above, with slightly different formulae and more work.
- Cases 20–23 are harder. We don't have a nice diagonal expression, and the conjectured asymptotics show that analysis will be trickier.

Possible future work

- Higher dimensions: $d = 3$ has been studied empirically by Bostan, Bousquet-Mélou, Kauers & Melczer. The orbit sum method appears to work rather rarely, however.

Possible future work

- Higher dimensions: $d = 3$ has been studied empirically by Bostan, Bousquet-Mélou, Kauers & Melczer. The orbit sum method appears to work rather rarely, however.
- Higher dimensions: weaken the condition of MM2014, but keep it nice enough that results for general dimension can be derived.

Appendix: why not use the diagonal method?

- For general $a_{pn,qn,rn}$ we could try to compute the diagonal GF $F_{pqr}(z) := \sum_{n \geq 0} a_{pn,qn,rn} z^n$ using the diagonal method as in Stanley.

Appendix: why not use the diagonal method?

- For general $a_{pn,qn,rn}$ we could try to compute the diagonal GF $F_{pqr}(z) := \sum_{n \geq 0} a_{pn,qn,rn} z^n$ using the diagonal method as in Stanley.
- However the diagonal is D-finite and there are major computational challenges in computing asymptotics.

Appendix: why not use the diagonal method?

- For general $a_{pn,qn,rn}$ we could try to compute the diagonal GF $F_{pqr}(z) := \sum_{n \geq 0} a_{pn,qn,rn} z^n$ using the diagonal method as in Stanley.
- However the diagonal is D-finite and there are major computational challenges in computing asymptotics.
- See Raichev & Wilson (2007), "A new diagonal method ...".