Higher Dimensional Lattice Walks: Connecting Combinatorial and Analytic Behavior
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Mark C. Wilson
University of Auckland

References

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- R. Pemantle & M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. Draft available on my website. https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/
- Sage implementations by Alex Raichev: https://github.com/araichev/amgf.

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$$b_n \sim (2\sqrt{2})^n \frac{\theta(n)}{\pi n^2}$$

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- Such constrained walk questions have been very actively studied in the last decade. They yield many natural examples of D-finite sequences.

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- We keep track of the endpoint, and also the length. This gives a trivariate sequence $a_{r,s,n}$ with generating function (GF)

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- Summing over r, s gives a univariate series $C(1, 1, t) := f(t) = \sum_n f_n t^n$.
- We seek in particular the asymptotics of f_n.

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- 23 classes of walks confined to a quadrant — D-finite functions — satisfy a linear ODE with polynomial coefficients — reasonably well understood.
- 56 quadrant classes, steps that are not small — non D-finite functions — poorly understood.

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- Bostan & Kauers (2010) explicitly showed that for the 23rd case (Gessel walks), $f(t)$ is algebraic (and hence D-finite).
- In the other 56 cases, $f(t)$ is indeed apparently not D-finite. So there are 23 nice inequivalent cases to discuss now.

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- Open: proof of asymptotics of f_n for 15 cases. We solve that here via a unified approach.

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- Solution 2: work hard.

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- We write $S(\mathbf{z}) = \sum_{\mathbf{i}\in\mathcal{S}} w_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} = z_d B + Q + \overline{z}_d A$ where $\overline{z} = z^{-1}$ and A, B, Q are independent of z_d.

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- The drift is the difference $B(\mathbf{1}) - A(\mathbf{1})$ between the weight of positive and negative steps in the asymmetric direction.

Theorem

$$F(\mathbf{1}, t) = \Delta \left(\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} \right),$$

where

$$G(\mathbf{z}, t) = (1 + z_1) \cdots (1 + z_{d-1}) \left(1 - t z_1 \cdots z_d \left(Q + 2 z_d A\right)\right)$$
$$H(\mathbf{z}, t) = (1 - z_d)\Big(1 - t z_1 \cdots z_d \overline{S}(\mathbf{z})\Big)\Big(1 - t z_1 \cdots z_d \left(Q + z_d A\right)\Big),$$

and

$$\overline{S}(\mathbf{z}) = S(\mathbf{z}_{\hat{d}}, \overline{z}_d) = \overline{z}_d B\left(\mathbf{z}_{\hat{d}}\right) + Q\left(\mathbf{z}_{\hat{d}}\right) + z_d A\left(\mathbf{z}_{\hat{d}}\right).$$

Theorem (Positive Drift Asymptotics)

Let

$$b_k = \sum_{\mathbf{i}\in\mathcal{S}, i_k=1} w_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathcal{S}, i_k=-1} w_{\mathbf{i}}.$$

for $1 \leq k < d$. *Then*

$$f_n \sim S(\mathbf{1})^n \cdot n^{\frac{-(d-1)}{2}} \cdot \left[\left(1 - \frac{A(\mathbf{1})}{B(\mathbf{1})}\right) \left(\frac{S(\mathbf{1})}{\pi}\right)^{\frac{d-1}{2}} \frac{1}{\sqrt{b_1 \cdots b_{d-1}}} \right].$$

Theorem (Negative Drift Asymptotics)

Let $\rho = \sqrt{\frac{A(\mathbf{1})}{B(\mathbf{1})}}$, *let* $b_k(\mathbf{z}_{\hat{k}}) := [z_k]S(\mathbf{z}) = [z_k^{-1}]S(\mathbf{z})$ *and let*

$$C_\rho := \frac{S(\mathbf{1},\rho)\,\rho}{2\,\pi^{d/2}\,A(\mathbf{1})(1-1/\rho)^2} \cdot \sqrt{\frac{S(\mathbf{1},\rho)^d}{\rho\, b_1(\mathbf{1},\rho)\cdots b_{d-1}(\mathbf{1},\rho)\cdot B(\mathbf{1})}}.$$

- *If* $Q \neq 0$ *then*

$$f_n \sim S(\mathbf{1},\rho)^n \cdot n^{-d/2-1} \cdot C_\rho.$$

- *If* $Q = 0$ *then*

$$f_n \sim n^{-d/2-1} \cdot \Big[S(\mathbf{1},\rho)^n \cdot C_\rho + S(\mathbf{1},-\rho)^n \cdot C_{-\rho}\Big].$$

Example

Consider the model defined by $\mathcal{S} = \{(1,0), (-1,0), (0,1), (0,-1)\}$, where the south step $(0,-1)$ has weight $a > 0$ and the north step $(0,1)$ has weight $b > 0$ (when a and b are integers we can think of having multiple copies of each step with different colours). Then

$$A(x) = a \qquad Q(x) = \overline{x} + x \qquad B(x) = b$$

and

$$s_n \sim \begin{cases} \left(2 + 2\sqrt{ab}\right)^n \cdot n^{-2} \cdot \frac{2a^{1/4}\left(1+\sqrt{ab}\right)^2}{\pi b^{3/4}\left(\sqrt{a}-\sqrt{b}\right)^2} & : b < a \\ (2+2a)^n \cdot n^{-1} \cdot \frac{2(1+a)}{\sqrt{a}\,\pi} & : b = a \\ (2+a+b)^n \cdot n^{-1/2} \cdot \frac{(a+b)\sqrt{2+a+b}}{b\sqrt{\pi}} & : b > a \end{cases}$$

with the different cases corresponding to negative drift, zero drift, and positive drift.

$\mathcal{S}$	Asymptotics	$\mathcal{S}$	Asymptotics	$\mathcal{S}$	Asymptotics
	$\frac{4}{\pi} \cdot \frac{4^n}{n}$		$\frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{\sqrt{n}}$		$\frac{A_n}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	$\frac{2}{\pi} \cdot \frac{4^n}{n}$		$\frac{4}{3\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}}$		$\frac{B_n}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2}$
	$\frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n}$		$\frac{\sqrt{5}}{3\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{C_n}{\pi} \cdot \frac{(2\sqrt{6})^n}{n^2}$
	$\frac{8}{3\pi} \cdot \frac{8^n}{n}$		$\frac{\sqrt{5}}{2\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{\sqrt{8}(1+\sqrt{2})^{7/2}}{\pi} \cdot \frac{(2+2\sqrt{2})^n}{n^2}$
	$\frac{2\sqrt{2}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{2\sqrt{3}}{3\sqrt{\pi}} \cdot \frac{6^n}{\sqrt{n}}$		$\frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi} \cdot \frac{(2+2\sqrt{3})^n}{n^2}$
	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{\sqrt{7}}{3\sqrt{3\pi}} \cdot \frac{7^n}{\sqrt{n}}$		$\frac{\sqrt{570-114\sqrt{6}}(24\sqrt{6}+59)}{19\pi} \cdot \frac{(2+2\sqrt{6})^n}{n^2}$
	$\frac{\sqrt{6\sqrt{3}}}{\Gamma(1/4)} \cdot \frac{6^n}{n^{3/4}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}}$		$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$
	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \cdot \frac{4^n}{n^{2/3}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{6^n}{n^{3/2}}$		

Table: Asymptotics for the 23 D-finite models.

$$A_n = \begin{cases} 24\sqrt{2} & : n \text{ even} \\ 32 & : n \text{ odd} \end{cases}, \quad B_n = \begin{cases} 12\sqrt{3} & : n \text{ even} \\ 18 & : n \text{ odd} \end{cases}, \quad C_n = \begin{cases} 12\sqrt{30} & : n \text{ even} \\ 144/\sqrt{5} & : n \text{ odd} \end{cases}$$

Extensions

- Small modifications yield results for walks constrained to return to an axis or the origin.
- Walks in Weyl chambers can be treated in this way.

Publication reform

- Pressure is building for complete conversion of the journal system to open access (e.g. Plan S from European research funders)
- Large commercial publishers have incentives not aligned with scholarship or the interests of readers and authors, and provide overall low quality service for very high prices.
- The journal market is dysfunctional (not properly competitive).
- I am associated with several organizations aiming to improve this: MathOA, Free Journal Network, Publishing Reform Forum. If you would like to help or learn more, please contact me.