Analytic Combinatorics in Several Variables

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Lecture I

Motivation, review, overview

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Recommended books for univariate asymptotics

- H. Wilf, *generatingfunctionology*, `http://www.math.upenn.edu/~wilf/DownldGF.html`

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- P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, http://algo.inria.fr/flajolet/Publications/AnaCombi/anacombi.html
- A. Odlyzko, *Asymptotic Enumeration Methods*, www.dtc.umn.edu/~odlyzko/doc/enumeration.html..

Main references for all lectures

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/

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- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.

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- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.
- Sage implementations by Alex Raichev: https://github.com/araichev/amgf.

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- Exercises are of varying levels of difficulty. We can discuss some in the problem sessions. Those marked (C) involve probably publishable research, for which I am seeking collaborators, and should be accessible to PhD students.

Table of Contents

Lecture 1: Overview

- In one variable, starting with a sequence $a_{\mathbf{r}}$ of interest, we form its generating function $F(\mathbf{z})$. Cauchy's integral theorem allows us to express $a_{\mathbf{r}}$ as an integral. The exponential growth rate of $a_{\mathbf{r}}$ is determined by the location of a dominant singularity $\mathbf{z}_*$ of F. More precise estimates depend on the local geometry of the singular set $\mathcal{V}$ of F near $\mathbf{z}_*$.

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- In the multivariate case, all the above is still true. However, we need to specify the direction in which we want asymptotics; we then need to worry about uniformity; the definition of "dominant" is a little different; the local geometry of $\mathcal{V}$ can be much nastier; the local analysis is more complicated.

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- The aperiodic case: $a_{\mathbf{r}}$ is not supported on a proper sublattice of $\mathbb{N}^d$.

From sequence to generating function and back

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- Example: (Fibonacci)

$$\begin{aligned} a_r &= a_{r-1} + a_{r-2} \quad \text{if } r \geq 2 \\ a_0 &= 0, a_1 = 1 \end{aligned}$$

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- Our focus this week is on the next step: deriving a formula (usually asymptotic approximation) for a_r, given a nice representation of F. This is coefficient extraction.

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- Further analysis depends on the type of singularity.

From singularities to asymptotic expansions

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- if ρ is algebraic/logarithmic, use singularity analysis (Flajolet-Odlyzko 1990);
- if ρ is essential, use the saddle point method.

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- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.
- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.
- Since there are no more poles, we can push the contour of integration to ∞ in this case, so the error in the approximation decays faster than any exponential function of r.

Univariate rational functions: general solution

- Given a rational function $p(z)/q(z)$ with $q(0) = 1$, factor it as $q(z) = \prod_i (1 - \phi_i z)^{n_i}$ with all ϕ_i distinct.

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- For example, Fibonacci yields $a_r \sim 5^{-1/2}[(1 + \sqrt{5})/2]^r$.
- Repeated roots provide polynomial correction to the exponential factor. For example, $1/(1 - 2z)^3 = \sum_r \binom{r+2}{2} 2^r z^r$.

Example (Essential singularity: saddle point method)

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- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.
- The integral over C_n has most mass near $z = n$, so that

$$
\begin{aligned}
a_n &= \frac{F(n)}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) \frac{F(ne^{i\theta})}{F(n)}\, d\theta \\
&\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp\left(-in\theta + \log F(ne^{i\theta}) - \log F(n)\right)\, d\theta.
\end{aligned}
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Example (Saddle point example continued)

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- This recaptures Stirling's approximation, since $n! = 1/a_n$:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

Multivariate asymptotics — some quotations

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- (Odlyzko 1995) “A major difficulty in estimating the coefficients of mvGFs is that the geometry of the problem is far more difficult. . . . Even rational multivariate functions are not easy to deal with.”
- (Flajolet/Sedgewick 2009) “Roughly, we regard here a bivariate GF as a collection of univariate GFs . . . .”

Finding multivariate GFs

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- Linear recursions with polynomial coefficients yield linear PDEs, which can be hard to solve, certainly harder than the ODEs in the univariate case.
- We will not deal with this issue in these lectures - we assume that the GF is given in explicit form (say rational or algebraic) and concentrate on extraction of Maclaurin coefficients.

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- We can compute, for some circle γ_x around $t = 0$,

$$\begin{aligned} F_{1,1}(x) &= [t^0]F(x/t, t) \\ &= \frac{1}{2\pi i} \int_{\gamma_x} \frac{F(x/t, t)}{t}\, dt \\ &= \sum_k \mathrm{Res}(F(x/t, t)/t; t = s_k(x)) \end{aligned}$$

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  where the $s_k(x)$ are the singularities satisfying $\lim_{x \to 0} s_k(x) = 0$.
- If F is rational, then $F_{1,1}$ is algebraic.

Example (Delannoy lattice walks)

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- How to compute a_{rs} for large r,s?
- For example, what does $a_{7n,5n}$ look like as $n \to \infty$?

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  - If $d > 2$, diagonals will not be algebraic in general, even if F is rational.
  - Fancier methods exist (based on holonomic or D-finite theory), but again computational complexity is a major obstacle.

Our plan

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- Amazingly little was known even about rational F in 2 variables. We aimed to create a general theory.

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- Analysis: the (Leray) residue formula is much harder to use.

Outline of results

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- There is an asymptotic expansion $\mathrm{formula}(\mathbf{z}_*)$ for $a_{\mathbf{r}}$, where $\mathrm{formula}(\mathbf{z}_*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.

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where the expansion is uniform on compact subsets of directions, provided the geometry does not change.
- The set $\mathrm{crit}(\overline{\mathbf{r}})$ is computable via symbolic algebra.
- To determine the dominant point requires a little more work, but usually not much. (*)

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- How does our method compare with others?
- How does it all work? (I want to see the details)

Exercises: finding GFs

- Find (a defining equation for) the GF for the sequence (a_n) defined by $a_0 = 0; a_n = n + (2/n)\sum_{0 \le k < n} a_k$ for $n \ge 1$.

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$$p(n, j) = \frac{2n - 1 - j}{2n - 1} p(n - 1, j) + \frac{j - 1}{2n - 1} p(n - 1, j - 1)$$

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- Express the GF for the sequence given by the recursion

  $$f(r, s) = f(r - 1, s) + f(r, s - 1) - \frac{(r + s - 1)}{(r + s)} f(r - 1, s - 1)$$

  $$f(0, s) = 1, f(r, 0) = 1$$

  as explicitly as you can.

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- Challenge for D-finiteness experts: for Delannoy walks, what is the largest $p+q$ (where $\gcd\{p,q\}=1$) for which you can compute an asymptotic approximation of $a_{pn,qn}$, with an error of less than 0.01% when $n=10$?

Lecture II

Smooth points in dimension 2

Lecture 2: Overview

- If the dominant singularity is a smooth point of $\mathcal{V}$, the local geometry is simple. In the generic case, the local analysis is also straightforward. We can derive explicit results that apply to a huge number of applications. In dimension 2, these are even more explicit.

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- We first consider the case where the dominant singularity is strictly minimal, meaning that F is analytic on the open polydisc D defined by z_*, which is the only singularity on $\overline{D}$. In this case we can use univariate residue theory accompanied by elementary deformations of the contour of integration.

Table of Contents

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- We focus here on the $d-1=1$ case but everything works in general dimension.

Reduction step 1: localization

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- Note that this is because of strict minimality: off N, the function $F(z, \cdot)$ has radius of convergence greater than σ, and compactness allows us to do everything uniformly.

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- Clearly $|z_*^r I'| \to 0$, and hence
$$a_{rs} \approx (2\pi i)^{-1} \int_N z^{-r} v(z)^s \Psi(z) \, dz.$$

Reduction step 3: Fourier-Laplace integral

- We make the substitution

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where D is a small neighbourhood of $0 \in \mathbb{R}$.

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  - $\operatorname{Re} f \geq 0$; the phase f and amplitude A are analytic.
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- Such integrals are well known in many areas including mathematical physics. Potential difficulties in analysis: interplay between exponential and oscillatory decay of f, degeneracy of f, boundary issues.

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- If 0 is an isolated stationary point and the boundary terms can be neglected, then we have a good chance of computing an asymptotic expansion for the integral.
- If furthermore $f''(0) \neq 0$ (the nondegeneracy condition), we have the nicest formula: the standard Laplace approximation for the leading term is

$$I(\lambda) \sim A(0)\sqrt{\frac{2\pi}{\lambda f''(0)}}.$$

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- Furthermore
$$f''(0) = \frac{z_*^2 v''(z_*)}{v(z_*)} + \frac{z_* v'(z_*)}{v(z_*)} - \left(\frac{z_* v'(z_*)}{v(z_*)}\right)^2.$$
- So given (z_*, w_*), for this value of α we can derive asymptotics using the Laplace approximation as above.

Converting back to the original data

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\begin{aligned}
f'(0) &= i\frac{r}{s} - i\frac{zH_z}{wH_w} \\
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  where these are evaluated at (z_*, w_*).
- The residue can also be computed in terms of H. We can now put everything together to give an explicit formula in terms of original data.

Generic smooth point asymptotics in dimension 2

- Suppose that $F = G/H$ has a strictly minimal simple pole at $\mathbf{p} = (z^*, w^*)$.
  If $Q(\mathbf{p}) \neq 0$, then when $s \to \infty$ with $(rwH_w - szH_z)_{|\mathbf{p}} = 0$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{2\pi}}\sqrt{\frac{-wH_w(\mathbf{p})}{sQ(\mathbf{p})}} + O(s^{-3/2})\right].$$

  The apparent lack of symmetry is illusory, since $wH_w/s = zH_z/r$ at $\mathbf{p}$.

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- This, the simplest multivariate case, already covers hugely many applications.
- Here $\mathbf{p}$ is given, which specifies the only direction in which we can say anything useful. But we can vary $\mathbf{p}$ and obtain asymptotics that are uniform in the direction.

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Important special case: Riordan arrays

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- In this case, if we define

$$\mu(x) := xv'(x)/v(x)$$
$$\sigma^2(x) := x^2 v''(x)/v(x) + \mu(x) - \mu(x)^2$$

the previous formula boils down (under extra assumptions) to

$$a_{rs} \sim (x_*)^{-r} v(x_*)^s \frac{\phi(x_*)}{\sqrt{2\pi s \sigma^2(x_*)}}$$

where x_* satisfies $\mu(x_*) = r/s$.

Example (Delannoy walks)

- Recall that $F(x,y) = (1-x-y-xy)^{-1}$. This is Riordan with $\phi(x) = (1-x)^{-1}$ and $v(x) = (1+x)/(1-x)$. Here $\mathcal{V}$ is globally smooth and for each (r,s) there is a unique solution to $\mu(x) = r/s$.

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- Compare Panholzer-Prodinger, Bull. Aust. Math. Soc. 2012.

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- Aside: this formula gives interesting sum of squares identities..

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- Aside: we can proceed analogously for arbitrary $d \geq 2$.
- See M.C. Wilson, *Diagonal asymptotics for products of combinatorial classes*, Combinatorics, Probability and Computing (Flajolet memorial issue).

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- A horizontally convex polyomino (HCP) is a union of cells $[a, a+1] \times [b, b+1]$ in the two-dimensional integer lattice such that the interior of the figure is connected and every row is connected.

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- The GF for horizontally convex polyominoes ($k =$ rows, $n =$ squares) is

$$F(x, y) = \sum_{n,k} a_{nk} x^n y^k$$
$$= \frac{xy(1-x)^3}{(1-x)^4 - xy(1 - x - x^2 + x^3 + x^2 y)}.$$

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- More on this example in Lecture 4.

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- (C) Use the formula for b_n above to systematically derive identities involving sums of squares that are not in OEIS.

Lecture III

Higher dimensions, other geometries

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- We derive explicit formulae for multiple points. The residue computations can be done in terms of residue forms, which enables us to derive stronger results.

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$$I(\lambda) \sim A(\mathbf{0})\sqrt{\frac{1}{\lambda \det \frac{\mathrm{Q}}{2\pi}}}.$$

- There are technical issues involved in proving this, because the phase f is neither purely real nor purely imaginary. See Chapter 5.

Smooth formulae for general d

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$$\nabla_{\log} H(\mathbf{z}) := (z_1 H_1, \ldots, z_d H_d) \text{ is parallel to } \mathbf{r}.$$

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- This specializes when $d = 2$ to the previous formula.

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- Our hypotheses are satisfied: smooth, combinatorial, aperiodic. For each $\overline{\mathbf{r}}$, there is a dominant point in the positive orthant.

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- For the diagonal direction we have $\mathbf{z}_*(\bar{\mathbf{1}}) = (2^{1/d} - 1)\mathbf{1}$ (by symmetry), so the number of "square" alignments satisfies

$$a(n, n \ldots, n) \sim (2^{1/d} - 1)^{-dn} \frac{1}{(2^{1/d} - 1)2^{(d^2-1)/2d}\sqrt{d(\pi n)^{d-1}}}$$

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- Confirms a result of Griggs, Hanlon, Odlyzko & Waterman, *Graphs and Combinatorics* 1990, with less work, and extends to generalized alignments.

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Logarithmic domain

- Recall U is the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence.

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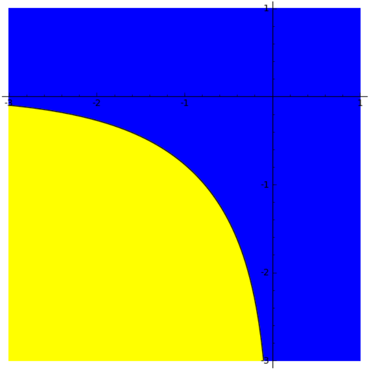
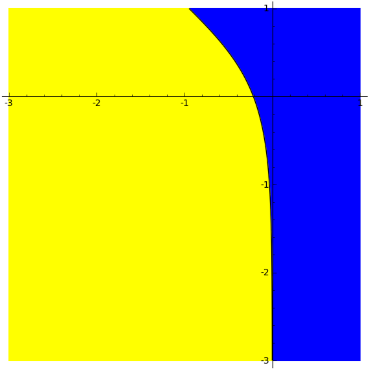
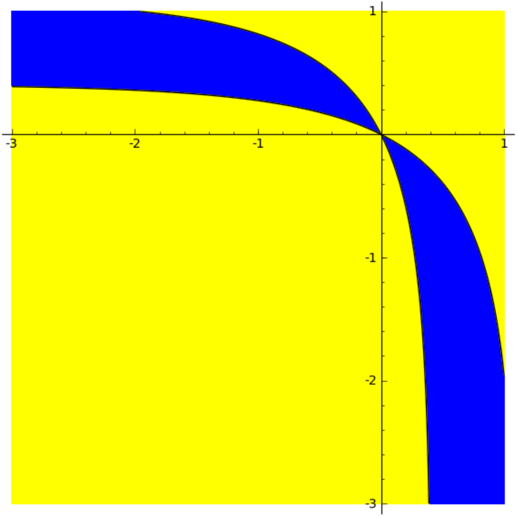
- The cone spanned by normals to supporting hyperplanes at $\mathbf{x}^* \in \log \mathcal{V}$ we denote by $\mathrm{K}(\mathbf{z}_*)$.
- If $\mathbf{z}_*$ is smooth, this is a single ray determined by the image of $\mathbf{z}_*$ under the logarithmic Gauss map $\nabla_{\log} H$.

$\log \mathrm{U}$ for smooth Delannoy and polyomino examples





$\log \mathrm{U}$ for nonsmooth example



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- If $\mathcal{V}$ is smooth everywhere, then asymptotics in all directions are supplied by such points.
- The quantity Q is essentially the Gaussian curvature of $\log \mathcal{V}$.

Alternative smooth point formula

- $$a_{\mathbf{r}} \sim {\mathbf{z}_*}^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2}\kappa(\mathbf{z}_*)}} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|}$$

  where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.

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- For multiple points that are not transverse, we also have results.
- We also have some results for cone points (Chapter 11, very difficult, not presented this week).

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- Step 3 (Fourier-Laplace integral): the resulting integral is more complicated, with a nastier domain and more complicated phase function.
- However in the generic (transverse) case we automatically obtain a nondegenerate stationary point in dimension $n + d - 2$, and can use a modification of the Laplace approximation (which deals with boundary terms).

Generic double point in dimension 2

- Suppose that $F = G/H$ has a strictly minimal pole at $\mathbf{p} = (z_*, w_*)$, which is a double point of $\mathcal{V}$ such that $G(\mathbf{p}) \neq 0$. Then as $s \to \infty$ for r/s in $\mathrm{K}(\mathbf{p})$,

$$a_{rs} \sim (z_*)^{-r} (w_*)^{-s} \left[\frac{G(\mathbf{p})}{\sqrt{(z_* w_*)^2 \, \mathrm{Q}(\mathbf{p})}} + O(e^{-c(r+s)}) \right]$$

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  - the expansion holds uniformly over compact subcones of K;
  - the hypothesis $G(\mathbf{p}) \neq 0$ is necessary; when $d > 1$, can have $G(\mathbf{p}) = H(\mathbf{p}) = 0$ even if G, H are relatively prime.

Example (Queueing network)

- Consider

$$F(x,y) = \frac{\exp(x+y)}{(1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{2y}{3} - \frac{x}{3})}$$

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- Note we say nothing here about the boundary of the cone.

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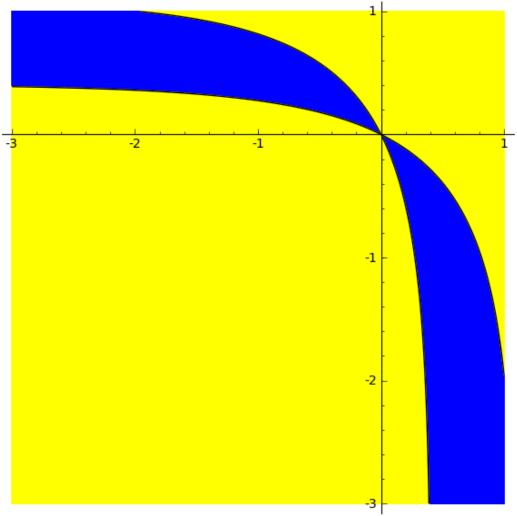

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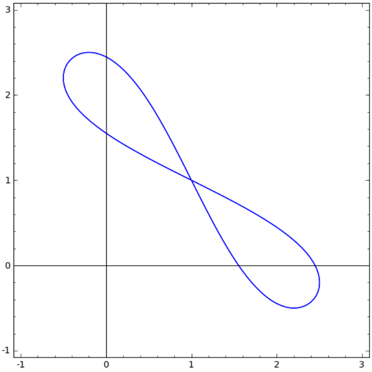
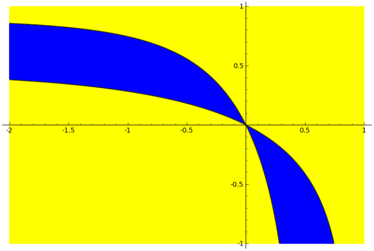
- Note that H factors locally at $(1, 1)$ but not globally.

$\mathcal{V}$ and $\log \mathrm{U}$ for lemniscate





Multiple points: generic shape of $\mathrm{formula}(\mathbf{z}_*)$

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$$\mathbf{z}_*{}^{-\mathbf{r}} G(\mathbf{z}_*) P\left(\frac{r_1}{z_1^*}, \ldots, \frac{r_d}{z_d^*}\right),$$

P a piecewise polynomial of degree $n - d$.

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- When $n > d$, we first preprocess (see Lecture 4) to reduce to the case $n \leq d$.

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- For example, $a_{3t,3t,2t} \sim (48\pi t)^{-1/2}$ with relative error less than 0.3% when $n = 30$.

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- Compare with the exact result when $d = 6, n = 10$.

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- Which method do you prefer?
- Which method can say something about asymptotics on the boundary of the cone?

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- Derive asymptotics for a_{rs} when $1/2 < r/s < 2$.

Lecture IV

Computational aspects

Asymptotics of Fourier-Laplace integrals

Higher order terms

Computations in rings

Local factorizations

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- The algebraic computations are usually best carried out using defining ideals, rather than explicit formulae.

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- Multiple point with $n = 2, d = 1$ gives integral like

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Simplex corners now intrude, continuum of critical points.

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- The answer is
$$I(\lambda) \sim \frac{2\pi}{|v_1'(1) - v_2'(1)|\lambda}$$

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- Consider
$$I(\lambda) = \int_{-\varepsilon}^{\varepsilon} \int_0^1 e^{-\lambda\phi(p,t)}\, dp\, dt$$
where $\phi(p,t) = i\lambda t + \log\left[(1-p)v_1(e^{it}) + pv_2(e^{it})\right]$.
- This arises in the simplest strictly minimal double point situation. Recall the v_i are the inverse poles near the double point.
- The answer is
$$I(\lambda) \sim \frac{2\pi}{|v_1'(1) - v_2'(1)|\lambda}$$
- This doesn't satisfy the hypotheses of the last slide, and so we needed to derive the analogue of the Laplace approximation.

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- Applications of higher order terms:
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  - When leading term is zero because of numerator.
  - Better numerical approximations for smaller indices.

Hörmander's explicit formula

For an isolated nondegenerate stationary point in dimension d,

$$I(\lambda) \sim \left(\det\left(\frac{\lambda f''(\mathbf{0})}{2\pi}\right)\right)^{-1/2} \sum_{k\geq 0} \lambda^{-k} L_k(A, f)$$

where

$$\underline{f}(t) = f(t) - (1/2)tf''(0)t^T$$

$$\mathcal{D} = \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b}(-\mathrm{i}\partial_a)(-\mathrm{i}\partial_b)$$

$$\tilde{L}_k(A, f) = \sum_{l\leq 2k} \frac{\mathcal{D}^{l+k}(A\underline{f}^l)(0)}{(-1)^k 2^{l+k} l!(l+k)!}.$$

$\tilde{L}_k$ is a differential operator of order $2k$ acting on A at 0 (considering the order $3m$ zero of $\underline{f}^m$), whose coefficients are rational functions of $f''(0), \ldots, f^{(2k+2)}(0)$.

Example (nonoverlapping patterns)

- Given a word over alphabet $\{a_1, \ldots, a_d\}$, players alternate reading letters. If the last two letters are the same, we erase the letters seen so far, and continue.

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- For example, in $abaabbba$, there are two occurrences.
- How many such snaps are there, for random words?
- Answer: let ψ_n be the random variable counting snaps in words of length n. Then as $n \to \infty$,

$$\mathbb{E}(\psi_n) = (3/4)n - 15/32 + O(n^{-1})$$
$$\sigma^2(\psi_n) = (9/32)n + O(1).$$

Example (snaps continued)

- The details are as follows. Consider W given by

$$
\begin{aligned}
W(x_1, \ldots, x_d, y) &= \frac{A(x)}{1 - yB(x)} \\
A(x) &= 1/[1 - \sum_{j=1}^{d} x_j/(x_j + 1)] \\
B(x) &= 1 - (1 - e_1(x))A(x) \\
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$$e_1(x) = \sum_{i=j}^{d} x_j.$$

- The symbolic method shows that $[x_1^n \ldots x_d^n, y^s]W(\mathbf{x}, y)$ counts words with n occurrences of each letter and s snaps.

Example (snaps continued)

We extract as usual. Note the first order cancellation in the variance computation. For $d = 3$,

$$\begin{aligned}
\mathbb{E}(\psi_n) &= \frac{[x^{n\mathbf{1}}]\frac{\partial W}{\partial y}(x,1)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (3/4)n - 15/32 + O(n^{-1}) \\
\mathbb{E}(\psi_n^2) &= \frac{[x^{n\mathbf{1}}]\left(\frac{\partial^2 W}{\partial y^2}(x,1) + \frac{\partial W}{\partial y}(x,1)\right)}{[x^{n\mathbf{1}}]W(x,1)} \\
&= (9/16)n^2 - (27/64)n + O(1) \\
\sigma^2(\psi_n) &= \mathbb{E}(\psi_n^2) - \mathbb{E}(\psi_n)^2 = (9/32)n + O(1).
\end{aligned}$$

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- We know the asymptotics of these are of order $n^{-3/2}$. This is consistent, because the numerator of F vanishes at $(1/2, 1/2)$.
- Our general formula yields
$$a_{nn} \sim 4^n \left(\frac{1}{4\sqrt{\pi}} n^{-3/2} + \frac{3}{32\sqrt{\pi}} n^{-5/2} \right).$$

Computing numerical approximations

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- To compute the kth term naively using Hörmander requires at least d^{3k} $d \times d$ matrix computations.
- There is surely a lot of room for improvement here.

Example (Snaps with $d = 3$)

n	1	2	4	8
$\mathbb{E}(\psi)$	0	1.000	2.509	5.521
$(3/4)n$	0.7500	1.500	3	6
$(3/4)n - 15/32$	0.2813	1.031	2.531	5.531
one-term relative error	undefined	0.5000	0.1957	0.08685
two-term relative error	undefined	0.03125	0.008832	0.001936
$\mathbb{E}(\psi^2)$	0	1.8000	7.496	32.80
$(9/16)n^2$	0.5625	2.250	9	36
$(9/16)n^2 - (27/64)n$	0.1406	1.406	7.312	32.63
one-term relative error	undefined	0.2500	0.2006	0.09768
two-term relative error	undefined	0.2188	0.02449	0.005220
$\sigma^2(\psi)$	0	0.8000	1.201	2.320
$(9/32)n$	0.2813	0.5625	1.125	2.250
relative error	undefined	0.2969	0.06294	0.03001

Example (2 planes in 3-space)

Using the formula we obtain

$$a_{3t,3t,2t} = \frac{1}{\sqrt{3\pi}}\left(\frac{1}{4}t^{-1/2} - \frac{25}{1152}t^{-3/2} + \frac{1633}{663552}t^{-5/2}\right) + O(t^{-7/2}).$$

The relative errors are:

rel. err. vs t	1	2	4	8	16	32
$k=1$	-0.660	-0.315	-0.114	-0.0270	-0.00612	-0.00271
$k=2$	-0.516	-0.258	-0.0899	-0.0158	-0.000664	0.00000780
$k=3$	-0.532	-0.261	-0.0906	-0.0160	-0.000703	-0.00000184

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  - find the critical point $\mathbf{z}_*(\mathbf{r})$;
  - compute a rational function of derivatives of H, evaluated at $\mathbf{z}_*$.
- The first can be solved by, for example, Gröbner basis methods.
- The second can cause big problems if done naively, leading to a symbolic mess, and loss of numerical precision. It is best to deal with annihilating ideals.

Example (Why ideals are better)

- Suppose x is the positive root of $p(x) := x^3 - x^2 + 11x - 2$, and we want to compute $g(x) := x^5/(867x^4 - 1)$.

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- If we compute x symbolically and then substitute into g, we obtain a huge mess involving radicals, which evaluates numerically to 0.193543073868354.
- If we compute x numerically and then substitute, we obtain 0.193543073867096.
- Instead we can compute the minimal polynomial of $y := g(x)$ by Gröbner methods. This gives

$$11454803y^3 - 2227774y^2 + 2251y - 32 = 0$$

and evaluating numerically yields 0.193543073868734.

Example (Polyomino computation)

- Recall the GF for horizontally convex polyominoes is

$$F(x,y) = \frac{xy(1-x)^3}{(1-x^4) - xy(1-x-x^2+x^3+x^2y)}.$$

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- The ideal in $\mathbb{C}[x, y]$ defined by $\{sxH_x - ryH_y, H\}$ has a Gröbner basis giving a quartic minimal polynomial for $x_*(\lambda)$, and $y_*(\lambda)$ is a linear function of $x_*(\lambda)$ (also satisfies a quartic).

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- Specifically, the elimination polynomial for x is

$$(1+\lambda)x^4+4(1+\lambda)^2x^3+10(\lambda^2+\lambda-1)x^2+4(2\lambda-1)^2x+(1-\lambda)(1-2\lambda).$$

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- However, for example when $r = 2s$ there is major simplification: the minimal polynomials for x and y respectively are $3x^2 + 18x - 5$ and $75y^2 - 288y + 256$, etc.
- Now given (r, s), solving numerically for C as a root gives a more accurate answer than if we had solved for x_*, y_* above and substituted.

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  - compute (derivatives of) the factors H_i near $\mathbf{z}_*$.
- Unfortunately, computations in the local ring are not effective (as far as we know). If a polynomial factors as an analytic function, but the factors are not polynomial, we can't deal with it algorithmically (yet).
- Smooth points are easily detected. There are some sufficient conditions, and some necessary conditions, for $\mathbf{z}_*$ to be a multiple point. But in general we don't know how to classify singularities algorithmically.

Example (local factorization of lemniscate)

- Let $H(x,y) =$
  $19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$,
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- At $(1, 1)$, changing variables to $h(u, v) := H(1 + u, 1 + v)$, we see that $h(u, v) = 4u^2 + 10uv + 4v^2 + C(u, v)$ where C has no terms of degree less than 3.

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- The quadratic part factors into distinct factors, showing that $(1,1)$ is a transverse multiple point.
- Note that our double point formula does not require details of the individual factors. However this is not the case for general multiple points.

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- If we have repeated factors, we can reduce to the case of distinct factors using exactly the same idea.
- If this is not done, we arrive at Fourier-Laplace integrals with non-isolated stationary points, which are hard to analyse.
- However after doing the above we always reduce to the case of an isolated point, which we can handle.

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$$F = \frac{1}{H_1 H_2 H_3} = \cdots = \frac{2-y}{(1-y)(1-xy)^2)} + \frac{1}{(1-x)(1-xy)^2}.$$

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- The next step, reducing the multiplicity of factors can be done at the residue stage (residue for higher order pole) or by other methods, and is both easy and algorithmic.
- Thus we can reduce to a (possibly large) sum of (polynomial multiples of) transverse double point asymptotic series.

Exercises

A computer algebra system will help for some of these.

- Use Hörmander's formula to compute L_0, L_1, L_2 for $F(x, y) = (1 - x - y)^{-1}$, at the minimal point $(1/2, 1/2)$. This gives asymptotics for the main diagonal coefficients $\binom{2n}{n}$.

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- The small change from $y(1 - 2y)/(1 - x - y)$ to $(1 - 2y)/(1 - x - y)$ should make no difference to our basic computational procedure. Show that, nevertheless, the results are very different. Explain.

Exercises

A computer algebra system will help for some of these.

- Use Hörmander's formula to compute L_0, L_1, L_2 for $F(x,y) = (1-x-y)^{-1}$, at the minimal point $(1/2, 1/2)$. This gives asymptotics for the main diagonal coefficients $\binom{2n}{n}$.
- The small change from $y(1-2y)/(1-x-y)$ to $(1-2y)/(1-x-y)$ should make no difference to our basic computational procedure. Show that, nevertheless, the results are very different. Explain.
- Compute the expectation and variance of the number of snaps in a standard deck of cards (no asymptotics required).

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- Compute the expectation and variance of the number of snaps in a standard deck of cards (no asymptotics required).
- Carry out the polyomino computation in detail.

Lecture V

Extensions

Overview

- We first look in turn at some of our standard assumptions in force over the last few lectures, and discuss what happens when each is weakened.

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- Removing the combinatorial assumption leads to topological issues which we address in the framework of stratified Morse theory.
- The Fourier-Laplace integrals arising from the reductions can be more complicated that those previously studied.
- We then look at going beyond the class of rational (meromorphic) singularities.

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- A toral point is one for which every point on its torus is a minimal singularity (such as $1/(1 - x^2y^3)$. These occur in quantum random walks. A routine modification.
- If the dominant point is smooth but H is not locally squarefree, then we obtain polynomial corrections that are easily computed. A routine modification.

Example (Periodicity)

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$$\mathbf{p} = \left(\frac{r}{\sqrt{r^2 - s^2}}, \sqrt{\frac{r - s}{r + s}} \right)$$
.
- There is also a dominant point at $-\mathbf{p}$. Adding the contributions yields
$$a_{rs} \sim \sqrt{\frac{2}{\pi}} (-1)^{(s-r)/2} \left(\frac{2r}{\sqrt{s^2 - r^2}} \right)^{-r} \left(\sqrt{\frac{s - r}{s + r}} \right)^{-s} \sqrt{\frac{s + r}{r(s - r)}}$$
when $r + s$ is even and zero otherwise.

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- If this occurs because there are too many sheets, the reduction from Lecture 4 works.
- If it occurs because the dimension of the space spanned by normals is just too small, then it is a little harder to deal with.
- Each term in our expansions depends on finitely many derivatives of G and H, so if sheets have contact to sufficiently high order, the results are the same as if they coincided. Thus if we can reduce in the local ring, all is well. Otherwise we may need to attack the F-L integral directly.

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- When $d_0 = d_1$ this gives the same result as a single repeated smooth factor.

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- If the phase of the Fourier-Laplace integral vanishes to order more than 2, more complicated behaviour ensues.
- If the order of vanishing is 2 everywhere except for 3 at a certain direction, for example, we obtain a phase transition and Airy phenomena.

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  where $\psi(z) = (z/3)(1 - z/3)^2$ and $\phi(z) = 3(1 + z)^2$.
- In directions away from $n = 3k$, our ordinary smooth point analysis holds. When $n = 3k$ we can redo the F-L integral easily and obtain asymptotics of order $n^{-1/3}$.
- Determining the behaviour as we approach this diagonal at a moderate rate is harder (Manuel Lladser PhD thesis), and recovers the results of Banderier-Flajolet-Schaeffer-Soria 2001.

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- When $d = 2$, this has been implemented algorithmically, but not for higher d.
- There is a lesser known version of Morse theory due to Whitney, called stratified Morse theory, which deals with singularities. There is substantial discussion of this in the book.

Cauchy integral formula is homological

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
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- It is natural to try a saddle point/steepest descent approach.

Stratified Morse theory

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  where C_i is a quasi-local cycle for ${\mathbf{z}_*}^{(i)} \in \mathrm{crit}(\mathbf{r})$.
- Key problem: find the highest critical points with nonzero n_i. These are the dominant ones.

Bicolored supertrees

Example

- Consider

$$F(x,y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$

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- In fact $(2, 1/8)$ dominates. The analysis is a substantial part of the PhD thesis of Tim DeVries (U. Pennsylvania).
- The answer:

$$a_{nn} \sim \frac{4^n\sqrt{2}\Gamma(5/4)}{4\pi} n^{-5/4}.$$

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Inverting diagonalization

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- Conversely, every univariate algebraic GF is the diagonal of some rational bivariate GF (next slide).
- The latter result does not generalize strictly to higher dimensions, but something close to it is true. Our multivariate framework means that increasing dimension causes no difficulties in principle, so we can reduce to the rational case.
- The elementary diagonal of $F(z_0, \ldots, z_d) = \sum_{r_0,\ldots,r_d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ is

$$\operatorname{diag} F := f(z_1, \ldots, z_d) = \sum_{r_1,\ldots,r_d} a_{r_1,r_1,\ldots,r_d} z_1^{r_1} \ldots z_d^{r_d}.$$

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
where $u(0, \underline{0}) \neq 0$ and $1 \leq k \in \mathbb{N}$.

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- Define
$$R(z_0, \mathbf{z}) = \frac{z_0^2 P_1(z_0, z_0 z_1, z_2, \dots)}{kP(z_0, z_0 z_1, z_2, \dots)}$$
$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \dots z_d).$$

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$$\tilde{R}(w, \mathbf{z}) = R(w, z_1/w, z_2, \dots z_d).$$
- The Argument Principle shows that $F = \operatorname{diag} R$:
$$\frac{1}{2\pi i} \int_C \tilde{R}(w, \mathbf{z}) \frac{dw}{w} = \sum \operatorname{Res} \tilde{R}(w, \mathbf{z}) = F(\mathbf{z}).$$

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$$\frac{1}{2\pi i} \int_C \tilde{R}(w, \mathbf{z}) \, \frac{dw}{w} = \sum \operatorname{Res} \tilde{R}(w, \mathbf{z}) = F(\mathbf{z}).$$
- Higher order terms are essential: the numerator of $\tilde{R}$ always vanishes at the dominant point. The Catalan example from Lecture 4 was created using this method.

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- Definition: Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have $d+1$ variables and let M be a $d \times d$ matrix with nonnegative entries. The M-diagonal of F is the formal power series in d variables whose coefficients are given by $b_{r_2,\ldots r_d} = a_{s_1,s_1,s_2,\ldots s_d}$ and $(s_1, \ldots, s_d) = (r_1, \ldots, r_d)M$.

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- Theorem: Let f be an algebraic function of d variables. Then there is a unimodular integer matrix M with positive entries and a rational function F in $d+1$ variables such that f is the M-diagonal of F.

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- Theorem: Let f be an algebraic function of d variables. Then there is a unimodular integer matrix M with positive entries and a rational function F in $d+1$ variables such that f is the M-diagonal of F.
- The example $x\sqrt{1-x-y}$ shows that the elementary diagonal cannot always be used.

Example (Narayana numbers)

- The bivariate GF $F(x, y)$ for the Narayana numbers

$$a_{rs} = \frac{1}{r}\binom{r}{s}\binom{r-1}{s-1}$$

satisfies $P(F(x, y), x, y) = 0$, where

$$\begin{aligned} P(w, x, y) &= w^2 - w\,[1 + x(y-1)] + xy \\ &= [w - F(x, y)]\left[w - \overline{F}(x, y)\right]. \end{aligned}$$

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- Using the above construction we obtain the lifting

$$G(u, x, y) = \frac{u(1 - 2u - ux(1-y))}{1 - u - xy - ux(1-y)}.$$

Example (Narayana numbers continued)

- The above lifting yields asymptotics by smooth point analysis in the usual way. The critical point equations yield

$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$

and we obtain asymptotics starting with s^{-2}. For example

$$a_{2s,s} \sim \frac{16^s}{8\pi s^2}.$$

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- Interestingly, specializing $y = 1$ commutes with lifting (and yields the shifted Catalan numbers as in Lecture 4). Is this always true?

Technical issues

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- Dominant singularities can be at infinity.
- There are other lifting procedures, some of which go from dimension d to $2d$. They seem complicated, and we have not yet tried them in detail.
- However in some cases they work better - for example $2xy/(2+x+y)$ is a lifting of $x\sqrt{1-x}$, whereas Safonov's method appears not to work easily.

Possible research projects

- Systematically compare the diagonal method and our methods.

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- In the Cauchy integral for $\sqrt{1-x}$, make a substitution to convert to an integral of a rational function. How general is this procedure?