Lattice path asymptotics via Analytic Combinatorics in Several Variables

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General references

- S. Melczer & M. C. Wilson, *Asymptotics of lattice walks via analytic combinatorics in several variables*. http://arxiv.org/abs/1511.02527.

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- G. Fayolle, R. Iasnogorodski, V. Malyshev, *Random Walks in the Quarter Plane*, 1999.

Example (A test problem)

- How many n-step nearest neighbour walks are there, if walks start from the origin, are confined to the first quadrant, and take steps in $\{S, NE, NW\}$? Call this a_n.

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- These kinds of lattice walks have many applications. They model physical and chemical structures. Their random analogues are important in queueing theory.

Overview — walks

- Consider nearest-neighbour walks in $\mathbb{Z}^d$, defined by a set $S \subseteq \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ of short steps. Define

$$S_j = \{i : (i, j) \in S\} \qquad \text{for each } j \in \{-1, 0, 1\}.$$

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- We can keep track of the endpoint, and also the length. This gives a $d+1$-variate sequence $a_{\mathbf{r},n}$ with generating function (GF) $\sum_{\mathbf{r},n} a_{\mathbf{r},n} \mathbf{x}^{\mathbf{r}} t^n$.

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- Summing over $\mathbf{r}$ gives a univariate series $f(t) = \sum_n f_n t^n$.
- We seek in particular the asymptotics of f_n, the number of walks of a given length.

A hierarchy of GFs arising in these problems

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- Worse functions
- Unrestricted and halfspace walks with short steps yield rational/algebraic GFs via the kernel method and are well understood (Bousquet-Mélou & Petkovsek).
- We concentrate today on 2-dimensional walks in the nonnegative quadrant.

Previous work on walks in the quadrant, I

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- They used finiteness to show for 22 cases that F is D-finite. For 19 of these, used the orbit sum method and for 3 more, the half orbit sum method.
- Bostan & Kauers (2010): for $d = 2$, explicitly showed the 23rd case (Gessel walks) has algebraic f.

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- Open: proof of asymptotics of f_n for cases 5–16. We solve that here.

Table of All Conjectured D-Finite $F(t;1,1)$ [Bostan & Kauers 2009]

	OEIS	$\mathfrak{S}$	alg	equiv		OEIS	$\mathfrak{S}$	alg	equiv
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

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- The ultimate justification involves Morse theory, but convex analysis often suffices in the combinatorial case.
- We deal in particular with multiple points (locally a transverse intersection of k smooth factors). If $1 \leq k \leq d$, formulae are of the form

$$a_{\mathbf{r}} \sim {\mathbf{z}_*}^{-\mathbf{r}} \sum_{l \geq 0} b_l ||\mathbf{r}||^{-(d-k)/2-l}.$$

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- The GF for walks restricted to the quarter plane has the form

$$f = \operatorname{diag} \frac{xyP(x^{-1}, y^{-1})}{(1 - txyS(x^{-1}, y^{-1}))\,(1-x)(1-y)}$$

where S and P are polynomials:

$$S(x,y) = \sum_{(i,j) \in S} x^i y^j$$

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- The trivariate GF is rational but the diagonal is only D-finite.

Singularities

- The factor $H_1 := 1 - txyS(x^{-1}, y^{-1})$ is a polynomial. Then

$$\begin{aligned}\nabla_{\log} H_1 &:= (x\partial H_1/\partial x, y\partial H_1/\partial y, t\partial H_1/\partial t)\\ &= (-1 + ty\partial S/\partial x, -1 + tx\partial S/\partial y, -1)\end{aligned}$$

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  and thus this factor is everywhere smooth.
- Other singularities come from factors of $(1 - x), (1 - y)$ and possibly from clearing denominators of $xyP(x^{-1}, y^{-1})$.
- When F is combinatorial, there is a dominant singularity for direction $\mathbf{1}$ lying in the positive orthant.

Critical points

- H_1 contains a smooth critical point (x, y, t) for the direction $(1, 1, 1)$ if and only if $\nabla S(x^{-1}, y^{-1}) = 0$.

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- If S has a vertical axis of symmetry, then $(x^2 - 1) \sum_j y^j = 0$.

Structure of G

- Write

$$S(x,y) = y^{-1}A_{-1}(x) + A_0(x) + yA_1(x)$$
$$= x^{-1}B_{-1}(y) + B_0(y) + xB_1(y).$$

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- G is generated by the involutions (considered as algebra homomorphisms)

$$(x,y) \mapsto \left(x^{-1}\frac{B_{-1}(y)}{B_1(y)}, y\right)$$
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- If S has vertical symmetry then $B_1 = B_{-1}$, these maps commute, and G has order 4.

Vertical axis of symmetry, I

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- Otherwise, there is a double point ($k=2$) at $(1,1,1/|S|)$. Its contribution is nonzero if and only if the numerator does not vanish and the direction $(1,1,1)$ lies in a certain cone.

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- The direction lies in the cone iff $\partial S/\partial x(1, 1) \geq 0$, iff $|S_1| \geq |S_{-1}|$ (happens in Cases 1–10).
- Thus for Cases 5–10 we have leading term $C|S|^n n^{-1}$.

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- This holds in Cases 11–16.

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- This happens in all cases 11–16. The numerator simplifies at the smooth point to $(1+x)(1-y^2|S_{-1}|/|S_1|)$, which is zero from the critical point equation for y.
- The leading term asymptotic is $C(|S_0| + 2\sqrt{|S_1||S_{-1}|})^n n^{-2}$.

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- If the drift is nonpositive, asymptotics come from the highest smooth point.

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- If this is positive, there are more possible walks that don't cross the boundary, so the quarter plane restriction is encountered less often. Asymptotics come from the double point $(1, 1, 1/|S|)$.
- If the drift is nonpositive, asymptotics come from the highest smooth point.
- This explains Cases 1–16 in a unified way. We could derive higher order asymptotics too (e.g. using Sage package implementing Raichev & Wilson papers).

Other cases

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- Cases 17–19 also follow as above, with slightly different formulae and more work.
- Cases 20–23 are harder. We don't have a nice diagonal expression, and the conjectured asymptotics show that analysis will be trickier. However the GFs are known to be algebraic and 1-dimensional methods can be used.

Extensions

- We can derive similar expressions for the number of walks returning to the x-axis, the y-axis, or the origin. A very similar analysis proves recently conjectured asymptotics of Bostan, Chyzak, van Hoeij, Kauers, and Pech.

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- Usually, the asymptotics are changed by a factor of n or $\sqrt{n}$. Sometimes the exponential rate changes, depending on the shape of the step set.
- Our approach allows for unified analysis of rational trivariate GFs, which provides results and insight, rather than ad hoc analysis of complicated univariate GFs, which provides results sometimes and no insight.

Possible future work

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- Walks in a Weyl chamber (Gessel & Zeilberger) yield very similar generating functions, analysable in the same way.

Appendix: problems with univariate methods

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- Although such a representation should, in principle, allow one to rigorously determine asymptotics, in practice this depends on computing hard integrals of hypergeometric functions.