Algebras of my acquaintance

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Preliminaries

- R is an associative ring/algebra over a field K (usually $\mathbb{C}$)
- G is a group (finite, discrete, algebraic, Lie)

On algebra

Chevalley: Algebra plays within mathematics the role which mathematics plays for physics. Algebra offers a language in which to express mathematical facts and a variety of patterns of reasoning, put in a standard form. Algebra is not an end in itself; it has to listen to outside demands issued from various parts of mathematics.

Shafarevich: Algebra is the study of structures arising in “measurement” or “coordinatization” of mathematical objects.

Galileo: Measure everything that is measurable, and make measurable everything that is not so.

"Every ring is a coordinate ring"

- Descartes et seq.: irreducible algebraic variety V $\leftrightarrow R = \mathbb{C}[x_1, \ldots, x_n]/P$ for some prime ideal P. Recapture V from R via space of maximal ideals.
- Gelfand: every commutative C* algebra has the form $C(X)$ for some compact Hausdorff X and vice versa
- Grothendieck: we interpret EVERY commutative ring as the coordinate ring of something ("schemes"). Particularly useful in number theory, allowing geometric insight.

Noncommutative geometry

- M. Artin's programme: extend commutative results to certain noncommutative rings (cf Connes, noncommutative *differential* geometry) and try to think geometrically about "noncommutative spaces".
- Uses projective, not affine geometry.
- Module-theoretic, very homological ("connected graded Noetherian domains of dimension 3", "Auslander regular, Gorenstein rings").

Quantum plane

- $\mathbb{C}\langle x, y \mid xy = qyx\rangle, q \in \mathbb{C}^*$, "coordinate ring of quantum plane".
- Primitive ring for generic q, unlike $q = 1$ case.
- Try to recapture some geometry by analyzing prime, primitive and maximal ideals.
- Behaviour depends crucially on whether q is root of 1.
- Higher dimensional "quantum algebras" also occur.

“Every ring is a ring of operators”

- Module $\equiv$ representation $\equiv$ homomorphism $R \to \mathrm{End}(V)$. We lose information if representation is not faithful, but considering ALL representations together compensates.
- Representation theory: interplay between abstract structure and concrete realizations. Essential for applications (physics); very useful even for structure theory ($p^a q^b$ theorem, semisimple Lie algebras).
- Basic problems: given an algebra, find all its representations and show how to decompose an arbitrary one into “nicer” ones.

Algebras in representation theory

General strategy: if it's not a ring, make it one!

- G finite: form group algebra $K[G]$, use theory of f.d. algebras.
- G Lie group: form Lie algebra L, and then universal enveloping algebra $U(L)$.
- Also have enveloping algebras of Lie superalgebras, colour algebras (arose in superstring theory) and "quantized" enveloping algebras $U_q(L)$.
- Many other examples: Hecke algebras etc (connections to knots, physics)

Weyl algebra

$$A_1(\mathbb{C}) = \mathbb{C}\langle x, y \mid xy - yx = 1\rangle$$

- $y \mapsto t, x \mapsto d/dt$ gives faithful representation as linear operators on space of smooth functions.
- Representations used in harmonic oscillator problem, "down-up" algebras.
- Algebra of differential operators on algebraic variety $\mathbb{C}$.
- Module theory used to solve Gelfand's ICM54 problem on meromorphic extension of analytic function.

Weyl algebra II

- A_1 is a simple algebra, so all modules are infinite-dimensional. Distinguish them by *GK-dimension*.
- Basis $\{x^i y^j \mid i, j \geq 0\}$, computation fairly easy.
- A_1 is ubiquitous: every primitive factor of $U(L)$, L nilpotent, is a Weyl algebra.
- In more variables, can form A_n in same way.
- An analogue: $A_1(\mathbb{C}; q) = \mathbb{C}\langle x, y \mid xy - qyx = 1\rangle$; representation using q-difference operator; no longer simple.

Primitive ideals

- Irreducible representations (external) $\leftrightarrow$ simple modules $\leftrightarrow$ primitive ideals (internal)
- A huge theory (Dixmier et seq.) for $U(L)$, L a Lie algebra
- Jacobson radical $J(R) = \bigcap\{\text{primitive ideals}\}$ — the obstruction to understanding R in terms of simple modules
- I have worked on describing $J(U(L))$ when L is a f.d. Lie superalgebra

Hopf algebras abound

- $K[G]$ (group actions) and its dual (group gradings).
- $U(L)$, L Lie algebra and $u(L)$, L restricted Lie.
- $\mathcal{O}(G)$ algebra of regular functions on group.
- Incidence Hopf algebra of a locally finite poset; correct setting for generating functions, umbral calculus, difference operators, etc.
- Invariants of physical models, knots, links, etc.
- "Measure" of extensions of von Neumann algebras and in Galois theory.

Everybody knows what a Hopf algebra is

- Formal definition annoyingly complicated.
- An algebra H whose dual H^* is also an algebra, with a connection between the two structures
- H has a counit ("trivial representation") and a comultiplication $\Delta : H \to H \otimes H$ describing how an element acts on a product

Examples: $\Delta(g) = g \otimes g$ for $g \in G \subset K[G]$ gives $g \cdot (ab) = (g \cdot a)(g \cdot b)$; $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in L \subset U(L)$ (Leibniz rule!)

“Every Hopf algebra comes from a group”

- G Lie group, $\mathcal{O}(G)$ algebra of regular functions on G has Hopf algebra structure, and G can be recaptured from it.
- Deform $\mathcal{O}(G)$ as Hopf algebra to get $\mathcal{O}_q(G)$. Then $U_q(L)$ is the (continuous) dual of this.
- A noncommutative, noncocommutative Hopf algebra is called (by some) a “quantum group” and its modules “quantum spaces”.
- $U_q(L)$ used by Drinfeld to solve QYBE.

Finite-dimensional Hopf algebras over $\mathbb{C}$

- Currently much activity — Kaplansky's 1975 conjectures falling one by one.
- A move is afoot to classify the semisimple ones (group algebras are "trivial").
- Generalizing many facts on group algebras (e.g. class equation, Burnside's theorem on irreducible representations).
- Galois theory with Hopf algebras (e.g. normal basis theorem).

Hopf Galois extensions

- Common generalization of classical Galois extensions of fields/rings, differential Galois extensions, crossed products. Setup is ring extension $R \subseteq A$ with $R = A^H$, invariants.
- There are Galois field extensions which are trivial for $K[G]$ but which are H-Galois for another H.
- Given an extension $R \subset A$, understand how primitive ideals in one ring relate to the other (my recent work). Applications to quantum groups and affine algebraic groups.

Summary

- Ring and module concepts give coherence, unity, elegance.
- Hopf algebras are finally worth the effort.
- Noncommutative “everything” is the way of the future.
- New and interesting algebras are sprouting up all the time (“the ring-of-the-month club”).

The future

- Au revoir et à bientôt
- Auf wiedersehen
- Adiău
- Khodaa haafez

Collaboration welcome!