The Mathematics of Elections

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Overview

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- Choosing a single alternative as a group requires aggregating individual preferences.
- There are many methods for doing this, and some can lead to very paradoxical outcomes, many of them connected to other areas such as statistics.
- Choosing a representative parliament can also lead to weird outcomes, in particular power imbalances between parties.

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- However 6 people like Indian least, so perhaps we want to allow people to choose which type they don't want. Under the veto rule, T wins.
- However, Mexican has substantial support, so perhaps we should award points to each type, say 2, 1, 0 in order of preference. Under the Borda rule, M wins.

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- Instant runoff: rank candidates; repeatedly eliminate candidate with lowest number of first places (in our previous example, Mexican wins).
- Copeland rule: compare candidates pairwise as in a tournament; a beats b if a majority of voters prefer a over b; highest total wins.

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- We can find a decision by voting on 2 candidates at a time, but the result depends on the order we choose (common source of dirty tricks in parliamentary settings).
- What should be "the" winner here?

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- Furthermore b is the Condorcet winner, but c is the Condorcet loser.
- It has been proved that every Condorcet method (one that elects the Condorcet winner whenever it exists) is vulnerable to this effect.

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- We can use this to compute the probability of such a paradox occurring, for reasonably nice distributions of preferences.

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- Theorem: Every social choice function that is not a dictatorship allows strategic voting in some election instance when the number of candidates is at least 3 and the number of voters at least 2.
- Note that if we allow ties to be unbroken, we could always elect all candidates. So there is a tradeoff between decisiveness and the various types of non-paradoxical behaviour that we instinctively desire.

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- Popular methods among theorists are approval voting, Borda's rule, and some Condorcet consistent rules.

Early researchers

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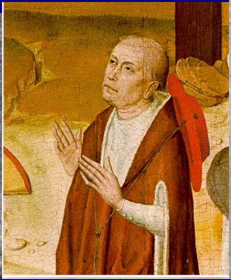
- Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet: 18th century French philosopher and mathematician. Championed use of pairwise voting. Active in progressive and revolutionary politics, and served in National Assembly. Died mysteriously in prison during the Reign of Terror.

Pictures of early researchers


Weighted voting games

- If we assume bloc voting (strict party discipline), each party has a fixed weight w_i equal to its number of seats. There is a fixed quota q (usually just over half the total size of Parliament) for a bill to pass. Denote this weighted voting game $[q; w_1, \ldots, w_k]$.

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- Similar voting setups hold in corporation (weights determined by amount of stock owned).
- All that is important is the list of winning coalitions: those who can force passage of a bill/motion.

Example: NZ Parliament

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- Note that parties 2-5, and parties 6-9, are equivalent in that they belong to the same minimal winning coalitions.

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- Treaty of Nice (currently in force) uses weights (totalling 345) but has more conditions. A coalition is winning iff it has at least 50% of the countries, 74% of the weights, 62% of the population.
- Treaty of Lisbon (from 2014): coalition wins iff it has at least 55% of countries and 65% of population. This method is easily implemented if new members join, and avoids complex negotiations over weights.

Single winner district systems

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- Could use Borda, ... but almost no one does.
- These methods can magnify small margins locally into large margins globally. For example, under plurality it is possible to have 49.999% support overall, and win zero districts.

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- The seat distribution in Parliament: Lib 177; Ref 52; PC 2; BQ 54; NDP 9; Independent 1.
- Not only is seat distribution not an increasing function of popular vote, there are some huge disproportionalities. Every party other than Lib belongs to no minimal winning coalitions: Lib is a dictator.

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- The apportionment problem: given a fixed size N of Parliament, how to allocate seats to parties as proportionally as possible, given that we must use an integer number of seats?
- At least 5 methods have been suggested. They have all been used in the context of allocating states numbers of representatives to the US Congress, and in other contexts. Some have alternative names, e.g. Webster method is equivalent to St Lagüe method used for MMP in NZ.

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Hill Same as Webster, but round using geometric mean instead of arithmetic.

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- Repeat with $660000, 245100, 104900$: Hamilton gives $65, 24, 11$.

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- Suppose 3 parties A,B,C with votes 657000, 237000, 106000 respectively, and 100 seats to distribute.
- Here $D = 10000$ and the standard allocations $65.7, 23.7, 10.6$.
- Hamilton and Jefferson give $66, 24, 10$; Adams, Webster, Hill $65, 24, 11$.
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- Repeat with house size 101: Hamilton gives $66, 25, 10$.

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- Repeat with house size 101: Hamilton gives $66, 25, 10$.
- Hamilton's method can allow the Alabama paradox and population paradox.

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- Finding paradoxical situations systematically is again a matter of solving equations and inequalities.
- Theorem (Balinski and Young): at least one paradox is unavoidable.

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- Several numerical measures of power have been proposed. None are without controversy. The most famous are the Shapley-Shubik index and the Banzhaf index.
- Most but not all indices assign more power to those with larger weight.

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- The Banzhaf measure is the probability that i is decisive for a uniformly randomly chosen coalition containing i.
- This measure was discovered first by Penrose and rediscovered by Banzhaf (1965) in the context of a court case over the Nassau County Board of Supervisors (weighted voting game $[16; 9, 9, 7, 3, 1, 1]$ - spot the dummies).

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- Example: Coleman (1971) introduced the measure $C = |W|/2^n$, the probability that a uniformly chosen random coalition is winning. The basic model is of each voter deciding Yes or No independently with probability $1/2$.
- Example: EUCM under Treaty of Nice has $C \approx 0.02$. Very hard to pass any motion, hence the need for reform in Treaty of Lisbon (which currently has $C \approx 0.13$).

NZ Parliament example revisited

- Power vector for Shapley-Shubik:
  $\sigma = (0.611, 0.090, 0.090.0.090, 0.090, 0.008, 0.008, 0.008, 0.008)$.
- (Normalized) power vector for Banzhaf:
  $\beta = (0.657, 0.074, 0.074.0.074.0.074, 0.011, 0.011, 0.011, 0.011)$.
- In reality not all coalitions of a given size are equally likely. However with almost any power index, it is clear that the largest party has very large power.
- Coleman index: $C = 0.492$. This is large.

More recent researchers

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- Martin Shubik: American economist (Yale University).

Pictures of more recent researchers




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- `http://www.referendum.org.nz/` contains much information about the various systems, much of it purely qualitative.

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- (not a hard constraint) Under STV, "It is likely the 120 MPs would be divided between 24 and 30 electorates, each with 3 to 7 MPs."

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- The various single winner district systems are unlikely to produce very different seat allocations.
- There is a very large difference in the proportionality of the various systems.
- Comparing power distribution of parties under each system is important, not just seat allocations.

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- Try it out!

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