### 2011 referendum options simulator

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# Key references

FM1998 D. Felsenthal, M. Machover. The Measurement of Voting Power, Edward Elgar, 1998.



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- DNW1981 P. Dubey, A. Neyman, R. J. Weber. Value theory without efficiency. Mathematics of Operations Research 6 (1981), 122–128.



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# Key motivating examples of simple games

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- Disequilibrium games: for a given noncooperative game and fixed profile of actions, declare a subset to be winning if is a witness to the profile not being a strong Nash equilibrium. Examples: voting rules with the sincere profile.

# Basic concepts of TU games and simple games

monotonicity 
$$S \subseteq T \implies v(S) \le v(T)$$
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We usually assume monotonicity for simple games, in which case we need only specify the minimal winning coalitions in order to specify the game. A dummy is not an element of any minimal winning coalition.





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- ▶ Treaty of Nice (currently in force) uses weights (totalling 345) but has more conditions. A coalition is winning iff it has at least 50% of the countries, 74% of the weights, 62% of the population.
- Treaty of Lisbon (from 2014): coalition wins iff it has at least 55% of countries and 65% of population. This method is easily implemented if new members join, and avoids complexity of Audulin negotiations over weights.

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Dummy  $\xi_i(G) = 0$  if i is a dummy in G.



# The Shapley value

► Shapley proved that there is a unique efficient value satisfying Anonymity, Dummy and Linearity. Explicitly it is given by

$$\sigma_i(G) = \frac{1}{n!} \sum_{S \subset X} (n - |S|)!(|S| - 1)! [v(S) - v(S \setminus \{i\})].$$



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- ➤ This can be interpreted in terms of a coalition-formation or bargaining model, not without controversy.
- ▶ The idea is to consider all possible orders of players with equal probability, and give player *i* its expected marginal contribution.





# Beyond efficiency

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where  $p(n,k) \ge 0$  and the following identities hold

$$\sum_{k} {n-1 \choose k-1} p(n,k) = 1$$
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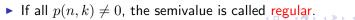
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- ▶ Regular semivalues satisfy Young sensibility: if the marginal contribution to each S is strictly higher in one game than another, then the  $\xi_i$  have the same relation.
- The class of probabilistic values is even more general the coefficients p can depend on S and not just on |S|.



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$$\Phi(G) - \Phi(G_{-\{i\}}) = \xi_i(G)$$

for all  $G=(X,v)\in \mathcal{G}$  such that  $X\neq \emptyset$ . Here  $G_{-\{i\}}$  is the game with player set  $X\setminus \{i\}$  and the same v.





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- ▶ The initial condition  $\Phi(\emptyset, v) = 0$  is usually assumed.
- ► There is a unique efficient value having a potential function, and it is the Shapley value. Explicitly, the potential looks like

$$\Phi(G) = \sum_{k} \frac{1}{k \binom{n}{k}} \sum_{|S|=k, S \subset X} v(S).$$



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- The answer: ξ has a potential if and only if it satisfies Myerson's balanced contributions axiom:

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In particular, every semivalue has a potential function. Explicitly:

$$\Phi(G) = \sum_{k} p(n,k) \sum_{|S|=k} v(S)$$

the expected value of a coalition chosen randomly according to the weights p(n,k).

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- The underlying idea is to measure the extent to which a player is important for winning coalitions. A key observation is that for weighted majority games, the relative power of the players can vary dramatically from the relative weights.
- Much has been written, but no standard definition of a power measure/index has been agreed. There are many conceptual confusions in the literature and some controversy.

### Concepts of power in simple games

▶ Felsenthal and Machover: there are at least two kinds of "power" and previous authors have conflated them. P-power deals with distribution of the spoils of power; I-power deals with decisiveness. The former may not be well-defined, but the latter is. The former is always relative, but the latter is absolute.





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- Laruelle and Valenciano: there are at least two kinds of situation, and previous authors have conflated them. Take it or leave it committees must only vote; pure bargaining committees involve complex negotiations. In the first case, decisiveness is not as important as "success". P-power in fact is related to decisiveness via bargaining committees.

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- The basic idea of I-power is to measure how often each player is decisive.
- ► The most common measure is the Banzhaf measure, the specialization of the Banzhaf semivalue: the probability that i is decisive for a uniformly randomly chosen coalition containing i.
- This measure was discovered first for simple games by Penrose and rediscovered by Banzhaf (1965) in the context of a court case over the Nassau County Board of Supervisors (weighted voting game [16; 9, 9, 7, 3, 1, 1]).

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- Example: EUCM under Treaty of Nice has  $C \approx 0.02$ . Very hard to pass any motion, hence the need for reform in Treaty of Lisbon (which currently has  $C \approx 0.13$ ).

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- ▶ Idea: define an individual decisiveness measure to be the restriction of a semivalue to SG. Note: such functions satisfy Anonymity, Positivity, Dummy, and Modularity (the replacement for Linearity).

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- ▶ Efficiency is an obvious requirement if a fixed prize is being divided, and this usually leads to the Shapley value (Shapley-Shubik index). Otherwise efficiency is meaningless and should be dropped.
- Collective decisiveness certainly is important, so individual power measures do measure something important, even if it is not "power".

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- ▶ Let  $\overline{Q}$  be the expected number of queries required. Then  $n+1-\overline{Q}$  is essentially the potential of the Shapley value of the manipulation game.
- ▶ If each voter can have a different cost to recruit (as in bribery), a TU (cost) game is more appropriate than a simple game, but similar ideas should work.

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- This is a substantial advance in the theory of measures of manipulability. There has been no definition of what such a measure should be, and no desirable axioms listed. Previous measures have been rather crude.
- ► Some (not all) of the previously used measures can be interpreted as semivalues, but not always regular ones. Our new approach allows a principled choice of measure for a given situation.

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- ▶ Assuming risk-neutral voters, there are two types of minimal winning coalitions: a single cba (respectively bca) votes for b (respectively c). Note that this game is not strong.
- ▶ The winning coalitions are those containing at least one bca or cba. The Coleman index is 15/16. The relative Banzhaf (or Shapley-Shubik) index of each cba or bca is 1/4, and abc voters are dummies.

