

PRIMENESS OF THE ENVELOPING ALGEBRA OF HAMILTONIAN SUPERALGEBRAS

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ABSTRACT. In 1990 Allen Bell presented a sufficient condition for the primeness of the universal enveloping algebra of a Lie superalgebra. Let Q be a nonsingular bilinear form on a finite-dimensional vector space over a field of characteristic zero. In this paper we show that Bell's criterion applies to the Hamiltonian Cartan type superalgebras determined by Q , and hence obtain some new prime noetherian rings.

1. INTRODUCTION

Let $L = L_+ \dot{+} L_-$ be a finite-dimensional Lie superalgebra over a field of characteristic zero, and let $U(L)$ be its universal associative enveloping (super)algebra. In [Bel90] Bell gave the following simple criterion for primeness of $U(L)$. Let $\{f_1, \dots, f_n\}$ be a basis for the odd part L_- of L . Form the *product matrix* $M = ([f_i, f_j])$, considered as a matrix over the symmetric algebra $S(L_+)$. If $\det M \neq 0$ then $U(L)$ is prime.

The primeness question for enveloping algebras of the classical simple Lie superalgebras has been settled completely in [Bel90] and [KK96]. An investigation into the applicability of Bell's criterion to the Cartan type Lie superalgebras was begun in [Wil96], continued in [WPW] and is concluded in this paper and [WP].

Here it is shown that the Hamiltonian algebras $H(Q)$ and $\widetilde{H}(Q)$ satisfy Bell's criterion. This immediately gives

Theorem. *Let K be a field of characteristic zero, let $n \geq 4$ and let Q be a nonsingular bilinear form on a K -vector space of dimension n . Then $U(H(Q))$ and $U(\widetilde{H}(Q))$ are prime. \square*

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As a consequence of the results of the above-mentioned papers we have the following theorem.

Theorem. *Let L be a finite-dimensional simple Lie superalgebra over an algebraically closed field of characteristic zero. Then L satisfies Bell's criterion, and hence $U(L)$ is prime, unless L is of one of the types: $b(n)$ for $n \geq 3$; $W(n)$ for odd $n \geq 5$; $S(n)$ for odd $n \geq 3$. \square*

2. THE HAMILTONIAN SUPERALGEBRAS

Good references for basic facts about Lie superalgebras are [Kac77] and [Sch79].

Let K be a field of characteristic zero, n a positive integer and V an n -dimensional K -vector space. Let $\Lambda = \Lambda(V)$ be the Grassmann algebra of V . Recall that Λ is an associative \mathbb{Z} -graded superalgebra. Fix a basis $\{v_1, \dots, v_n\}$ for V . For each ordered subset $I = \{i_1, i_2, \dots, i_r\}$ of $N = \{1, 2, \dots, n\}$ with $i_1 < i_2 < \dots < i_r$, let v_I be the product $v_{i_1}v_{i_2}\cdots v_{i_r}$. The set of all such v_I forms a basis for Λ , where we interpret $1 = v_\emptyset$ as the empty product, and the homogeneous component Λ_r is spanned by the v_I with $|I| = r$. The anticommutativity of multiplication in Λ implies that

$$(1) \quad v_I v_J = \begin{cases} \pm v_{I \cup J} & \text{if } I \cap J = \emptyset, \\ 0 & \text{if } I \cap J \neq \emptyset. \end{cases}$$

The algebra $W = W(V)$ is the \mathbb{Z} -graded Lie superalgebra consisting of all superderivations of Λ . Every element of W maps V into Λ and since it is a superderivation it is completely determined by its action on the generating subspace V . It follows that W can be identified with $\Lambda \otimes_K V^*$ and we shall henceforth do so.

Under this identification the map $\partial_i = \partial/\partial v_i$ corresponds to the dual of v_i which we shall also denote ∂_i . The set of all $v_I \otimes \partial_i$ is then a homogeneous basis for W , the degree of such an element being equal to $|I| - 1$.

For each symmetric bilinear form Q on V there are subalgebras of W denoted by $H(Q)$ and $\widetilde{H}(Q)$. Their (rather complicated) definition can be found in [Sch79, page 194] or [Kac77, section 3.3.2]. If we extend K to its algebraic closure then all such algebras become isomorphic to the algebra $\widetilde{H}(n)$ (respectively $H(n)$) defined below. Since Bell's criterion holds over a given field if and only if it holds over the algebraic closure of that field, it is sufficient to verify the criterion for the algebras $\widetilde{H}(n)$ and $H(n)$.

We now recall some basic facts about the Hamiltonian superalgebras. The subspace of W spanned by all superderivations of the form

$$D_\lambda = \sum_{i \in N} \partial_i(\lambda) \otimes \partial_i,$$

where $\lambda \in \Lambda$, is a Lie superalgebra called $\widetilde{H} = \widetilde{H}(n)$. \widetilde{H} inherits a natural \mathbb{Z} -grading from W and we have

$$\widetilde{H} = \bigoplus_{r=-1}^{n-2} H_r.$$

The subalgebra $H = H(n) = \bigoplus_{r=-1}^{n-3} H_r = [\widetilde{H}, \widetilde{H}]$ is a simple Lie superalgebra of Cartan type.

The homogeneous component H_r is isomorphic as a vector space (in fact as an H_0 -module) to Λ_{r+2} via $D_\lambda \mapsto \lambda$. Thus the superderivations $x_I = D_{v_I}$, where $\emptyset \neq I \subseteq N$, form a basis for \widetilde{H} , and $\dim H_r = \binom{n}{r+2}$.

3. COMPUTATION

It is known that the multiplication in H satisfies

$$[D_\lambda, D_\mu] = \pm D_{\{\lambda, \mu\}}$$

where $\{\lambda, \mu\} = \sum_i \partial_i(\lambda) \partial_i(\mu)$. Note that this differs slightly from the notation in [Kac77], and that the exact multiplication formula is not needed for our purposes.

It follows from (1) that $\partial_i(v_I) \partial_i(v_J) = 0$ unless $I \cap J = \{i\}$, whence

$$(2) \quad [x_I, x_J] = \begin{cases} \pm x_{I \Delta J} & \text{if } |I \cap J| = 1, \\ 0 & \text{otherwise} \end{cases}$$

where Δ denotes the symmetric difference (Boolean sum). Since Δ is the addition in the usual Boolean ring structure on the power set of N , this implies that for a given $A, I \subseteq N$, the equation $[x_I, x_J] = \pm x_A$ has at most one solution for J . This solution exists precisely when $A \Delta I \neq \emptyset$, that is when $I \not\subseteq A$ and $A \not\subseteq I$. Furthermore if $|I|$ is odd and $|A|$ even then $|J| = |A| + |I| - 2|I \cap A|$ is necessarily odd. Thus every even x_A appears (perhaps with a minus sign) in the product matrix, and each such x_A appears at most once in each row or column.

3.1. n even.

Theorem 3.1. *Let $n \geq 4$ be an even integer. Then $H(n)$ and $\widetilde{H}(n)$ satisfy Bell's criterion.*

Proof. Write $n = 2m$. The highest odd degree occurring in \widetilde{H} and H is $n - 3$. It follows that if we group the basis elements x_I by increasing degree, then the product matrices for both H and \widetilde{H} are the same and that this common matrix has the block reverse triangular structure

$$\begin{pmatrix} H_{-1,-1} & H_{-1,1} & \dots & \dots & H_{-1,n-3} \\ H_{1,-1} & H_{1,1} & \dots & H_{1,n-5} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{n-3,-1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

where $H_{r,s}$ is the block formed by the products of elements of degree r with those of degree s . Furthermore each block on the reverse diagonal is square, since if $r + s = n - 4$ then $\dim H_r = \binom{n}{r+2} = \binom{n}{s+2} = \dim H_s$. The product matrix is nonsingular if and only if each of these blocks is nonsingular.

Fix such a reverse diagonal block $H_{r,s}$ corresponding to products of elements of degree r by those of degree $s = n - 4 - r$. Using the identification of H_{n-4} with Λ_{n-2} we can index the basis elements of H_{n-4} by their (ordered) 2-element complements, for example $y_{13} = x_{N \setminus \{1,3\}}$. We now make the specialization which sends y_{ij} to 0 unless $j - i = m$, and call the m remaining variables $z_1 = y_{11'}, \dots, z_m = y_{mm'}$. For each i let $i' = i + m \pmod{n}$. Note that $(i')' = i$ and $z_i = z_{i'}$. The image B of the block $H_{r,s}$ under this specialization is a matrix whose only possibilities for nonzero entries are $\pm z_i$ for some i .

We shall obtain a further block decomposition of B . Replacing all nonzero elements of B by 1's we obtain a $(0,1)$ matrix which is the adjacency matrix of a unique graph $G = G(B)$. In other words, G has vertices the x_I and an edge joining x_I and x_J if and only if the product $[x_I, x_J]$ remains nonzero under our specialization above. If for simplicity we label the vertex corresponding to x_I by I , there is an edge in G joining I to J if and only if $[x_I, x_J] = \pm z_i$ for some i . We shall say that in this case I and J are joined by an edge of *colour* i .

Finding a block decomposition of B is equivalent to decomposing G into disjoint subgraphs, which we now proceed to do. Fix $i \in N$. We determine exact conditions on I and J for there to exist an edge of colour i joining them. It follows from (2) that this occurs if and only if either $I \cap J = \{i\}$ and $I \cup J = N \setminus \{i'\}$, or $I \cap J = \{i'\}$ and $I \cup J = N \setminus \{i\}$. Thus there is an edge of colour i joining I and J if and only if $|I| + |J| = n$, one of i or i' belongs to both I and J and the other belongs to neither. Furthermore, for a given $I \neq N$, there is at most one edge of a given colour at the vertex I . Also there is at least one edge of some colour at the vertex I : since $I \neq N$, for some i we

must have $i \in I$ and $i' \notin I$.

We now obtain a further block decomposition of B by showing that the set of colours occurring at a given vertex of $G(B)$ is constant on each component. To this end, we first show that vertices distance 2 apart have the same colours. Suppose that I and J are linked by an edge of colour i . Then without loss of generality $I \cap J = \{i\}$ and $I \cup J = N \setminus \{i'\}$. Let K be linked to J . If J and K are linked by an edge of colour j then either $\{i, i'\} = \{j, j'\}$, in which case $K = I$, or $\{i, i'\} \cap \{j, j'\} = \emptyset$. In the latter case we can assume $J \cap K = \{j\}$ and $J \cup K = N \setminus \{j'\}$. Thus $i' \in K$ since $i' \in J \cup K$ but $i' \notin J$. Let $X = J \cup \{i', j'\} \setminus \{i, j\}$. Then $|X| = |J|$, $K \cap X = \{i'\}$, $K \cup X = N \setminus \{i\}$ and so K and X are linked by an edge of colour i . Thus every colour occurring at I also occurs at K , and by symmetry I and K have the same colours.

It follows that if I and J are joined by an edge then they have the same colours, since if an edge of some colour i joins I and L , then J and L have the same colours by above and so the colour i occurs at J . By induction on the length of a path joining two vertices, the set of colours occurring at a vertex is constant on components. This decomposes $G(B)$ into a union of disjoint subgraphs, each corresponding to a given set of colours. Hence B decomposes as a direct sum of smaller blocks, each of which is parametrized by some nonempty subset of the set of colours.

Now fix such a block corresponding to a given set of colours. This matrix is such that in every row and column, each variable which is present occurs exactly once, perhaps with a minus sign. Then specializing all but one of these variables to zero we obtain a nonsingular monomial matrix. This shows that the original product matrix for $H(n)$ and $\widetilde{H}(n)$ is nonsingular. \square

The fact that the noetherian rings $R = U(H)$ and $S = U(\widetilde{H})$ are simultaneously prime is not a surprise. The component H_{n-2} is 1-dimensional, spanned by x say. Since $[x, H] \subseteq [\widetilde{H}, \widetilde{H}] = H$, $\text{ad } x$ stabilizes R . When n is even then $\text{ad } x$ is an ordinary derivation and so S is the differential polynomial ring $R[x; \text{ad } x]$. It is a well-known fact (see for example [Row88, Proposition 8.3.32]) that in this situation R is prime if and only if S is.

3.2. n odd. This case reduces rather easily to the previous one.

Theorem 3.2. *Let $n \geq 5$ be odd. Then $H(n)$ and $\widetilde{H}(n)$ satisfy Bell's criterion.*

Proof. Let M, \widetilde{M} be the product matrices for $H(n), \widetilde{H}(n)$ respectively. The top degree $n - 2$ occurring in $\widetilde{H}(n)$ is odd, and $\dim H_{n-2} = 1$. Thus \widetilde{M} is obtained from M by adding another row and column. Since this procedure either leaves the rank unchanged or increases the rank by 1, it suffices to show that \widetilde{M} is nonsingular.

We decompose \widetilde{M} into 4 blocks as follows. Group the rows indexed by those I for which $n \in I$ together and follow them by the rows for which $n \notin I$. Do the same for the columns. This gives an obvious 2×2 block structure. Make the specialization which sets all even x_I with $n \in I$ to zero. Then \widetilde{M} specializes to a matrix of the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. It suffices to show that X and Y are nonsingular.

The matrix Y has entries which are the pairwise products of the x_I with $I \subseteq \{1, \dots, n-1\}$ and hence is just a product matrix for $H(n-1)$. Thus Y is nonsingular by Theorem 3.1.

Now choose I with $n \in I$. Since $I \neq N$, both $I \not\subseteq N \setminus \{n\}$ and $N \setminus \{n\} \not\subseteq I$ hold and so there is precisely one J with $n \in J$ for which $[x_I, x_J] = \pm x_{N \setminus \{n\}}$. Thus in X every row and column has precisely one occurrence of $\pm x_{N \setminus \{n\}}$, so specializing to zero all variables except this one yields a nonsingular monomial matrix. \square

It is not as obvious *a priori* that the rings $R = U(H)$ and $S = U(\widetilde{H})$ should be simultaneously prime. Let x span H_{n-2} . Then $[x, x] = 2x^2 = 0$ and so $S = R[x; \delta]/I$, where δ is the skew derivation $\text{ad } x$ and I is the ideal generated by x^2 . Obviously S prime implies R prime but the converse for extensions of this type requires some hypothesis on δ , usually that it be outer in some way. By arguments similar to those in [BM86], assuming that δ is outer on the symmetric Martindale quotient ring of R should suffice. However it is not clear how to easily verify this property.

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