PRIMENESS OF THE ENVELOPING ALGEBRA OF A CARTAN TYPE LIE SUPERALGEBRA

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Abstract. We show that a primeness criterion for enveloping algebras of Le superalgebras discovered by Bell is applicable to the Cartan type Lie superalgebras $W(n)$, n even. Other algebras are considered but there are no definitive answers in these cases.

Allen Bell has shown in [B] that if L is a finite-dimensional Lie superalgebra over a field of characteristic zero, then the primeness of the universal enveloping algebra $U(L)$ is implied by the nonsingularity of the *product matrix* $([f_i, f_j])$. Here $\{f_1, \ldots, f_s\}$ is a basis for the odd part L_1 of L, and the matrix is defined over the polynomial algebra $S(L_0)$. Bell used this result to show that the universal enveloping algebra of a finite-dimensional classical simple Lie superalgebra is prime except possibly in the case of algebras of type $b(n)$. This outstanding case was settled in the negative by a direct argument ([KK], [Z]). An obvious next step is to consider the simple algebras of Cartan type.

Our main (new) result here is that for even $n \geq 4$, $U(W(n))$ is prime.

A good basic reference for the properties of Cartan type Lie superalgebras is [S].

1. $W(n)$

1.1. Basics. Let K be a field of characteristic zero and let $\Lambda = \Lambda(V)$ be the exterior algebra of the vector space $V = K^n$. Then Λ is an associative superalgebra of dimension 2^n where the $\mathbb{Z}_2$-grading is induced by the $\mathbb{Z}$-grading given by degree. If $\{v_1, \ldots, v_n\}$ is a basis for V, then a basis for Λ is given by all $v_I = v_{i_1} v_{i_2} \cdots v_{i_s}$ where $I = \{i_1, \ldots, i_s\}$ is an ordered subset of $N = \{1, \ldots, n\}$. Of course the v_i anticommute and the centre of Λ is the span of all v_I with $|I|$ even. Let $W = W(n) = D(\Lambda)$ where D denotes the Lie superalgebra of superderivations. Then $W = \oplus_{r=-1}^{n-1} W_r$ is naturally $\mathbb{Z}$-graded and this induces the $\mathbb{Z}_2$-grading. Here the graded component W_r consists of all superderivations which map V into Λ_{r+1}. For homogeneous $\partial \in W$ and $x, y \in \Lambda$, we have $\partial(xy) = \partial(x)y \pm x\partial(y)$ where the $-$ occurs if and only iff both x and ∂ are odd. Every element of W restricts to a linear

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map $V \longrightarrow \Lambda$. Conversely every element of W arises in this way and we have the isomorphism of vector spaces $W \cong \Lambda \otimes_K V^*$. We shall use this identification for computations.

Fix an ordered basis $\{v_1, \dots, v_n\}$ for V and for $1 \leq i \leq n$ let ∂_i be the element of W such that $\partial_i(v_j) = \delta_{ij}$. An explicit basis for $\Lambda \otimes V^*$ is then given by the set of all $v_I \otimes \partial_i$, where here we identify ∂_i with its restriction to V. We use the isomorphism above to transfer the grading and multiplication from W to $\Lambda \otimes V^*$. It follows that the basis elements $v_I \otimes \partial_i$ belonging to to the graded component of degree r are those with $|I| = r+1$ and so $\dim W_r = n\binom{n}{r+1}$ and $\dim W = n2^n$. We obtain the multiplication formula for *odd* elements

$$(*) \qquad [v_I \otimes \partial_i, v_J \otimes \partial_j] = v_I \partial_i(v_J) \otimes \partial_j + v_J \partial_j(v_I) \otimes \partial_i.$$

Note that it is immediate from this formula that the product is zero if $|I \cap J| \geq 2$.

1.2. Computation. From now on we restrict to the case where n is even. The aim is to show that the determinant of the product matrix, which is a homogeneous polynomial in $n2^{n-1}$ variables, is not the zero polynomial. Even for $n = 4$ it is hopeless to compute the determinant directly. We rely heavily on finding a good specialization, that is we want to assign values in K to some of the variables and show that the determinant of the matrix resulting from this is nonzero. Of course this is not necessary — any homomorphism from the polynomial ring in $n2^{n-1}$ variables would suffice — but it is the most obvious method. The specialization we employ below is a very simple one (we just set some variables to zero and others to 1).

We first consider the specialization which sets all the even components of W to zero except the component of maximal degree $n-2$. Then the product matrix specializes to a block monomial matrix P. The nonzero blocks are the product submatrices P_r formed by W_r and W_{n-2-r} with r odd, $-1 \leq r \leq n-1$. Note that

$$\dim W_r = n\binom{n}{r+1} = n\binom{n}{n-r-1} = n\binom{n}{(n-r-2)+1} = \dim W_{n-2-r},$$

so the P_r are square. We make the further specialization which sends $v_I \otimes \partial_i$ to zero if $i \in I$. The remaining n variables are $x_k = v_{N\setminus\{k\}} \otimes \partial_k$, and we specialize these all to 1. Thus the product matrix specializes to a block monomial matrix Q over K. We shall show that each block Q_r (the specialization of P_r) is nonsingular.

We aim to decompose further each Q_r. Say that (I, i) is *linked* to (J, j) if the image of the product $[v_I \otimes \partial_i, v_J \otimes \partial_j]$ remains nonzero in Q. This has an obvious graph-theoretical interpretation. We now calculate explicitly conditions on (I, i) and (J, j) which are equivalent to their being linked.

We see that the product in (*) is nonzero in W only if $i \in J$ or $j \in I$. Also it is clear that for the product in (*) to remain nonzero under our specialization it must lie in the span of x_i and x_j.

First suppose that $i = j$. The first term on the right side in (*) remains nonzero under our specialization if and only if $i \in J$ and

$$I \cup (J \setminus \{i\}) = N \setminus \{i\}.$$

Since $|I| + |J| = n$, this is equivalent to $I \cap J = \emptyset$ and $I \cup J = N$. Thus $i \notin I$. Similarly, if the second term remains nonzero then $i \in I$, $i \notin J$, $I \cap J = \emptyset$ and

$I \cup J = N$. Hence at most one term on the right side of (*) remains nonzero, and the corresponding entry of Q equals ± 1.

Now suppose that $i \neq j$. The first term in (*) remains nonzero if and only if $i \in J$ and

$$I \cup (J \setminus \{i\}) = N \setminus \{j\}.$$

This is equivalent to the conditions $I \cap J = \{i\}$, $I \cup J = N \setminus \{j\}$. Similarly the second term remains nonzero if and only if $j \in I$, $I \cap J = \{j\}$, $I \cup J = N \setminus \{i\}$. Note that again both terms cannot remain nonzero simultaneously and so the product in (*) specializes to 0 or ± 1.

Thus (I, i) and (J, j) are linked if and only if exactly one of the following conditions is satisfied:

(1) $i \in I, j \notin J, I \setminus \{i\}$ and $J \cup \{j\}$ are mutually complementary in N
(2) $i \notin I, j \in J, I \cup \{i\}$ and $J \setminus \{j\}$ are mutually complementary in N

In each case the corresponding entry in Q is just ± 1.

We now determine the components of the graph alluded to above and thereby obtain a further block decomposition.

Say a pair (I,i) is of type (I,r) if $i \notin I$ and $|I| = r + 1$, and of type (II,r) if $i \in I$ and $|I| = r + 1$. If $r = -1$ then all the (I, i) are of type I, and if $r = n - 1$ all are of type II. Otherwise both types of variables occur. Now variables of the same type are not linked, and so for $1 \leq r \leq n - 3$ the matrix Q_r is, up to a reordering of rows and columns, the direct sum of two square blocks. The symmetry of the product matrix means that we need only consider the blocks formed by the product of type I by type II variables. If (I, i) is of type (I,r) then by the above $A = I \cup \{i\}$ has size $r + 2$, and $B = N \setminus A$ has size $n - 2 - r$. Conversely, given mutually complementary A and B with respective sizes $r + 2$ and $n - 2 - r$, let i and j be elements of A. Then $(A \setminus \{i\}, i)$ and $(B \cup \{j\}, j)$ are linked and of type (I,r) and (II,n-2-r) respectively. It follows that each component of the graph consists of all the (I, i) and (J, j) determined by a given A. Thus after reordering rows and columns if necessary, each Q_r can be taken to be block diagonal, where the blocks have size $r + 2$ and each block has every entry either 1 or -1.

Now we fix such a block M of size $r + 2$. It suffices to prove M nonsingular. For this, we need to determine the exact placement of the $\pm$ signs in M, which is facilitated by a slight change of basis. For each type (II,n-r) variable (J, j), write $J' = J \setminus \{j\}$ and order J so that $J' < j$ and J' is ordered naturally. This changes the basis element v_J to an element w_J which differs from v_J by a factor of ± 1. If we replace the basis elements v_J corresponding to the (J, j) of type (II,n-r) by the w_J, then all of the above block decompositions hold for the new basis just formed. Thus it is no loss of generality to assume that $v_J = w_J$ for these (J, j) and we shall do so from now on.

We need a few calculations to simplify the work below. From the way that we have ordered J, the fact that $|J|$ is even and the fact that ∂_j is odd it follows that

$$\partial_j(v_J) = \partial_j(-v_j v_{J'}) = -v_{J'}.$$

Also if $j \in I$ then

$$v_j \partial_j(v_I) = v_I$$

(since if j is in an even position in I then $\partial_j(v_I)$ incurs a minus sign, but then v_j requires an odd number of interchanges to get to its proper position — the other case is similar).

For each (I, i) of type (I,r) , if $J = I^c$ and $j = i$ we have

$$[v_I \otimes \partial_i, v_J \otimes \partial_j] = v_I \partial_i(v_J) \otimes \partial_i = -v_I v_{J'} \otimes \partial_i,$$

whereas for the other (J, j) we have

$$[v_I \otimes \partial_i, v_J \otimes \partial_j] = v_{J'} v_j \partial_j(v_I) \otimes \partial_i = v_{J'} v_I \otimes \partial_i = v_I v_{J'} \otimes \partial_i,$$

the last equality holding because $|I|$ is even so v_I is central in Λ. Thus by reordering the rows or columns of M and multiplying columns or rows by -1 if necessary, we can arrange so that the only -1 entries occur along the leading diagonal and the other entries are all 1, i.e. M can be taken to be

$$\begin{pmatrix} -1 & 1 & \dots & 1 \\ 1 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -1 \end{pmatrix}$$

It is well known (and straightforward to show) that such a matrix is nonsingular if its dimension is not 2×2. Since $r \neq 0$ (it is odd), M is nonsingular. This shows that Q, and hence P, is nonsingular.

Combining the above with Bell's results yields

Theorem. If n is even and $n \geq 2$, then $U(W(n))$ is prime. □

2. Other Cartan type algebras

Among the subalgebras of $W(n)$ which are also simple Lie superalgebras is the special algebra $S(n)$. The exact definition need not concern us here. The essential properties required are that $S(n)$ has a $\mathbb{Z}$-grading

$$S(V) = \bigoplus_{r=-1}^{n-2} S_r,$$

and that $\dim S_r = (n - r - 1)\binom{n+1}{r+1}$. Choosing bases for S consistent with this grading gives a natural block structure to the product matrix. If $n \geq 3$ is odd then this matrix looks like

$$\begin{pmatrix} 0 & S_0 & \dots & \dots & S_{n-3} \\ S_0 & S_2 & \dots & S_{n-3} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{n-3} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The block in the southwest corner labelled S_{n-3} is formed from products of basis elements of S_{n-2} by those from S_{-1} and is therefore of size $\binom{n+1}{n-1} \times (n - 2)$. Thus it has more rows than columns, and it follows immediately that the product matrix is singular (in fact every term in the full expansion of the determinant is zero).

3. Comments

It is clear that the method used above for $W(n)$ will not work for odd n. For the first block decomposition, both r and $n-2-r$ must be odd and this means n must be even. Investigation of the cases $n=3$ and $n=5$ using a computer has yielded the fact that the product matrix is nonsingular for $n=3$; however I have as yet no systematic approach in the odd case. The other outstanding Cartan type superalgebras ($S(n)$ for even n, $H(n)$ and $\tilde{S}(2n)$) seem more difficult. Computer investigation shows that the product matrix for $S(4)$ is nonsingular, but again a general argument has not been forthcoming.

The root space method which Bell used for the classical simple algebras fails here since if λ is an odd root, $-\lambda$ need not be an odd root, and even if it is, the corresponding root spaces may have different dimensions. Since the dimensions of the Cartan type algebras such as $W(n)$ grow exponentially with n, and all obvious decomposition methods fail to reduce this problem substantially, it seems necessary to employ specializations setting many variables to zero. The one used above for $W(n)$ sets all but n variables to zero and the resulting matrix is manageable. However it would be nice to have a more structural understanding of why that specialization works so well, since a similar one which appeared promising in preliminary calculations proved ultimately much less suitable.

The converse of Bell's theorem has no known counterexample. Perhaps the examples of the last section will provide one, though it seems a very difficult task to show the enveloping algebra to be prime by ring-theoretic means.

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References

[B] Allen D. Bell, "A criterion for primeness of enveloping algebras of Lie superalgebras", J. Pure and Appl. Alg. 69 (1990), 111-120.

[KK] Ellen Kirkman and James Kuzmanovich, "Minimal prime ideals in enveloping algebras of Lie superalgebras", preprint 1994.

[S] M. Scheunert, *The Theory of Lie Superalgebras*, Springer LNM 716, Berlin 1979.

[Z] Efim Zelmanov, personal communication.