POWER MEASURES DERIVED FROM THE SEQUENTIAL QUERY PROCESS

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ABSTRACT. We study a basic sequential model for the formation of winning coalitions in a simple game, well known from its use in defining the Shapley-Shubik power index. We derive in a uniform way a family of measures of collective and individual decisiveness in simple games, and show that, as for the Shapley-Shubik index, they extend naturally to measures for TU-games. These individual measures, which we call weighted semivalues, form a class whose intersection with that of the class of weak semivalues yields the class of all semivalues.

We single out the simplest measure in this family for more investigation, as it is new to the literature as far as we know. Although it is very different from the Shapley value, it is closely related in several ways, and is the natural analogue of the Shapley value under a nonstandard, but natural, definition of simple game. We illustrate this new measure by calculating its values on some standard examples.

1. INTRODUCTION

Many authors have discussed the value theory of cooperative TU-games and its counterpart for simple games, the theory of power measures. The material in the present paper arises from a generalization of a particular measure of manipulability of voting rules (below called $\overline{Q}$) circulated in preprint form by the present authors. We realized that our original arguments generalize greatly, and yield a general construction for TU-games that leads directly to a large class of allocations including all semivalues. Amongst these, the simplest one yields a semivalue with several attractive properties.

1.1. **Our contribution.** We explore the (Shapley-Shubik) sequential model for the formation of winning coalitions in a simple game, and define (Section 3 and 4) a family of decisiveness measures Q_F^* and individual power measures q_F^* for simple games. These measures satisfy many desirable properties, and extend naturally to the class of TU-games. It turns out that this construction generates the class of what we call weighted semivalues. We give several interpretations of these values in Section 5. In this framework the simplest F is affine and yields a weighted semivalue we call q_0^*, which we single out for further attention. Although it differs substantially from the Shapley-Shubik index, it is closely related as we see in Section 5.3. We illustrate the new measures by applying them to some well-known games, including games derived from the study of strategic manipulation of voting rules, our original motivating examples.

1.2. **Basic definitions.** The definitions of simple game and TU-game are not entirely standardized. We use the most common definitions found in the literature. However, in Section 5.3, we shall drop some of the assumptions made here.

A *TU-game* on a finite set X is defined by its *characteristic function* $v : 2^X \to \mathbb{R}$, such that $v(\emptyset) = 0$. We denote the class of all TU-games on X by $\mathcal{G}(X)$. The class of finitely additive TU-games consists of those games satisfying $v(S \cup T) + v(S \cap T) = v(S) + v(T)$ for all $S, T \subseteq X$, and is denoted $\mathcal{AG}(X)$. A TU-game is *monotonic* if $A \subseteq B \subseteq X$ implies $v(A) \leq v(B)$.

An *allocation* on X is a function $\psi : \mathcal{G}(X) \to \mathcal{AG}(X)$.

Remark. *An allocation is often called a* value assignment *or simply* value. *An additive game is completely specified by the value of v on singletons, so an allocation is just a way of associating a*

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nonnegative real number with each player (we prefer not to use a vector, in order to avoid choosing an arbitrary ordering on players).

A *simple game* [12] $G = (X, W)$ on a finite set X is defined by a collection W of subsets of X (called *winning coalitions*), such that $\emptyset \notin W$. Equivalently, it is a TU-game on X where v takes only the values 0 and 1 (the value 1 corresponding to the property "winning", whilst coalitions with value 0 are called *losing*). Note that we do not require that the game be nonempty — that is, we may have $W = \emptyset$. The class of all simple games on X is denoted $\mathscr{SG}(X)$ and we define $\mathscr{SG}$ analogously to $\mathscr{G}$. A special class of game is the (weak) *unanimity game* $\mathscr{U}_S$ defined by S, where a coalition is winning if and only if it contains S. When $|S| = 1$ this is called a *dictatorial game*.

2. The random query process

Let $G = (X, W) \in \mathscr{SG}$. Consider the following stochastic process. We choose elements of X sequentially without repetition, at each step choosing uniformly from the set of elements not yet chosen, until the set of elements seen so far first becomes a winning coalition. This is the same process considered by Shapley and Shubik [11] in defining their power index (see Section 5.2 for more details).

We first consider the random variable equal to the number of queries required.

Definition 2.1. *Let $V_1, \dots, V_n$ be elements sampled without replacement from X, where $n = |X|$. Equivalently, $\pi := (V_1, \dots, V_n)$ is a uniformly random permutation of X, representing the order in which elements are to be chosen. Let*

$$Q_\pi(G) = \min\{k : \{V_1, \dots, V_k\} \text{ contains a winning coalition}\}.$$

Remark. *If the game is empty we will not find a winning coalition. In this case we define $Q_\pi(G)$ to have the value $n+1$. If the game is monotone, in Definition 2.1 the word "contains" can be replaced by "is".*

Definition 2.2. *The quantity $\overline{Q}(G)$ is defined to be the expectation of $Q(G)$ with respect to the uniform distribution on permutations of X.*

2.1. Non-sequential interpretation.

The sequential nature of the process is only apparent, once we have averaged over all possible orders. Thus we ought to be able to find a representation of $\overline{Q}(G)$ that does not mention order of players. In order to do this, we assume from now on that the game is monotone.

Definition 2.3. *For each natural number k, define the probability measure m_k to be the uniform measure on the set of all subsets of X of size k. Thus each subset of X of size k is equally likely to be chosen, with probability $\binom{n}{k}^{-1}$.*

For each natural number k, we let W_k (respectively, L_k) denote the set of all winning (respectively, losing) coalitions of size k.

Lemma 2.4. *For each k with $0 \le k \le n$,*

$$\Pr(Q(G) \le k) = \Pr(W_k)$$

where the latter probability is with respect to m_k.

In other words, the probability that we require at most k queries to find a winning coalition equals the probability that a randomly chosen k-subset is a winning coalition.

Proof. The event $Q(G) \le k$ means precisely that the initial subset $A(Q(G), k)$ formed by the first k queries contains a winning subset. Each subset of X of size k occurs with equal probability $\binom{n}{k}^{-1}$ as an initial subset of queries of the query sequence, so that $A(Q(G), k)$ is distributed as a uniform random sample from X_k. □

Remark. *The cumulative distribution function of $Q(G)$ can thus be computed by simply counting the number of winning coalitions of each fixed size.*

We can now derive a simple explicit formula for $\overline{Q}(G)$.

Lemma 2.5.
$$\overline{Q}(G) = n+1-\sum_{k=0}^{n} \frac{|W_k|}{\binom{n}{k}}.$$

Proof. For every π, $Q_\pi(G)$ is at most $n+1$. If G is empty then W_k is empty for all k and $\overline{Q}(G) = n+1$, as expected. Otherwise, W_n has a single element and $Q_\pi(G)$ is at most n. Thus by Lemma 2.4 we have

$$\begin{aligned}
\overline{Q}(G) &= E[Q(G)] \\
&= \sum_{k=0}^{n+1} k \Pr(Q(G) = k) \\
&= \sum_{k=0}^{n} \Pr(Q(G) > k) \\
&= \sum_{k=0}^{n} \frac{|L_k|}{\binom{n}{k}} \\
&= n+1-\sum_{k=0}^{n} \frac{|W_k|}{\binom{n}{k}}
\end{aligned}$$

□

Remark. *Note that the summation can start at $k = 1$ because the game is nontrivial. If we allowed trivial games, then the value of the formula for $\overline{Q}(G)$ would be 0, which agrees with intuition.*

3. Changes of variable and collective measures

The number of random queries made in order to find a winning coalition seems to us to be a fundamental quantity of a simple game. The quantity $\overline{Q}(G)$ intuitively seems to be a measure of inertia or *resistance* (as discussed in [5]): its value is large if winning coalitions are scarce, and small if they are plentiful. The rescaled quantity $1-\overline{Q}(G)/(n+1)$ looks like an index of what has been called *complaisance* [5, 9] and *decisiveness* [2]. We consider far more general rescalings of $\overline{Q}(G)$, with interesting consequences as will be seen below.

Definition 3.1. *Let $\mathscr{F}$ be the set of all real-valued functions on the nonnegative integer quadrant $\mathbb{N}^2$. Let $F \in \mathscr{F}$ satisfy*

(i) $F(n,k)$ is decreasing in k for each fixed n.
(ii) $F(n,0) = 1$ and $F(n,k) = 0$ whenever $k > n$.

We say that F is an admissible rescaling.

Remark. *We do not require that F be decreasing in n for each fixed k.*

We shall see below that there is a direct relationship between F and the function f obtained as follows.

Lemma 3.2. *There is a bijection $F \leftrightarrow f$ given by*

$$f(n,k) = \frac{F(n,k)-F(n,k+1)}{\binom{n}{k}} \tag{1}$$

$$F(n,k) = \sum_{j=k}^{n} f(n,j)\binom{n}{j}. \tag{2}$$

Note that F is admissible if and only if f is nonnegative and $\sum_{k=0}^{n} f(n,k)\binom{n}{k} = 1$.

There is a bijection $F \leftrightarrow \mu$ given by

$$F(n,k) = \sum_{j=k}^{n} \mu(n,j)$$
$$\mu(n,j) = F(n,k)-F(n,k+1)$$

Note that F is admissible if and only if for each n, $\mu(n,\cdot)$ is a probability measure on $\{0,\dots,n\}$. □

Remark. *We shall often write $\mu_n(k)$ for $\mu(n,k)$.*

We now define our candidate for a measure of decisiveness.

Definition 3.3. *Let $G = (X, W) \in \mathscr{SG}$. Define $Q^*_F(G) : \mathscr{SG} \to \mathbb{R}$ by*

$$Q^*_F(G) = E[F(Q(G))]$$

where the expectation is taken with respect to the uniform distribution on permutations of X as in Definition 2.2.

Proposition 3.4. *The function Q^*_F is a decisiveness index on $\mathscr{SG}$. Explicitly,*

$$Q^*_F(G) = \sum_{k=0}^{n} f(n,k)|W_k|$$

where f and F are linked as in (1).

Proof. Let f be as given in (1). Then

$$\begin{aligned}
\sum_{k=0}^{n} f(n,k)|W_k| &= \sum_{k=0}^{n} f(n,k)\binom{n}{k}\Pr(Q(G)\le k)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{k} f(n,k)\binom{n}{k}\Pr(Q(G)=j)\\
&= \sum_{j=0}^{n}\left(\sum_{k=j}^{n} f(n,k)\binom{n}{k}\right)\Pr(Q(G)=j)\\
&= \sum_{j=0}^{n} F(n,j)\Pr(Q(G)=j)\\
&= E[F(Q(G))].
\end{aligned}$$

Now (1) and the standing assumptions on F imply that Q^*_F takes values between 0 and 1 and these values are attained. Hence Q^*_F is a decisiveness index on $\mathscr{SG}$ according to the definition in [13]. □

Example 3.5. *Choosing $f(n,k) = 2^{-n}$ yields the Coleman index [3]. In this case μ_n is the binomial distribution with parameter* $1/2$*, and*

$$F(n,k) = 2^{-n}\sum_{j=k}^{n}\binom{n}{j}$$

which equals the probability that a uniformly randomly chosen subset has size at least k.

Example 3.6. *The simplest functional form of the construction above occurs when $F(n,k) = 1 - k/(n+1)$, in which case μ_n is the uniform distribution and*

$$Q^*_0(G) := Q^*_F(G) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}}|W_k| = 1 - \frac{\overline{Q}(G)}{n+1}.$$

There is a close connection between power and decisiveness measures for simple games and value theory of TU-games [13]. In view of that connection, it is natural to generalize to TU-games.

Definition 3.7. *For each admissible F, define a map $Q^*_F : \mathscr{G} \to \mathbb{R}$ as follows. For each game $G = (X, v)$,*

$$Q^*_F(G) := \sum_{k=0}^{n} f(n,k)\sum_{|S|=k, S\subseteq X} v(S) = \sum_{S\subseteq X} f(n,|S|)\,v(S).$$

We usually denote $Q^*_F(G)$ simply by Q^*_F when no confusion is likely.
This extended definition of Q^*_F yields a very general object, called a *collective value* in [13].

3.1. **The self-dual case.** We can derive some special formulae for Q_F^* in important special cases. We recall that the *dual* of a TU-game $G = (X, v)$ is the TU-game $G^* = (X, v^*)$ where $v^*(S) = v(X) - v(X \setminus S)$ for each S. In the case of simple games, winning coalitions become losing, and vice versa, when passing to the dual. A game is *self-dual* if $v^*(S) = v(S)$ for all S.

Proposition 3.8. *Let $G = (X, v) \in \mathscr{G}$ and suppose that F satisfies the identity*

$$F(n,k) - F(n,k+1) = F(n,n-k) - F(n,n+1-k). \tag{3}$$

If G is self-dual, then $Q_F^*(G) = v(X)/2$.

Remark. *The condition on F says that the probability measure μ_n is symmetric on $[0..n]$. For example, the Coleman index satisfies this property, and Proposition 3.9 was proved for that special case in [2, Proposition 3.4]. The index Q_0^* also satisfies the condition. This condition we call* self-duality *for the following reason. Given a collective value I, define another collective value I^* by $I^*(G) = I(G^*)$. The value is self-dual if $I = I^*$. It is easily seen that a collective value of the form Q_F^* is self-dual if and only if F satisfies the stated condition.*

Proof. Let $I = q_F^*$ satisfy the stated condition and let G be a self-dual game. Then

$$\begin{aligned} I(G) + I(G^*) &= \sum_S f(n,|S|)\, v(S) + \sum_S f(n,|S|)\, [v(X) - v(S)] \\ &= \sum_k f(n,k) \binom{n}{k} v(X) \\ &= v(X) \end{aligned}$$

while

$$\begin{aligned} I(G) - I(G^*) &= \sum_S f(n,|S|)\, v(S) - \sum_S f(n,|S|)\, [v(X) - v(S)] \\ &= \sum_S f(n,|S|)\, v(S) - \sum_S f(n,|X \setminus S|)\, v(X \setminus S) \\ &= 0. \end{aligned}$$

The result follows by solving for $I(G)$. □

In the case of simple games we can say a little more. Recall that a simple game is *proper* if the complement of each winning coalition is losing, and *strong* if the complement of every losing coalition is winning.

Proposition 3.9. *Let $G = (N, W)$ be a simple game and suppose that F satisfies the identity* (3).

(i) If G is proper and strong then $Q_F^*(G) = 1/2$.
(ii) If G is proper and not strong then $Q_F^*(G) < 1/2$.
(iii) If G is strong and not proper then $Q_F^*(G) > 1/2$.

Proof. For each k we define four types of subset: D_k (respectively C_k) consists of those which are winning, and whose complement is not (respectively is), whereas Q_k (respectively P_k) consists of those which are losing, and whose complement is not (respectively is). Complementation yields a map from G to G^* such that $D_k \Leftrightarrow P_{n-k}, C_k \Leftrightarrow C_{n-k}, Q_k \Leftrightarrow Q_{n-k}$. Thus $Q_F^*(G) + Q_F^*(G^*) = 1$ as in the proof of the previous proposition, and

$$\begin{aligned} Q_F^*(G) - Q_F^*(G^*) &= \sum_k f(n,k)\, (|D_k| + |C_k|) - \sum_k f(n,k)\, (|Q_k| + |P_k|) \\ &= \sum_k f(n,k)\, (|D_k| + |C_k|) - \sum_k f(n,k)\, (Q_k + D_k) \\ &= \sum_k f(n,k)\, (|C_k| - |Q_k|)\,. \end{aligned}$$

G is proper if and only if $C_k = 0$ for all k, while G is strong if and only if $Q_k = 0$ for all k (a simple game is proper and strong if and only if it is self-dual). The results follow by solving for $Q_F^*(G)$. □

Example 3.10. *Next, we consider the unweighted qualified majority voting game. The winning coalitions are precisely those of size at least k_0, for some fixed k_0 (depending on n). The value of Q_F^* on such a game equals $\sum_{k=k_0}^{n} f(n,k)\binom{n}{k} = F(n,k_0)$. Thus if n is odd and $k_0 = (n+1)/2$ (the straight majority game), Q_F^* has value $F(n,(n+1)/2$. If furthermore F satisfies the symmetry condition* (3), *then direct computation shows that Q_F^* takes the value* 1/2. *This is to be expected, since the game in question is proper and strong.*

4. Individual measures

In this section we discuss properties of the *marginal function* of Q_F^*, which we denote q_F^*. Explicitly, $q_{F,i}^*(G) = Q_F^*(G) - Q_F^*(G_{-\{i\}})$. We first review some properties of semivalues and generalizations.

4.1. **Semivalues and related concepts.** Several classes of allocations have been discussed in the literature. They can be given by axiomatic characterizations, but explicit formulae are more useful for our purposes.

Definition 4.1. *Let X be a finite set.*

A weighted weak semivalue *on X is an allocation on X that has the form*

$$\psi_i(v) = \sum_{S\subseteq X} p(S)\,[v(S) - v(S\setminus\{i\}]$$

where $p(S) \geq 0$.

A weak semivalue *on X is a weighted weak semivalue for which $\sum_{S:i\in S} p(S) = 1$ for each $i \in X$.*

Remark. *The above concepts were introduced in [1] in axiomatic terms. An equivalent formulation is that an allocation is a weighted weak semivalue if and only if it satisfies the standard axioms of Linearity, Positivity, Projection and Balanced Contributions.*

Definition 4.2. *A* weighted semivalue *on X is a weighted weak semivalue for which $p(S)$ depends only on $|S|$, and thus has the form*

$$\psi_i(v) = \sum_{k=0}^{n} f(n,k) \sum_{|S|=k, S\subseteq X} [v(S) - v(S\setminus\{i\}] \tag{4}$$

where $f(n,k) \geq 0$ for all n,k.

A semivalue *on X is a weighted semivalue on X that in addition satisfies the normalization condition*

$$\sum_{k=1}^{n} \binom{n-1}{k-1} f(n,k) = 1. \tag{5}$$

Remark. *The definitions have the unfortunate consequence that a semivalue is a weighted weak semivalue that is both a weak semivalue and a weighted semivalue! The term* weighted semivalue *is formally used in the present paper for the first time, to our knowledge.*

In [4] it was proven that semivalues are precisely the allocations satisfying the standard axioms Linearity, Positivity, Projection and Anonymity. The last says that if $\pi : X \to X$ is a permutation, then $\psi(\pi i) = \pi\psi(i)$ for each $i \in X$. Note that because of Anonymity, we may assume that $X = X_n := \{1,2,\dots,n\}$, where $n = |X|$. Let $\mathcal{G}$ denote the union $\bigcup_n \mathcal{G}(X_n)$. A *semivalue* on $\mathcal{G}$ is a function that for each n restricts to a semivalue on X_n. In [4] it was shown that in addition to the explicit form (5), the recursion

$$f(n,k) = f(n+1,k) + f(n+1,k+1) \tag{6}$$

is necessary and sufficient for such an extension.

The particular weighted semivalue we have in mind is the marginal function of Q^*. We first show that rescaling is needed in all but the most trivial cases.

Proposition 4.3. *Let F be an admissible rescaling, let f be related to F as in Lemma 3.2 and let ψ be defined as in* (4). *Then*

(i) *ψ gives a weighted semivalue on X_n for each n.*
(ii) *Let $c_n = \sum_{k=1}^n \binom{n-1}{k-1} f(n,k)$. Then $\tilde{\psi}$ defined by $\tilde{\psi}_i(v) = \psi_i(v)/c_n$ gives a semivalue on X_n for each n.*
(iii) *ψ gives a semivalue on X_n if and only if $f(n,k) = 0$ for $1 \le k \le n-1$ and $f(n,n) = 1$.*

Proof. Note that ψ gives a weighted semivalue on $\mathcal{G}_n$ because F is admissible, hence $f(n,k) \ge 0$ for all n,k. Dividing by c_n ensures that the normalization condition (5) is satisfied. Finally, suppose that $c_n = 1$ and let $a_{nk} = \binom{n}{k} f(n,k)$. By admissibility, $a_{nk} \ge 0$ and $\sum_k a_{nk} = 1$. By hypothesis, $1 = c_n = \sum_k \frac{k}{n} a_{nk}$. Subtracting these two equalities yields the result. □

4.2. **The marginal function of Q_F^*.** The marginal function of a decisiveness index is often interpreted as a power index (the analogue for TU-games is the relationship between a potential and an allocation [13, 6]). We now explore this direction.

Proposition 4.4. *Let F be an admissible rescaling. Then*
(i) *Q_F^* is the potential function of a function q_F^* that for each n induces a weighted semivalue on $\mathcal{G}_n$, given by*

$$\begin{aligned} q_{F,i}^* &= \sum_{k=0}^{n} f(n,k) \sum_{|S|=k} [v(S) - v(S \setminus \{i\})] \\ &= \sum_{S: i \in S} f(n,|S|) D_i(S) \end{aligned}$$

Here F and f are related as in Lemma 3.2.
(ii) *Let*

$$c_n := \frac{1}{n} \sum_{k=1}^{n} k \left[F(n,k) - F(n,k+1) \right].$$

Then the normalized quantity q_F^*/c_n is a semivalue on X_n for each n.
(iii) *q_F^* is a weighted semivalue on $\mathcal{G}$ if and only if F satisfies the recursion identity*

$$F(n,k) - F(n,k+1) = \frac{n+1-k}{n+1} F(n+1,k) + \frac{2k-n}{n+1} F(n+1,k+1) - \frac{k+1}{n+1} F(n+1,k+2). \tag{7}$$

If, furthermore, c_n is independent of n, then the normalized quantity q_F^*/c_n is a semivalue on $\mathcal{G}$, given by

$$\sum_{S: i \in S} \frac{f(n,|S|)}{Q_F^*(\mathcal{U}_1)} D_i(S)$$

Proof. The first part follows from Proposition 4.3 because q_F^* has exactly the form stated. The other results follow from the basic characterization of semivalues, translating the formulae for f into those for F. The coherence recursion (6) translates into

$$\begin{aligned} F(n,k) - F(n,k+1) = &\frac{n+1-k}{n+1} \left[F(n+1,k) - F(n+1,k+1) \right] \\ &+ \frac{k+1}{n+1} \left[F(n+1,k+1) - F(n+1,k+2) \right]. \end{aligned}$$

Note that c_n is precisely the value $Q_F^*(\mathcal{U}_{\{1\}})$ of Q_F^* on the dictatorial game with n players. The result now follows by algebraic simplification. □

The construction above is in fact universal.

Proposition 4.5. *There is a bijection between probability measures on $\{0,1,\ldots,n\}$ and weighted semivalues on $\mathcal{G}_n$ given by $\mu_n \leftrightarrow q_F^*$.*

Proof. From Proposition 4.4, q_F^* is a weighted semivalue. Conversely, given a weighted semivalue ξ we define $\mu_n(k) = \overline{f}(n,k) := f(n,k)/c_n$ where $c_n := \sum_k f(n,k)\binom{n}{k}$. Then defining F by (1) applied to $\overline{f}$ we have $q_F^* = \xi$. □

Remark. *Note that a weighted semivalue satisfies the normalization condition for a semivalue if and only if for each n, the mean of μ_n is exactly n.*

The recursion (6) *translates to*

$$\mu_n(k) = \left[1 - \frac{k}{n+1}\right]\mu_{n+1}(k) + \frac{k+1}{n+1}\mu_{n+1}(k+1).$$

Example 4.6. *For the Coleman index, the marginal function is given by $f(n,k) = 2^{-n}$. The associated semivalue (obtained by dividing by $Q_F^*(\mathscr{U}_1) = 1/2$) is the Banzhaf value.*

Example 4.7. *We now consider the Shapley value, given by*

$$\begin{aligned}\sigma_i(G) &= \sum_{\emptyset \subset S \subseteq X} \frac{(n-|S|)!(|S|-1)!}{n!}[v(S) - v(S\setminus\{i\})] \\ &= \sum_{\emptyset \subset S \subseteq X} \frac{|S|^{-1}}{\binom{n}{|S|}}[v(S) - v(S\setminus\{i\})] \\ &= \sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} \sum_{S\subseteq X, |S|=k} [v(S) - v(S\setminus\{i\})]\end{aligned}$$

It is the semivalue associated to q_F^* where $F(n,0) = 1$ and for $k \geq 1$

$$F(n,k) = \frac{H_n - H_{k-1}}{H_n} := \frac{\sum_{j=k}^{n} \frac{1}{j}}{\sum_{j=1}^{n}\frac{1}{j}}.$$

where H_n denotes as usual the nth harmonic number.

We single out the simplest case for special mention. Recall that a *regular* semivalue is one for which the weights $f(n,k)$ for $1 \leq k \leq n$ are all nonzero.

Proposition 4.8. *The formula*

$$\frac{q_{0,i}^*}{Q_0^*(\mathscr{U}_1)} = \frac{2}{n+1}\sum_{k=0}^{n}\sum_{S\subseteq X,|S|=k}\frac{[v(S) - v(S\setminus\{i\})]}{\binom{n}{k}} \tag{8}$$

defines a regular semivalue on $\mathscr{G}$.

Proof. This follows from Proposition 4.4 and Proposition 3.9, because $F(n,k) = 1-k/(n+1)$ satisfies the identity (3) and so q_0^* is self-dual. □

5. Interpretation of the measures

The collective value Q_F^* can be easily interpreted as a decisiveness index on simple games, which gives the probability of finding a winning coalition when coalitions are sampled first by choosing size according to μ_n and then choosing a coalition of that size uniformly at random. The individual value q_F^* can be interpreted in the usual way (not without controversy) as a power index [13]. In this section we consider some other interpretations.

5.1. **Coalition formation.** Consider the following model of coalition formation [7]. We assume that each possible coalition (subset S of X) forms with probability $p(S)$, and that only one coalition S will form. Consider the following two expectations. First, the *ex ante* expected marginal contribution of i to S is

$$E[D_i(S)] := E[v(S) - v(S\setminus\{i\})] = \sum_{S:i\in S} p(S)\,(v(S) - v(S\setminus\{i\})).$$

The *ex interim* expected marginal contribution of i to S, conditional on $i \in S$, is

$$\Phi_i(v,p) := E[D_i(S) \mid S \ni i] = \frac{E[D_i(S)]}{\Pr(S \ni i)}.$$

Then [7, Proposition 3] the maps $\Phi_i(\cdot, p)$ are in bijection with the set of all probability distributions on 2^X. Furthermore the map corresponding to p is precisely the *weighted weak semivalue* given by

$$\Phi_i(v,p) = \sum_{S:i\in S}\frac{p(S)}{\sum_{T:i\in T} p(T)} D_i(S).$$

Note that Φ is a weak semivalue if and only if $\sum_{T:i\in T} p(T)$ does not depend on i, and a semivalue if and only, if in addition, $p(S)$ depends only on $|S|$. Thus for weak semivalues, the ex ante and ex interim marginal contribution of i to S is the same.

We now apply the above framework to our measures q_F^*.

Proposition 5.1. *Let F be an admissible rescaling. Then under the coalition formation model above, $q_{F,i}^*$ gives the ex ante expected contribution of i to S, while the associated semivalue gives the ex interim expected marginal contribution of i to S, conditional on $i \in S$.* □

Remark. *The special case of the Shapley value was discussed in [7, Proposition 2], where essentially the same formula was derived.*

5.2. **Sequential interpretation.** The Shapley value σ can be defined sequentially as follows. Given a game $G = (X, v)$, follow the query process and at each step award $v(S) - v(S \setminus \{i\})$ to i, where S is the set of elements queried so far and i is the last element queried. The expected value with respect to the uniform distribution on permutations of X is the Shapley value $\sigma_i(G)$. For simple games, this means that we award 1 point to i whenever i is *pivotal*, and 0 otherwise.

We can generalize this interpretation to measures of the form q_F^*. We follow the query process, and award $k\mu_n(k)$ to the pivotal element if the process stops at k queries. For example, for q_0^* this means awarding $k/(n+1)$, the fraction of the maximum possible number of queries made (a possible interpretation is that we offer a higher price to pivotal elements to reveal themselves as we repeatedly fail to find them) . This corresponds to the nonsequential formula for q_F^* using the analogous computation to that for the Shapley value above.

5.3. **Another model of semivalues.** The standard model of simple game requires that the empty coalition never be winning (this is a consequence of the standard assumption that $v(\emptyset) = 0$ for all TU-games). For reasons of mathematical elegance, at least, we prefer that the class of simple games be closed under duality. This then requires that the grand coalition is always winning, which is a reasonable assumption for weighted voting games. However for many applications, such as to the study of manipulation, empty games naturally arise. Thus, by duality, we should admit trivial games to the class of simple games. This approach is followed, for example, by Taylor and Zwicker [12].

We denote by $\mathscr{S}\mathscr{G}^+(X)$ the class of simple games in this extended sense. The analogue for TU-games we denote $\mathscr{G}^+(X)$.

In this new model, the axiomatic definition of semivalue remains the same, since the efficiency, anonymity, positivity and projection axioms still make sense. However the characterization of [4] changes slightly.

Proposition 5.2. *A function ψ is a semivalue on $\mathscr{G}^+(X)$ if and only if it has the form*

$$\psi(i) = \sum_k f(n,k) \sum_{S\subseteq X, |S|=k} [v(S) - v(S \setminus \{i\})]$$

where $f(n,k) \geq 0$ and f satisfies the identity

$$\sum_k f(n,k)\binom{n}{k} = 1.$$

Such a function extends to a semivalue on $\mathscr{G}^+$ if and only if f also satisfies the identity

$$f(n,k) = f(n,k+1) + f(n+1,k+1).$$

Proof. The proof of [4] still works, with the only change being that the normalization condition is slightly different. This is because the vector space of all games now includes the empty game in the standard basis of unanimity games and so has dimension one more than before. □

Corollary 5.3. *Let F be admissible. Then q_F^* is a semivalue on $\mathscr{G}^+(X)$.*

Remark. *For example, q_0^* is the exact analogue of the Shapley value in this new context. The Shapley value is the semivalue that has equal weight on all coalition sizes from 1 to n, hence the formula $f(n,k) = [n\binom{n-1}{k-1}]^{-1}$, whereas q_0^* has equal weight on all coalition sizes from 0 to n.*

W^m	C	Q_0^*
1;2;3;4	0.9375	0.8000
1;2;3	0.8750	0.7500
1;2;34	0.8125	0.7000
1;2	0.7500	0.6667
1;23;24;34	0.7500	0.6500
1;23;24	0.6875*	0.6067
12;13;14;23;24;34	0.6875*	0.6000
12;13;14;23;24	0.6250	0.5833
1;23	0.6250	0.5500
1;234	0.5625*	0.5500
12;13;14;23	0.5625*	0.5333
12;13;24;34	0.5625*	0.5333
1	0.5000	0.5000
12;13;23	0.5000	0.5000
12;13;24	0.5000	0.5000
12;13;14;234	0.5000	0.5000
12;34	0.4375*	0.4667
12;13;234	0.4375*	0.4667
12;13;14	0.4375*	0.4500
12;13	0.3750	0.4167
12;134;234	0.3750	0.4033
123;124;134;234	0.3125*	0.4000
12;134	0.3125*	0.3833
123;124;134	0.2500	0.3500
12	0.2500	0.3333
123;124	0.1875	0.3000
123	0.1250	0.2500
1234	0.0625	0.2000
	0.0000	0.0000

TABLE 1. Values of Q_0^* and q_0 for simple games with 4 players.

6. THE MEASURES Q_0^* AND q_0^*

We recall that $Q_0^*(G)$ is the probability that a randomly sampled coalition is winning, when a coalition size is chosen uniformly and condition on that a particular coalition is chosen. Alternatively, it gives the expected fraction of the maximum possible number of queries saved when players are sampled uniformly at random without replacement until a winning coalition has been found.

Example 6.1. *We present the values of Q_0^* for all simple games with 4 players (up to isomorphism). Because of anonymity, we may call the players $1,2,3,4$ and we define each game by listing its minimal winning coalitions in the obvious way (the last line corresponds to the empty coalition). Table 1 is the analogue of the table in [2] for the Coleman index C, which we also list for comparison. We have reordered some rows from the table in [2], by permuting some rows with equal values of C. These rows are marked with *.*

Note that the values of C always (weakly) decrease going down the column, as do those of Q_0^*. Thus C and Q_0^* never disagree on the relative decisiveness of two games when they both agree that two games are not equally decisive. Even though the range of the values of C (excluding the empty game) is much larger than the range of values of Q_0^*, the latter never has equal values on two games when the former does not, but the former sometimes has equal values when the latter does not. Thus Q_0^* appears to discriminate better between games.

The next type of example was our original motivation for the study of Q_0^*. The quantity $\overline{Q}$ seems to us a compelling way to measure the effort taken by an external agent to change the outcome of the election via manipulation. Assuming that the voting situation is known but not the complete profile (we may know from polling data how many voters of each type there are, but not their identity), the agent incurs a unit cost to determine each voter's type. The sequential model occurs naturally here.

Example 6.2. *Consider a 3-candidate election where voters submit complete and total preference orders. Suppose that the voting situation is as follows: 2 voters have preference order abc, 1 has bac and another has cba. A manipulation is a change of vote by some subset that causes a preferred outcome for those voters, assuming the other voters continue to vote sincerely. We define a winning coalition to be a subset containing a subset that can manipulate coalitionally. Here for concreteness we break ties uniformly at random, and assume risk-averse manipulators (see [10] for more details).*

Using a positional scoring rule that awards $1,\alpha,0$ points to the first, second, third ranked candidates, we see that the scores of a,b,c respectively are $2+\alpha, 1+3\alpha, 1$.

There is no manipulating coalition which can make c win, since the last voter cannot help c overtake both a and b, while the other voters have no incentive to do so.

If $\alpha \leq 1/2$ then a wins (solely, unless $\alpha = 1/2$ in which case a and b tie). The cba voter can change to bca, and this allows b to win, so is preferred by that voter. Thus a manipulating coalition of size 1 exists. Furthermore, if $\alpha \geq 1/3$, then the bac voter also has the power to make b win by voting bca. It follows that for $1/3 \leq \alpha \leq 1/2$, the winning coalitions are those containing either of the last two voters, while for $0 \leq \alpha < 1/3$, the winning coalitions are those containing the last voter. In the former case, a winning coalition is found by the random query process after 3 queries for 4 of the 24 possible query sequences, after 2 queries for 8 query sequences and after 1 query for 12 query sequences. Thus $\overline{Q} = 40/24$ and $Q_0^* = 1 - \overline{Q}/5 = 2/3$. Similarly, in the latter case the relevant voter is found after 1,2,3, or 4 queries in each of 6 query sequences, leading to $\overline{Q} = 60/24$ and $Q_0^* = 1/2$.

If $\alpha > 1/2$ then b is the sole winner. Either of the abc voters can make a win by switching to acb. In this case we have by the same argument as above that $Q_0^* = 2/3$. As described in detail above, these values of Q_0^* give a measure of the probability of finding a manipulating coalition when coalitions are sampled according to a particular probability distribution.

In the case $0 \leq \alpha < 1/3$, the individual measure q_0^* has the value 0 for all but the last voter, and $1/2$ for that voter. When $1/3 \leq \alpha \leq 1/2$, the values are 0 for the first two voters, and $2/15$ for each of the last two, whereas when $1/2 < \alpha$, the roles of the first pair and last pair are reversed. These numbers represent the ex ante probability of being critical in a winning coalition, if coalitions form according to the particular probabilistic model used here.

6.1. **A bargaining model.** Laruelle and Valenciano [8] present a model of bargaining intended to help give a noncooperative foundation to the theory of power indices and values. They discuss a setup where a proposer suggests an initial allocation of payoffs to a winning coalition containing the proposer. Let p be a map that for each set X of players, takes each simple game on X to a probability distribution over $X \times 2^X$. The idea is that $p_G(i,S)$ is the probability that i will be the proposer with the support of S. They impose a dummy axiom which leads to the condition that $p_G(i,S) = 0$ unless i swings S. Anonymity is also a reasonable assumption.

They discuss nonsequential ("first choose S, then i") and sequential ("choose i and S simultaneously") approaches. In the former case, given a probability distribution over coalitions where the probability of S depends only on $|S|$, in the first round we choose S and then choose choosing $i \in S$ uniformly at random. If i swings S, then i is the proposer, otherwise we repeat rounds until a proposer is found). Thus we are in the arena of weighted semivalues and we write $p(n,s)$ for $p(S)$ when $|X| = n$.

Proposition 6.3. *Let F be admissible and consider the nonsequential protocol above, where S is chosen according to the probability given by μ_F. Let π_i be the probability that i is eventually chosen as the proposer, and let r_i be the probability that i is chosen as proposer in the first step. Then there is c so that $\pi_i = cr_i$ for all i.*

In particular, for $F = F_0$, π_i equals the Shapley-Shubik index of i.

Proof. Let r_i denote the probability that i is chosen as proposer in the first step. Then $\pi_i = r_i / \sum_i r_i$. Now

$$\begin{aligned} r_i &= \sum_{S: S\in W, i\in S, S\setminus\{i\}\notin W} \frac{p(S)}{|S|} \\ &= \sum_{S\neq\emptyset} p(S)\frac{D_i(S)}{|S|} \\ &= \sum_{k=1}^{n} \frac{f(n,k)}{k} \sum_{|S|=k} D_i(S) \end{aligned}$$

When $F = F_0$, $f(n,k)/k$ is precisely $[n+1]^{-1}$ times the coefficient found in the Shapley-Shubik index, namely $[k\binom{n}{k}]^{-1}$. Thus r_i equals $[n+1]^{-1}$ times the Shapley-Shubik index σ_i of i. Thus since $\sum_i r_i$ is independent of i (in fact it equals $[n+1]^{-1}$), the ratios π_i / r_i are constant, and so the normalized probability π_i equals σ_i. □

Remark. *This result was stated (in other words) without proof in [8, p. 124].*

In the sequential scenario, the query process seems a very natural one. If the pivotal voter is always chosen as the proposer, then the probability of being the proposer is again the Shapley-Shubik index. However other choices are possible. For example, for each admissible F, if we weight the pivotal voter by $k\mu_n(k)$ every time it is pivotal in position k, and then compute the overall probability accordingly, we obtain the normalized version of q_F^*.

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