Asymptotics of multivariate generating functions

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A simple, nontrivial example

Let $a_{r,s}$ be the number of paths in $\mathbb{Z}^2$ from $(0,0)$ to (r,s) which go only north, east, or northeast.

- Find rate of growth of $a_{r,s}$ as $d := \sqrt{r^2+s^2} \to \infty$

$$a_{r,s} \sim \left[\frac{d-s}{r}\right]^{-r} \left[\frac{d-r}{s}\right]^{-s} \sqrt{\frac{rs}{2\pi d(r+s-d)^2}}.$$

- Estimate $a_{100,100}$.

$$a_{100,100} \cong \frac{(1+\sqrt{2})^{201}}{10 \cdot 2^{5/4}\sqrt{\pi}} \quad \text{(accurate to within 0.1\%.)}$$

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The basic setup

- Given an ordinary power series generating function F in d variables, $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ for $\mathbf{r} \in \mathbb{N}^d$.
- Coefficient extraction: "compute" its Mclaurin coefficients. Realistically, we must usually settle for deriving asymptotic approximations for $a_{\mathbf{r}}$ as $|\mathbf{r}| \to \infty$.
- We assume that F defines an analytic function in a neighbourhood of $\mathbf{0} \in \mathbb{C}^d$.

Coefficient asymptotics when $d = 1$

- Behaviour of GF near its dominant singularities determines asymptotics of coefficients.
- Suppose ξ, $|\xi| = 1$, is the unique singularity of F of smallest modulus, and is a pole (other cases can also be treated).
- Cauchy: $a_n = (2\pi i)^{-1} \int_{(1-\epsilon)S^1} z^{-n-1} F(z)\, dz = (2\pi i)^{-1} \int_{(1+\epsilon)S^1} z^{-n-1} F(z)\, dz - \mathrm{Res}(z^{-n-1}F; \xi)$.
- The integral is of smaller exponential order than the residue, which therefore dominates.
- Robust with respect to perturbation of coefficients.

Previous work

- For $d = 1$, software solves the problem in many instances including rational F.
- Relatively little work has been done in the case $d > 1$, as evinced by database search and survey articles.
- Much of what has been done breaks symmetry between variables in an undesirable way, is not very general and hard to generalize, yields only leading term asymptotics, or is not computationally effective.

Our project

- Thoroughly investigate the case $d > 1$. Emphasis is on meromorphic F. Amazingly little is known *even about rational F in 2 variables.*
- Goal 1: improve over all previous work in generality, ease of use, symmetry, computational effectiveness, uniformity of asymptotics. Create a theory!
- Goal 2: establish mvGFs as an area worth studying in its own right, a meeting place for many different areas, a common language. I am recruiting!

Notation

- $F = G/H$ meromorphic in nontrivial polydisc in $\mathbb{C}^d$.
- $\mathcal{V} = \{\mathbf{z} | \, H(\mathbf{z}) = 0\}$ the singular variety of F.
- $\mathcal{T}(\mathbf{z}), \mathcal{D}(\mathbf{z})$ the torus, polydisc centred at $\mathbf{0}$ and containing $\mathbf{z}$.
- A point $\mathbf{z} \in \mathcal{V}$ is *minimal* if $\mathcal{V} \cap \mathcal{D}(\mathbf{z}) \subset \mathcal{T}(\mathbf{z})$.
- Point of $\mathcal{V}$ is *strictly minimal* if $\mathcal{V} \cap \mathcal{T}(\mathbf{z}) = \{\mathbf{z}\}$, and *toral* if $\mathcal{V} \cap \mathcal{T}(\mathbf{z}) = \mathcal{T}(\mathbf{z})$.
- A minimal point can be a *smooth*, *multiple* or *cone* point, depending on local geometry of $\mathcal{V}$.

Prototype examples for each geometry

- (smooth points) $F(x, y) = (1 - x - y - xy)^{-1}$; Delannoy numbers (count planar lattice paths of earlier slide).
- (multiple points) $F(\mathbf{z}) = \prod_i (1 - \sum a_{ij} z_j)^{-1}$; normalization constants of queueing networks.
- (cone points) $F(x, y, z) = \frac{z/2}{(1-yz)P(x,y,z)}$, P a certain polynomial; probabilities of being in frozen region (Arctic circle theorem for planar lattice tilings by dominoes).

Features of our method

- We use Cauchy integral formula; residue in 1 variable; Fourier-Laplace integral in $d-1$ variables; stationary phase method.
- Must specify a direction $\bar{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ for asymptotics. Does method work for all $\bar{\mathbf{r}}$?
- Each minimal point $\mathbf{z} \in \mathcal{V}$ yields asymptotics in a certain cone $\boldsymbol{\kappa}$ of directions.
- For smooth points of $\mathcal{V}$, $\boldsymbol{\kappa}$ collapses to a single ray; for multiple points, $\boldsymbol{\kappa}$ is nontrivial. How do asymptotics piece together on boundaries of cones?

Fourier-Laplace integrals

We are led ($\mathbf{w} = \mathbf{z}e^{i\boldsymbol{\theta}}$) to analysis of integrals of the form

$$I(\lambda) = \int_D e^{i\lambda f(\mathbf{x})} \psi(\mathbf{x}) \, dV(\mathbf{x})$$

where: $\operatorname{Im} f \geq 0$; $f, \psi \in C^\infty(D, \mathbb{C})$; D is a compact "manifold with corners"; dV the volume element.

- Not covered by standard results. Challenges: range of f is 2-D; $\psi \neq 0$ on ∂D; $\operatorname{Im} f = 0$ at points of ∂D; f can be stationary on a set of positive dimension.
- All our results depend upon analysis of these integrals. I am looking for help!

Sample theorem – smooth point, $d = 2$

Theorem. Suppose that (z, w) is a smooth strictly minimal point of $\mathcal{V}$. Define

$$Q(z, w) = -a^2 b - ab^2 - a^2 z^2 H_{zz} - b^2 w^2 H_{ww} + abH_{zw}$$

where $a = wH_w, b = zH_z$. Then when $r/s = b/a$

$$z^r w^s a_{rs} \sim \frac{G(z, w)}{\sqrt{2\pi}} \sqrt{\frac{-a}{sQ(z, w)}}.$$

This is the generic case and the theorem includes much previous work as special cases. The apparent lack of symmetry is illusory, since $r/s = b/a$. It is uniform in compact sets of directions away from the axes.

Exemplifying the theorem

$F(x, y) = (1 - x - y - xy)^{-1}$. For r/s fixed, asymptotics are governed by the minimal point in the first quadrant satisfying $1 - x - y - xy = 0, x(1 + y)s = y(1 + x)r$. Using these relations and the theorem we obtain

$$a_{rs} \sim \left[\frac{d - s}{r}\right]^{-r} \left[\frac{d - r}{s}\right]^{-s} \sqrt{\frac{rs}{2\pi d(r + s - d)^2}}.$$

A complete series is descending powers of r or s or d can be obtained.

Sample theorem – multiple point, $d = 2$

Theorem. Suppose that (z, w) is a strictly minimal, double point of $\mathcal{V}$. Then for each subset K of the interior of the cone $\boldsymbol{\kappa}(z, w)$, bounded away from the walls, there are $c, C > 0$ such that for $(r, s) \in K$,

$$\left| z^r w^s a_{rs} - \frac{G(z, w)}{\sqrt{-z^2 w^2 \det H''(z, w)}} \right| \leq C e^{-c|(z,w)|}.$$

The simplest example (number of sheets n equals dimension d). We have analysed $n < d$ and $n > d$ also.

Exemplifying the theorems

Consider $a_{r,s}$ given by $a_{0,0} = 1$ and

$$6a_{r,s} = 5a_{r,s-1} + 7a_{r-1,s} - a_{r,s-1} - 3a_{r-1,s-1} - 2a_{r,s-2}.$$

Then $F(z,w) = \frac{6}{(w+z-2)(w+2z-3)}$ and $\mathcal{V}$ has a minimal, double point at $(1,1)$, governing asymptotics for $1/2 < r/s < 2$. All other minimal points are smooth. We obtain:

$$a_{r,s} \sim \begin{cases} \text{exp times poly in } s & \text{if } r/s \notin [1,2]; \\ 1/6 & \text{if } 1 < r/s < 2; \\ 1/12 & \text{if } r/s \in \{1,2\}. \end{cases}$$

Progress so far

- Complete expansions for smooth points of $\mathcal{V}$ (JCT A 2002). Leading term $a_{\mathbf{r}} \sim C|\mathbf{r}|^{-d/2}$.
- Complete expansions for generic multiple points (number of sheets equals dimension). Get $a_{\mathbf{r}} \sim C$ in a certain cone of directions.
- Almost finished more general analysis of multiple points (ask me for preprint).
- Proof that our method applies always for nonnegative bivariate sequences.

Outstanding problems

- How to find and classify minimal singularities algorithmically?
- Complete analysis of oscillatory integrals in general case.
- Asymptotics "in the gaps"; uniformity.
- Coordinate-free approach, symmetric formulae, exact formulae.
- Cone point analysis.