LONGEST ALTERNATING SUBSEQUENCES IN PATTERN-RESTRICTED PERMUTATIONS

GHASSAN FIRRO, TOUFIK MANSOUR, AND MARK C. WILSON

Abstract. Inspired by the results of Stanley and Widom concerning the limiting distribution of the lengths of longest alternating subsequences in random permutations, and results of Deutsch, Hildebrand and Wilf on the limiting distribution of the longest increasing subsequence for pattern-restricted permutations, we find the limiting distribution of the longest alternating subsequence for pattern-restricted permutations in which the pattern is any one of the six patterns of length three. Our methodology uses recurrences, generating functions, and complex analysis, and also yields more detailed information. Several ideas for future research are listed.

1. Introduction

Let S_n be the symmetric group of permutations of $1, 2, \dots, n$ and let $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$. An increasing subsequence in π of length ℓ is a subsequence $\pi_{i_1}\pi_{i_2} \dots \pi_{i_\ell}$ satisfying $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_\ell}$ (note that we are not considering sub*words*, so the indices $i_1, \dots, i_\ell$ need not be contiguous). Several authors have studied properties of the length of the longest increasing subsequence $is_n(\pi)$ of a permutation π and the associated random variable is_n given by taking an element of S_n uniformly at random. Logan and Shepp [10] and Vershik and Kerov [13] showed that the asymptotic expectation satisfies $E(is_n) = \frac{1}{n!}\sum_{\pi\in S_n} is_n(\pi) \sim 2\sqrt{n}$ when $n \to \infty$, and the limiting distribution of a suitably scaled and translated version of is_n was determined by Baik, Deift, and Johansson [4].

Recently, Stanley [12] developed an analogous theory for alternating subsequences, that is, subsequences $\pi_{i_1}\pi_{i_2} \dots \pi_{i_\ell}$ of π satisfying $\pi_{i_1} > \pi_{i_2} < \pi_{i_3} > \pi_{i_4} < \cdots \pi_{i_\ell}$. He proved that the mean of the random variable al_n whose value is the longest alternating subsequence of an element of S_n chosen uniformly at random is $\frac{4n+1}{6}$ for $n \geq 2$, and the variance of al_n is $\frac{8}{45}n - \frac{13}{180}$ for $n \geq 4$. Furthermore, Widom [14] showed that the limiting distribution as $n \to \infty$ of the normalized random variable $(al_n - 2n/3)/\sqrt{8n/45}$ is the standard normal distribution.

Note that we are considering "alternating subsequences starting with a fall". The alternative of "alternating subsequences starting with a rise" would yield the same results in this case by symmetry. However we will make use of both types of alternating subsequences in our situation described below, because the results are not the same in that case.

Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations. We say that π *avoids* τ, or is τ-*avoiding*, if there do not exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1} \dots \pi_{i_k}$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. It is well known that the number of τ-avoiding permutations of length n, $\tau \in S_3$, is given by $c_n = \frac{1}{n+1}\binom{2n}{n}$ the n-th Catalan number; see [8, 9].

Deutsch, Hildebrand and Wilf [5] connected pattern avoidance and longest increasing subsequences by deriving the limiting distribution of the longest increasing subsequences in τ-avoiding permutations, for each of the possible $\tau \in S_3$. Inspired by the results of the above authors, we complete the picture by studying the limiting distribution of the longest alternating subsequences for τ-avoiding permutations, for each possible $\tau \in S_3$.

Our main results may be formulated as follows.

Theorem 1.1. *Let $\tau \in S_3$. In the class of τ-avoiding permutations of length n, the length of the longest alternating subsequence has mean $\mu_\tau \sim n/2$ and variance $\sigma_\tau^2 \sim n/4$. The details are shown in Table 1.*

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τ	123	132, 231, 321	213, 312
μ_τ	$\frac{2n^2+5n-9}{2(2n-1)}$	$\frac{(n-1)(2n+5)}{2(2n-1)}$	$\frac{n+1}{2}$
σ^2_τ	$\frac{(n+1)(8n^3-50n^2+101n+9)}{4(2n-1)^2(2n-3)}$	$\frac{(n+1)(8n^3-42n^2+73n-15)}{4(2n-1)^2(2n-3)}$	$\frac{(n+1)(4n^2-15n+15)}{4(2n-1)(2n-3)}$

TABLE 1. Statistics for al_n on $S_n(\tau)$

Moreover, the (almost normalized) random variable $X_n^\tau = \frac{al_n - \frac{n}{2}}{\frac{1}{2}\sqrt{n}}$, *defined for all permutations of length* n *that avoid the pattern* τ, *converges in distribution to the standard normal distribution as* $n \to \infty$. *In other words,* al_n *satisfies a Gaussian Limit law.*

We note that as in Stanley's study, we obtain Gaussian limiting behaviour (but with a smaller mean and larger variance). The generating functions obtained by Stanley were rational whereas ours turn out to be algebraic but not rational. The Gaussian limit is obtained by using general results on limit laws for combinatorial classes presented in the forthcoming work [6].

The outline of the rest of the paper is as follows. In Section 2 we set up recurrences for the number of τ-avoiding permutations with longest alternating subsequence of a given size. This leads to a system of functional equations that we solve using the kernel method. From the explicit form of the generating functions we are able to derive probabilistic information including a Gaussian limit law.

2. Derivation of the generating functions

We introduce trivariate generating functions A_τ and B_τ as follows.

Let $bl(\pi)$ be the maximum length of an "alternating subsequence starting with a rise" in π. That is, the maximum length of subsequences $\pi_{i_1}\pi_{i_2}\dots\pi_{i_\ell}$ of π satisfying $\pi_{i_1} < \pi_{i_2} > \pi_{i_3} < \pi_{i_4} > \cdots \pi_{i_\ell}$.

Define $j(\pi) = \pi_1$.

$$A_\tau(x, v, q) = \sum_{\pi \in S(\tau)} x^{|\pi|} v^{j(\pi)-1} q^{al(\pi)} = \sum_{n,m,j} a_\tau(n, m, j) x^n v^{j-1} q^m$$
$$B_\tau(x, v, q) = \sum_{\pi \in S(\tau)} x^{|\pi|} v^{j(\pi)-1} q^{bl(\pi)} = \sum_{n,m,j} b_\tau(n, m, j) x^n v^{j-1} q^m$$

where $a_\tau(n, m, j)$ (respectively $b_\tau(n, m, j)$) denotes the number of elements of $S_n(\tau)$ having $al(\pi) = m$ (respectively $bl(\pi) = m$) and $j(\pi) = j$.

Various refinements will be used (the reader should take care to avoid confusion, as some different notations look similar). We write $a_\tau(n, m)$ instead of $a_\tau(n, 1, m)$. We then define

$$a_\tau(n) = a_\tau(n; q) = \sum_{m=1}^{n} a_\tau(n, m) q^m, \quad b_\tau(n) = b_\tau(n; q) = \sum_{m=1}^{n} b_\tau(n, m) q^m.$$

More generally, let $a_\tau^m(n; j_1, \dots, j_s)$ (resp. $b_\tau^m(n; j_1, \dots, j_s)$) be the number of τ-avoiding permutations π of length n such that $al(\pi) = m$ (resp. $bl(\pi) = m$) and $\pi_1 \dots \pi_s = j_1 \dots j_s$. For $n \geq m$ define

$$a_\tau(n; j_1 \dots, j_s) = a_\tau(n; j_1, \dots, j_s; q) = \textstyle\sum_{j=1}^{n} a_\tau^j(n; j_1, \dots, j_s) q^j,$$
$$b_\tau(n; j_1 \dots, j_s) = b_\tau(n; j_1, \dots, j_s; q) = \textstyle\sum_{j=1}^{n} b_\tau^j(n; j_1, \dots, j_s) q^j,$$

and for $n \geq 0$,

$$A_\tau(n; v) = A_\tau(n; v, q) = \textstyle\sum_{j=1}^{n} a_\tau(n; j; q) v^{j-1}, \quad A_\tau(x, v) = A_\tau(x, v, q) = \sum_{n \geq 0} A_\tau(n; v) x^n,$$
$$B_\tau(n; v) = B_\tau(n; v, q) = \textstyle\sum_{j=1}^{n} b_\tau(n; j; q) v^{j-1}, \quad B_\tau(x, v) = B_\tau(x, v, q) = \sum_{n \geq 0} B_\tau(n; v) x^n.$$

Our plan is to study the generating functions $A_\tau(x, v, q)$ and $B_\tau(x, v, q)$ for each pattern $\tau \in S_3$. We first show how to reduce the number of cases by using some standard involutions.

Lemma 2.1. *Let* $\tau = \tau_1\tau_2 \dots \tau_k \in S_k$ *be any pattern. Define*

$$\tau' = (k + 1 - \tau_1)(k + 1 - \tau_2) \dots (k + 1 - \tau_k) \in S_k.$$

Then $A_\tau(x, v, q) = vB_{\tau'}(xv, 1/v, q)$. *In particular* $A_\tau(x, 1, q) = B_{\tau'}(x, 1, q)$, *and* $al(\tau) = bl(\tau')$.

Proof. A straightforward application of the *complement map*

$$\pi_1\pi_2\dots\pi_n \mapsto (n+1-\pi_1)(n+1-\pi_2)\dots(n+1-\pi_n).$$

■

Proposition 2.2. *The following equalities hold.*

(i) The number of 231*-avoiding permutations* π *of length* n *having* $al(\pi)=m$ *equals the number of* 132*-avoiding permutations* π *of length* n *having* $al(\pi)=m$*. In other words,*
$$A_{231}(x,1,q)=A_{132}(x,1,q).$$
(ii) The number of 213*-avoiding permutations* π *of length* n *having* $al(\pi)=m$ *the same as the number of* 312*-avoiding permutations* π *of length* n *having* $al(\pi)=m$*. In other words,*
$$A_{213}(x,1,q)=A_{312}(x,1,q).$$
(iii) The number of 231*-avoiding permutations* π *of length* n *having* $al(\pi)=m$ *the same as the number of* 321*-avoiding permutations* π *of length* n *having* $al(\pi)=m$*. In other words,*
$$A_{231}(x,1,q)=A_{321}(x,1,q).$$

Proof. In each case we exhibit a bijection from $S_n(\tau)$ to $S_n(\tau')$ which does not change the value of $al(\pi)$.

(i) Let $\pi=\pi' n\pi''$ be any permutation in $S_n(231)$, where $\pi_j=n$. Note that π',π'' avoid 231 and $\pi_k<\pi_l$ for each pair of indices with $k<n<l$. Define the map α recursively as follows. Let $\alpha(\pi)$ be the permutation
$$(\beta'_1+n-j)(\beta'_2+n-j)\dots(\beta'_{j-1}+n-j)n\beta'',$$
where $\beta'=\alpha(\pi')$ and $\beta''=\alpha((\pi''_1-j+1)(\pi''_2-j+1)\dots(\pi''_{n-j}-j+1))$. For example, $\alpha(21534)=43521$. Note that $\alpha(\pi)$ avoids 132.

(ii) Let $\pi=\pi'1\pi''$ be any permutation in $S_n(213)$, where $\pi_j=1$. Then, $\pi'_a>\pi''_b$ for all $1\le a<j<b\le n$. Define $\alpha(\pi)$ to be the permutation
$$(\beta'_1+1)(\beta'_2+1)\dots(\beta'_{j-1}+1)1(\beta''_1+j)(\beta''_2+j)\dots(\beta''_{n-j}+j),$$
where $\beta'=\alpha((\pi'_1-n+j-1)(\pi'_2-n+j-1)\dots(\pi'_{j-1}-n+j-1))$ and $\beta''=\alpha((\pi''_1-1)(\pi''_2-1)\dots(\pi''_{n-j}-1))$. For example, $\alpha(6745132)=3425176$. Note that $\alpha(\pi)$ avoids 312.

(iii) Simion and Schmidt [11] introduced a simple bijection, say $f:\pi\mapsto\pi'$, between the $S_n(123)$ and $S_n(132)$ which fixes each element of $S_n(123)\cap S_n(132)$. Essentially, the inverse of f fixes the position of each left-right minimum, and fills the remaining positions in decreasing order. Recall that π_i is called a left-right minimum of π if there no $j<i$ such that $\pi_j<\pi_i$. It follows from the definition of f that if π_i,π_j (respectively π'_i and π'_j) are left-right minima with $i<j$ then $\pi_{i+1}>\pi_{i+2}>\dots>\pi_{j-1}$ (respectively $\pi'_{i+1}<\pi'_{i+2}<\dots<\pi'_{j-1}$).

Define $g=r\circ f\circ r$, where r is the reversal map ($r:\pi_1\pi_2\cdots\pi_n\to\pi_n\cdots\pi_2\pi_1$). From the definitions we obtain that g is a bijection between $S_n(321)$ and $S_n(231)$. The above properties of f imply that for each $\pi\in S_n(132)$, the length of the longest subsequence of the form
$$\dots>\pi_{i_4}<\pi_{i_3}>\pi_{i_2}<\pi_{i_1},\qquad (i_1>i_2\dots)$$
is preserved by f. Hence, $al(r(\pi))=al(r(f\pi))$.

■

Thus we need to consider in detail only the cases $\tau=123$ and $\tau=132$, which we now proceed to do.

2.1. **A system of functional equations for the generating functions.** In this section we find an explicit formula for the generating functions $A_\tau(x,1,q)$, where $\tau\in S_3$. To do that we first find recurrence relations for the generating functions $a_\tau(n;j)$ by using the scanning-elements algorithm as described in [7]. A rewriting of these relations automatically gives a system of functional equations satisfied by the multivariate generating function $A_\tau(x,v,q)$.

The argument in each case is similar. We consider first the case $\tau=123$. From the above definitions we have for all $n\ge3$,

$$\begin{aligned}
a_{123}(n;j) &= \sum_{i=1}^{j-1} a_{123}(n;j,i) + a_{123}(n;j,n) \\
&= q\sum_{i=1}^{j-1} b_{123}(n-1;i) + \sum_{i=1}^{j-1} a_{123}(n;j,n,i) + a_{123}(n;j,n,n-1) \\
&= q\sum_{i=1}^{j-1}(b_{123}(n-1;i) + b_{123}(n-2;i)) + a_{123}(n;j,n,n-1) \\
&= \cdots = q\sum_{i=1}^{j-1}(b_{123}(n-1;i) + \cdots + b_{123}(j,i)) + a_{123}(n;j,n,\ldots,j+1) \\
&= q\sum_{i=1}^{j-1}(b_{123}(n-1;i) + \cdots + b_{123}(j,i)) + \begin{cases} qb_{123}(j-1), & j>1 \\ q^2, & j=1, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
b_{123}(n;j) &= \sum_{i=1}^{j-1} b_{123}(n;j,i) + b_{123}(n;j,n) \\
&= \sum_{i=1}^{j-1} b_{123}(n-1;i) + \sum_{i=1}^{j-1} b_{123}(n;j,n,i) + b_{123}(n;j,n,n-1) \\
&= \sum_{i=1}^{j-1}(b_{123}(n-1;i) + q^2 b_{123}(n-2;i)) + b_{123}(n;j,n,n-1) \\
&= \cdots = \sum_{i=1}^{j-1}(b_{123}(n-1;i) + q^2 b_{123}(n-2,i) + \cdots + q^2 b_{123}(j,i)) + b_{123}(n;j,n,\ldots,j+1) \\
&= \sum_{i=1}^{j-1}(b_{123}(n-1;i) + q^2 b_{123}(n-2;i) + \cdots + q^2 b_{123}(j,i)) + \begin{cases} q^3, & j=1 \\ q^2 b_{123}(j-1), & j>1 \\ b_{123}(n-1), & j=n. \end{cases}
\end{aligned}$$

Hence, for all $n \geq 4$ and $j = 1, 2, \ldots, n$,

$$\begin{cases} a_{123}(n;j) - a_{123}(n-1;j) &= q\sum_{i=1}^{j-1} b_{123}(n-1;i), \\ b_{123}(n;j) - b_{123}(n-1;j) &= \sum_{i=1}^{j-1}(b_{123}(n-1;i) + (q^2-1)b_{123}(n-2;i)). \end{cases}$$

Multiplying by v^{j-1} and summing over $j = 1, 2, \ldots, n$ we obtain

$$\begin{aligned}
&A_{123}(n;v) - A_{123}(n-1;v) \\
&\quad = q\sum_{j=1}^{n} v^{j-1}\sum_{i=1}^{j-1} b_{123}(n-1;i) = q\sum_{i=1}^{n-1}\sum_{j=i}^{n-1} v^j b_{123}(n-1;i) = \tfrac{qv}{1-v}\sum_{i=1}^{n-1}(v^{i-1} - v^{n-1})b_{123}(n-1;i) \\
&\quad = \tfrac{qv}{1-v}(B_{123}(n-1;v) - v^{n-1}B_{123}(n-1;1)),
\end{aligned}$$

$$\begin{aligned}
&B_{123}(n;v) - B_{123}(n-1;v) \\
&\quad = \sum_{j=1}^{n} v^{j-1}\sum_{i=1}^{j-1}(b_{123}(n-1;i) + (q^2-1)b_{123}(n-2;i)) \\
&\quad = \sum_{i=1}^{n-1}\sum_{j=i}^{n-1} v^j(b_{123}(n-1;i) + (q^2-1)b_{123}(n-2;i)) \\
&\quad = \tfrac{v}{1-v}\sum_{i=1}^{n-1}(v^{i-1} - v^{n-1})b_{123}(n-1;i) + \tfrac{(q^2-1)v}{1-v}\sum_{i=1}^{n-2}(v^{i-1} - v^{n-1})b_{123}(n-2;i) \\
&\quad = \tfrac{v}{1-v}(B_{123}(n-1;v) - v^{n-1}B_{123}(n-1;1)) + \tfrac{(q^2-1)v}{1-v}(B_{123}(n-2;v) - v^{n-2}B_{123}(n-2;1)),
\end{aligned}$$

for all $n \geq 4$. Multiplying the above recurrence relations by x^n/v^n and summing over all $n \geq 4$, we obtain

$$\begin{aligned}
&A_{123}(x/v;v) - \sum_{i=0}^{3} A_{123}(i;v)\tfrac{x^i}{v^i} - \tfrac{x}{v}A_{123}(x/v;v) + x\sum_{i=0}^{2} A_{123}(i;v)\tfrac{x^i}{v^i}\\
&\qquad = \tfrac{qx}{1-v}\left(B_{123}(x/v;v) - \sum_{i=0}^{2} B_{123}(i;v)\tfrac{x^i}{v^i} - B_{123}(x;1) + \sum_{i=0}^{2} B_{123}(i;1)x^i\right)\\
&B_{123}(x/v;v) - \sum_{i=0}^{3} B_{123}(i;v)\tfrac{x^i}{v^i} - \tfrac{x}{v}B_{123}(x/v;v) + x\sum_{i=0}^{2} B_{123}(i;v)\tfrac{x^i}{v^i}\\
&\qquad = \tfrac{x}{1-v}\left(B_{123}(x/v;v) - \sum_{i=0}^{2} B_{123}(i;v)\tfrac{x^i}{v^i} - B_{123}(x;1) + \sum_{i=0}^{2} B_{123}(i;1)x^i\right)\\
&\qquad + \tfrac{(q^2-1)x^2}{v(1-v)}\left(B_{123}(x/v;v) - \sum_{i=0}^{1} B_{123}(i;v)\tfrac{x^i}{v^i} - B_{123}(x;1) + \sum_{i=0}^{1} B_{123}(i;1)x^i\right),
\end{aligned}$$

and using the initial conditions

$$\begin{aligned}
&A_{123}(0;v) = 1, && B_{123}(0;v) = 1,\\
&A_{123}(1;v) = q, && B_{123}(1;v) = q,\\
&A_{123}(2;v) = q(1+vq), && B_{123}(2;v) = q(q+v),\\
&A_{123}(3;v) = q^2(1+(1+q)v+(1+q)v^2), && B_{123}(3;v) = q(q^2+q(1+q)v+(1+q)v^2),
\end{aligned}$$

together with some rather tedious algebraic manipulations we conclude that

$$\begin{cases} \left(1-\frac{x}{v}\right)A_{123}(x/v,v) - \frac{xq}{1-v}B_{123}(x/v,v) = 1-(1-q)\frac{x}{v} - q(1-q)\frac{x^3}{v^3} - \frac{xq}{1-v}B_{123}(x,1);\\ \left(1-\frac{x+(q^2-1)x^2}{v(1-v)}\right)B_{123}(x/v,v) = 1+(q-1)(1+q\frac{x}{v}+q^2\frac{x^2}{v^2})\frac{x}{v} - \frac{xv+(q^2-1)x^2}{v(1-v)}B_{123}(x,1). \end{cases} \tag{2.1}$$

We next consider the case $\tau = 132$. From the definitions, for each $j = 1, 2, \ldots, n$ we have the system of recurrences

$$a_{132}(n;j) = \sum_{i=j+1}^{n} a_{132}(n;j,i) + \sum_{i=1}^{j-1} a_{132}(n;j,i) = a_{132}(n-1;j) + q\sum_{i=1}^{j-1} b_{132}(n-1;i),$$

$$b_{132}(n;j) = \sum_{i=j+1}^{n} b_{132}(n;j,i) + \sum_{i=1}^{j-1} b_{132}(n;j,i) = qa_{132}(n-1;j) + \sum_{i=1}^{j-1} b_{132}(n-1;i).$$

Multiplying by v^{j-1} and summing over all possible $j = 1, 2, \ldots, n$ we obtain, for each $n \geq 2$, the system

$$\begin{aligned}
A_{132}(n;v) &= \sum_{j=1}^{n} v^{j-1}a_{132}(n-1;j) + q\sum_{j=1}^{n-1} v^{j-1}\sum_{i=1}^{j-1} b_{132}(n-1;i)\\
&= A_{132}(n-1;v) + q\sum_{i=1}^{n-1}\sum_{j=i}^{n-1} v^j b_{132}(n-1;i)\\
&= A_{132}(n-1;v) + \tfrac{qv}{1-v}\sum_{i=1}^{n-1}(v^{i-1}-v^{n-1})b_{132}(n-1;i)\\
&= \tfrac{qv}{1-v}(B_{132}(n-1;v) - v^{n-1}B_{132}(n-1;1)) + A_{132}(n-1;v),\\
B_{132}(n;v) &= q\sum_{j=1}^{n} v^{j-1}a_{132}(n-1;j) + \sum_{j=1}^{n-1} v^{j-1}\sum_{i=1}^{j-1} b_{132}(n-1;i)\\
&= qA_{132}(n-1;v) + \sum_{i=1}^{n-1}\sum_{j=i}^{n-1} v^j b_{132}(n-1;i)\\
&= qA_{132}(n-1;v) + \tfrac{v}{1-v}\sum_{i=1}^{n-1}(v^{i-1}-v^{n-1})b_{132}(n-1;i)\\
&= \tfrac{v}{1-v}(B_{132}(n-1;v) - v^{n-1}B_{132}(n-1;1)) + qA_{132}(n-1;v).
\end{aligned}$$

Multiplying the above recurrence relations by x^n/v^n and summing over $n \geq 2$ while using the initial conditions $A_{132}(0;v) = B_{132}(0;v) = 1$ and $A_{132}(1;v) = B_{132}(1;v) = q$ we find that

$$\begin{aligned}
A_{132}(x/v;v) - 1 - q\tfrac{x}{v} &= \tfrac{qx}{1-v}\left(B_{132}(x/v;v) - B_{132}(x;1)\right) + \tfrac{x}{v}\left(A_{132}(x/v;v) - 1\right),\\
B_{132}(x/v;v) - 1 - q\tfrac{x}{v} &= \tfrac{x}{1-v}\left(B_{132}(x/v;v) - B_{132}(x;1)\right) + \tfrac{qx}{v}\left(A_{132}(x/v;v) - 1\right),
\end{aligned}$$

which is equivalent to

$$\left(1-\frac{x}{v}\right)A_{132}(x/v,v)-\frac{xq}{1-v}B_{132}(x/v,v)=1+(q-1)\frac{x}{v}-\frac{xq}{1-v}B_{132}(x,1);$$
$$-\frac{xq}{v}A_{132}(x/v,v)+\left(1-\frac{x}{1-v}\right)B_{132}(x/v,v)=1-\frac{x}{1-v}B_{132}(x,1).$$

We can eliminate $A_{132}(x/v;v)$ from the second equation by adding xq/v times the first equation to $(1-x/v)$ times the second. This yields the equivalent system

$$\begin{cases} \left(1-\frac{x}{v}\right)A_{132}(x/v,v)-\frac{xq}{1-v}B_{132}(x/v,v)=1+(q-1)\frac{x}{v}-\frac{xq}{1-v}B_{132}(x,1), \\ \left(1-\frac{x+(q^2-1)x^2}{v(1-v)}\right)B_{132}(x/v,v)=1+\frac{x(q-1)}{v}+\frac{x^2q(q-1)}{v^2}-\frac{xv+(q^2-1)x^2}{v(1-v)}B_{132}(x,1). \end{cases} \tag{2.2}$$

2.2. **Solution of the functional equations via the kernel method.** In both systems (2.1) and (2.2) the second equation can be written in the form

$$K(x/v,v,q)B_\tau(x/v,v,q)=R(x/v,q)+(K(x/v,v,q)-1+x/v)B_\tau(x,1,q),$$

where

$$K(x,v,q)=1-\frac{x+(q^2-1)vx^2}{1-v}$$
$$R_{123}(x,q)=1+(q-1)x(1+qx+q^2x^2)$$
$$R_{132}(x,q)=1+(q-1)x(1+qx).$$

This seemingly underdetermined system may be solved systematically using the *kernel method*, as described in [1]. If we set $K(x/v,v,q)$ to zero, then we have $B_\tau(x,1,q)=R(x/v,q)/(1-x/v)$ where (x,v,q) are linked by the equation $K(x/v,v,q)=0$. Near the origin we have

$$K(x,v,q)=\frac{(v-\xi_+(x,q))(v-\xi_-(x,q))}{v(1-v)}$$

where

$$\xi_\pm(x,q)=\frac{1\pm\sqrt{(1-2x)^2-4q^2x^2}}{2}=\frac{1\pm\sqrt{(1-4x+4(1-q^2)x^2}}{2}=\frac{1\pm\sqrt{(1-2(1+q)x)(1-2(1-q)x)}}{2}.$$

We must take $v=\xi_+(x,q)$ in order to obtain a power series solution. This then yields

$$B_{132}(x,1,q)=\frac{\xi_-(x,q)}{x}\frac{1+(q-1)x}{1+(q^2-1)x} \tag{2.3}$$

$$B_{123}(x,1,q)=\frac{(q-1)(x-1)}{1+(q^2-1)x^2}+\frac{\xi_-(x,q)}{x}\frac{q(1+(q-1)x)}{(1+(q^2-1)x)^2}. \tag{2.4}$$

Using the reductions above and Proposition 2.2 we obtain the following result. Since $A_\tau(x,1,q)$ and $B_\tau(x,1,q)$ each reduce to the Catalan generating function $C(x)=(1-\sqrt{1-4x})/(2x)$ when $q=1$, we expect them to be, at best, algebraic and not rational. As we now see, nothing worse happens — we obtain a nice deformation of C.

Theorem 2.3. *The generating functions for the number of τ-avoiding permutations π of length n with $al(\pi)=m$ all have the form*

$$A_\tau(x,1,q)=a(x,q)+b(x,q)\frac{1-\sqrt{(1-2x)^2-4x^2q^2}}{2x}$$

where a and b are as given in the table below.

τ	$a(x,q)$	$b(x,q)$
123	$\frac{(1-q^2)(q(1+q)(1-q)^2x^4-q(1-q)x^3+x^2(1-q)+x(q-2)+1)}{(1-x+xq^2)^2}$	$\frac{q^2(1-x+qx)}{(1-x+xq^2)^3}$
132, 231, 321	$\frac{(1-q)(1-x)}{1-x+xq^2}$	$\frac{q(1-x+xq)}{(1-x+xq^2)^2}$
312, 213	0	$\frac{1-x+xq}{(1-x+xq^2)}$

The corresponding results for B_τ are easily obtained from the reductions described earlier and so are not listed here.

3. Extraction of coefficients from the GF

A number of corollaries follow from Theorem 2.3. The first is an explicit expression for the number of permutations π of length n having $al(\pi) = m$. To do this we require the following lemma.

Lemma 3.1. *Let $f_s(x,q)$ be the generating function $\frac{1-x+xq}{2(1-x+xq^2)^s}(1-\sqrt{(1-2x)^2-4x^2q^2})$. Let $n \geq 3$ and $n \geq m \geq 1$. Then the x^nq^m coefficient in the generating function $f_s(x,q)$ is*

$$\sum_{j=0}^{s-2} \frac{(-1)^{\lfloor \frac{m}{2} \rfloor}}{j+1}\binom{2j}{j}\binom{\lfloor \frac{m}{2}\rfloor+s-2-j}{s-2-j}\binom{n+s-2j-4}{\lfloor \frac{m-1}{2}\rfloor+s-2-j} + \sum_{j=\lfloor \frac{m}{2}\rfloor+s-1}^{n} \frac{(-1)^{n-j-\lfloor \frac{m+3}{2}\rfloor}}{j+1}\binom{2j}{j}\binom{j+1-s}{\lfloor \frac{m}{2}\rfloor}\binom{j-s-\lfloor \frac{m-3}{2}\rfloor}{n-j-\lfloor \frac{m+3}{2}\rfloor}.$$

Proof. We have

$$u := f_s(x,q) - \frac{x(1-x+xq)}{(1-x+xq^2)^{s-1}} = \frac{(1-x+xq^2)^s}{1-x+xq}\left(u + \frac{x(1-x+xq)}{(1-x+xq^2)^{s-1}}\right)^2.$$

Hence by Lagrange inversion formula [16]

$$f_s(x,q) = (1-x+xq)\left(\sum_{j=0}^{s-2} \frac{x^{j+1}}{j+1}\binom{2j}{j}\frac{1}{(1-x+xq^2)^{s-1-j}} + \sum_{j\geq s-1} \frac{x^{j+1}}{j+1}\binom{2j}{j}(1-x+xq^2)^{j+1-s}\right).$$

Expanding the above generating function at $x = q = 0$ gives the desired result. ∎

Corollary 3.2. *Fix $n \geq 1$ and let $1 \leq m \leq n$. Then*

(i) The number of 123-avoiding permutations π of length $n \geq 3$ satisfying $al(\pi) = m$ is given by

$$\sum_{j=\lfloor \frac{m+2}{2}\rfloor}^{n+1} \frac{(-1)^{n-j-\lfloor \frac{m-1}{2}\rfloor}}{j+1}\binom{2j}{j}\binom{j-2}{\lfloor \frac{m-2}{2}\rfloor}\binom{j-\lfloor \frac{m+1}{2}\rfloor}{n-j-\lfloor \frac{m-1}{2}\rfloor}.$$

(ii) The number of 132-avoiding (or of 231-avoiding or 321-avoiding) permutations π of length n satisfying $al(\pi) = m$ is given by

$$\sum_{j=\lfloor \frac{m+1}{2}\rfloor}^{n+1} \frac{(-1)^{n-j-\lfloor \frac{m}{2}\rfloor}}{j+1}\binom{2j}{j}\binom{j-1}{\lfloor \frac{m-1}{2}\rfloor}\binom{j-\lfloor \frac{m}{2}\rfloor}{n-j-\lfloor \frac{m}{2}\rfloor}.$$

(iii) The number of 312-avoiding (or of 213-avoiding) permutations π of length n satisfying $al(\pi) = m$ is given by

$$\sum_{j=\lfloor \frac{m-1}{2}\rfloor}^{n+1} \frac{(-1)^{n-j-\lfloor \frac{m+1}{2}\rfloor}}{j+1}\binom{2j}{j}\binom{j}{\lfloor \frac{m}{2}\rfloor}\binom{j-\lfloor \frac{m-1}{2}\rfloor}{n-j-\lfloor \frac{m+1}{2}\rfloor}.$$

Proof. It is not hard to see that the generating function $A_{123}(x,1,q)$ can be written as

$$\begin{aligned} A_{123}(x,1,q) &= 1 - \tfrac{x^3-x}{1-x}q + \tfrac{x^3-x-1}{1-x}q^2 + \sum_{m\geq 3}(-1)^{\lfloor \frac{m+2}{2}\rfloor}\frac{x^{\lfloor \frac{m-1}{2}\rfloor}(x^2-x-\lfloor \frac{m}{2}\rfloor)}{(1-x)^{\lfloor \frac{m+1}{2}\rfloor}} \\ &+ \tfrac{q^2(1-x+qx)}{2x(1-x+xq^2)^3}(1-\sqrt{(1-2x)^2-4x^2q^2}) \end{aligned}$$

Lemma 3.1 yields the first formula and arguments similar to those used in the first case provide the others. ∎

Theorem 2.3 also provides us with the following asymptotics as $n \to \infty$.

Corollary 3.3. *Fix $m \geq 3$. Then for each τ we have as $n \to \infty$*

$$al_\tau(n,m) \sim 2^n(n/2)^m W(n)X(m)$$

where W and X are in $\Theta(1)$. More specifically we have the following asymptotic results.

(i) As $n \to \infty$, the number of 123-avoiding permutations π of length n satisfying $al(\pi) = m$ satisfies

$$\begin{aligned} a_{123}(n,1) &= 0 \\ a_{123}(n,2) &= n \\ a_{123}(n,3) &= \binom{n}{2} - 1 \\ a_{123}(n,m) &\sim \frac{n^{2\lfloor \frac{m}{2}\rfloor - 4}}{(\lfloor \frac{m}{2} - 1\rfloor)!(\lfloor \frac{m}{2}\rfloor - 2)!} \cdot 2^{n-2\lfloor \frac{m}{2}\rfloor + 5} \qquad \textit{for } m \geq 4. \end{aligned}$$

(ii) As $n \to \infty$, the number of 132-avoiding (also the number of 231-avoiding or of 321-avoiding) permutations π of length n satisfying $al(\pi) = m$ satisfies

$$\begin{aligned} a_{132}(n,1) &= 1 \\ a_{132}(n,2) &= n - 1 \\ a_{132}(n,m) &\sim \frac{n^{2\lfloor \frac{m+1}{2}\rfloor - 4}}{(\lfloor \frac{m-1}{2}\rfloor)!(\lfloor \frac{m-3}{2}\rfloor)!} \cdot 2^{n-2\lfloor \frac{m+1}{2}\rfloor + 4} \qquad \textit{for } m \geq 3. \end{aligned}$$

(iii) As $n \to \infty$, the number of 312-avoiding (also the number of 213-avoiding) permutations π of length n satisfying $al(\pi) = m$ satisfies

$$\begin{aligned} a_{312}(n,1) &= 1 \\ a_{312}(n,2) &= n - 1 \\ a_{312}(n,m) &\sim \frac{n^{2\lfloor \frac{m}{2}\rfloor - 2}}{(\lfloor \frac{m}{2}\rfloor)!(\lfloor \frac{m}{2}\rfloor - 1)!} \cdot 2^{n-2\lfloor \frac{m}{2}\rfloor + 1} \qquad \textit{for } m \geq 3. \end{aligned}$$

Proof. Let

$$f = A_\tau(x,1,q) = a(x,q) + b(x,q)\frac{1 - \sqrt{(1-2x)^2 - 4x^2q^2}}{2x}$$

as in Theorem 2.3. Then we can rewrite f as

$$\begin{aligned} f &= a(x,q) + b(x,q) + b(x,q)\frac{1 - 2x - \sqrt{(1-2x)^2 - 4x^2q^2}}{2x} \\ &= a(x,q) + b(x,q) + b(x,q)\frac{1 - \sqrt{1 - 4x^2q^2/(1-2x)^2}}{2x/(1-2x)} \\ &= a(x,q) + b(x,q)(1 + c(x,q)). \end{aligned}$$

We note that the subsitution $u = xq/(1-2x)$ gives

$$c(x,q) = qu\frac{1 - \sqrt{1-4u^2}}{2u^2} = \sum_{j\geq 0} c_j \frac{x^{2j+1}q^{2j+2}}{(1-2x)^{2j+1}}$$

where c_j is the j-th Catalan number $\frac{1}{j+1}\binom{2j}{j}$.

Since $a(x,q)$ and $b(x,q)$ are rational with denominator $(1-x+xq^2)^s = (1-x)^s(1+q^2(x/(1-x))^s$ for some $s > 0$, we see by geometric series expansion that for each m the coefficient $[q^m]a(x,q)$ and $[q^m]b(x,q)$ has a pole at $x = 1$ and no other singularities. Thus the above representation makes it clear that $[q^m]f$ has dominant singularity at $x = 1/2$, coming only from the square root term. Hence the exponential growth rate of $[q^m]f$ as $n \to \infty$ is 2. In fact the leading term asymptotic comes only from the term with highest degree j occurring.

We give more details only for the case $\tau = 321$ as the other cases are similar. Write $a = x/(1-x)$ and $b = x/(1-2x) = (1/2)/(1/2 - x)$. Then we have

$$A_{321}(x,1,q) = \frac{q}{1-x}\frac{1+aq}{(1+aq^2)^2}\left[1 + bq^2\sum_{j\geq 0} c_j(b^2q^2)^j\right].$$

For $l \geq 2$, consider the coefficient of q^{2l} in this last equation. The highest degree term in j occurring is when $j = l - 2$, and the coefficient of q^{2l} is equal to $ab^{2l-3}c_{l-2}/(1-x)$. A straightforward partial fraction argument (for example as in [6, p243]), followed by some algebraic simplification using the explicit formula for the Catalan numbers, gives us a leading order asymptotic in n of $2^n 2^{-(l-2)} n^{2l-4}/((l-1)!(l-2)!)$.

Similarly, consider the coefficient of q^{2l+1}. The highest degree term in j occurring is when $j = l-1$, and the coefficient of q^{2l+1} is then $b^{2l-1} c_{l-1}/(1-x)$. We obtain a leading order asymptotic of $2^n 2^{-(l-1)} n^{2l-2}/(l!(l-1)!)$.

We can combine the odd and even degree cases by observing that when $m = 2l$, $l - 2 = \lceil (m-3)/2 \rceil$, whereas when $m = 2l+1$, $l - 1 = \lceil (m-3)/2 \rceil$. This yields the displayed formula. ■

We might also consider different regimes for bivariate asymptotics. Robin Pemantle and coauthors [15] have an ongoing programme to derive asymptotics via analysis of multivariate generating functions. Their results are particularly well suited to the case where n and m are of the same asymptotic order, namely $m = \Theta(n)$ as $n \to \infty$. The particular generating functions we have derived do not fall under any of the cases which they have so far solved, however, and we proceed no further in this paper.

3.1. **Probabilistic considerations.** Recall that we consider the random variable al_n that gives the length of the longest alternating subsequence of a τ-avoiding permutation chosen uniformly. We can easily obtain the mean and variance.

Corollary 3.4. *The following results on mean and variance hold.*

(i) The mean and variance of the longest alternating subsequence in 123*-avoiding permutations of length n are given for all $n \geq 2$ by*

$$\mu_{123}(n) = \frac{2n^2 + 5n - 9}{2(2n-1)}$$
$$\sigma_{123}(n) = \frac{(n+1)(8n^3 - 50n^2 + 101n + 9)}{4(2n-1)^2(2n-3)}.$$

(ii) The mean and variance of the longest alternating subsequence in 132*-avoiding permutations (also the number of* 231*-avoiding permutations or* 321*-avoiding permutations) of length n are given for all $n \geq 2$ by*

$$\mu_{132}(n) = \frac{(n-1)(2n+5)}{2(2n-1)}$$
$$\sigma_{132}(n) = \frac{(n+1)(8n^3 - 42n^2 + 73n - 15)}{4(2n-1)^2(2n-3)}$$

(iii) The mean and variance of the longest alternating subsequence in 312*-avoiding permutations (also the number of* 213*-avoiding permutations) of length n are given for all $n \geq 2$ by*

$$\mu_{312}(n) = \frac{n+1}{2}$$
$$\sigma_{312}(n) = \frac{(n+1)(4n^2 - 15n + 15)}{4(2n-1)(2n-3)}.$$

Proof. The explicit formula for $\mu_{123}(n) = \frac{1}{c_n} \sum_{\pi \in S_n(123)} al(\pi)$ can be obtained from the fact that the generating function $\sum_{n \geq 0} \mu_{123}(n) c_n x^n$ is given by

$$\frac{\partial}{\partial q} A_{123}(x,1,q) \mid_{q=1} = \frac{(1-4x)(2-x)}{2x} - \frac{(2 - 13x + 16x^2)}{2x\sqrt{1-4x}} = x + \sum_{n \geq 2} \frac{2n^2 + 5n - 9}{2(2n-1)} c_n x^n.$$

Thus, $\mu_{123}(n) = \frac{2n^2+5n-9}{2(2n-1)}$, as claimed.

The second factorial moment $\mu'_{123}(n) = \frac{1}{c_n} \sum_{\pi \in S_n(123)} al(\pi)(al(\pi) - 1)$ can be obtained from the fact that the generating function $\sum_{n \geq 0} \mu'_{123}(n) c_n x^n$ is given by

$$\begin{aligned} \frac{\partial^2}{\partial q^2} A_{123}(x,1,q) \mid_{q=1} &= -\frac{4x^4 + 12x^3 - 32x^2 + 15x - 1}{x} - \frac{1 - 21x + 128x^2 - 292x^3 + 200x^4}{x\sqrt{1-4x}^3} \\ &= 2x^2 + \sum_{n \geq 3} \frac{n^4 + 3n^3 - 15n^2 - 2n + 45}{(2n-1)(2n-3)} c_n x^n. \end{aligned}$$

Using this expression, the variance $\sigma_{123}(n)$ is given by $\sigma_{123}(n) = \mu'_{123}(n) + \mu_{123}(n) - \mu_{123}(n)^2$ and so we have $\sigma_{123}(n) = \frac{(n+1)(8n^3-42n^2+73n-15)}{4(2n-1)^2(2n-3)}$.

The argument is the same for any other value of τ. ∎

We can now obtain the limiting distribution of the longest alternating subsequence for pattern-restricted permutations in which the pattern is any one of the six patterns of length three.

Theorem 3.5. *Let $\tau \in S_3$ and define the random variable X_n^τ on $S_n(\tau)$ by*

$$X_n^\tau = \frac{al_n - \frac{n}{2}}{\frac{1}{2}\sqrt{n}}.$$

Then X_n^τ converges in distribution to a standard normal as $n \to \infty$.

Proof. We give the argument for $\tau = 123$, the other cases being completely analogous. Theorem 2.3 gives that $A_{123}(x,1,q) = V(x,q) + U(x,q)\sqrt{1-2x-2xq}$, where

$$\begin{aligned} V(x,q) &= \tfrac{(1-q^2)(q(1+q)(1-q)^2x^4 - q(1-q)x^3 + x^2(1-q) + x(q-2)+1)}{(1-x+xq^2)^2} + \tfrac{q^2(1-x+qx)}{2x(1-x+xq^2)^3}, \\ U(x,q) &= -\tfrac{q^2(1-x+qx)}{2x(1-x+xq^2)^3}\sqrt{1-2x+2xq}. \end{aligned}$$

We apply a general result [6, Thm IX.10] on limit laws for algebraic singularities (which ultimately relies on the so-called Quasi-Power theorem). We need to verify several conditions. First, note that we can choose $\varepsilon > 0$ so that xU and V are analytic for $|x| \le r = 1/4 + \varepsilon$ and $|q-1| < \varepsilon$. Next, $xU(x,1)$ has a unique root $\rho = 1/4 < r$, and $\rho U(\rho, 1) \neq 0$. For q near 1, there is a nonconstant root $x = \rho(q)$ of the equation $1 - 2x - 2xq = 0$, namely $\rho(q) = [2(1+q)]^{-1}$, ρ is analytic in q, and $\rho(1) = \rho$. Finally, the variability condition is satisfied: the quantity $(\rho/\rho(1))'' + (\rho/\rho(1))' - [(\rho/\rho(1))']^2$ is nonzero, where ρ' denotes the derivative with respect to q.

The result now follows from the quoted theorem (note that the details are very similar to those in Example 17 following the theorem). ∎

We can ask more refined questions, such as whether a local limit theorem holds (the normalized probability density converges to the normal density). The result described in [6, Thm IX.13], and the example following, show that a local Gaussian limit law also holds in our case. Thus the normalized histogram of the values of al_n is well approximated by the standard normal density for large n. Similarly we could consider large deviation probabilities as in [6, Thm IX.14]. However, this is beyond the scope of our main interest here and we omit the details.

4. Comments and ideas for future investigation

We note that the mean and variance of al_n are asymptotically independent of the pattern and the same limit law is obtained in each case. The results given in [5] for the longest increasing subsequence show that for some patterns similar Gaussian limiting behaviour occurs, with the same asymptotic mean and variance. However for others a quite different behaviour ensues (the most extreme is the case $\tau = 123$ since if π avoids τ then its longest increasing subsequence is of length at most 2, but there are other more interesting cases). Thus, just as in the unrestricted case, considering longest alternating subsequences always leads to Gaussian limits, and this is markedly different from the situation with longest increasing subsequences.

The full history recurrences used to derive the functional equations for the generating functions A_τ and B_τ are somewhat cumbersome. Ideally one would like to derive the functional equation directly from some context-free grammar defining the combinatorial class in question ("the symbolic method"), as in [6].

The explicit formulae for the coefficients of $A_\tau(x,1,q)$ involving alternating multiple sums are rather complicated. While a simpler explicit formula of this type may not be possible, a simpler recurrence relation may be possible. Since A_τ is algebraic it is D-finite and satisfies a linear PDE with polynomial coefficients. Thus the values $a_\tau(n,m)$ also satisfy a linear recurrence with polynomial coefficients. We have not pursued this further, but it may yield a simpler method for calculating $a_\tau(n,m)$ for moderately large values of n and m.

By Theorem 2.3 we have $A_{123}(x,1,-1) = \frac{1-2x}{2x}(1-\sqrt{1-4x})$. This implies that the number of 123-avoiding permutations of length n with odd longest alternating subsequence equals $\frac{n+1}{3(n-1)}$ times the number of 123-avoiding permutations of length n with even longest alternating subsequence.

Similar results hold for the other values of τ but they are different: for example, when $\tau = 132$ the corresponding fraction is $\frac{n-2}{2n-1}$. It is not immediately obvious to us why this disparity should occur.

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Department of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: gferro@math.haifa.ac.il

Department of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: toufik@math.haifa.ac.il

Department of Computer Science, University of Auckland, Private Bag 92019 Auckland, New Zealand
E-mail address: mcw@cs.auckland.ac.nz