Measuring Partial Balance in Signed Networks

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Abstract

Is the enemy of an enemy necessarily a friend, or a friend of a friend a friend? If not, to what extent does this tend to hold? Such questions were formulated in terms of signed (social) networks and necessary and sufficient conditions for a network to be "balanced" were obtained around 1960. Since then the idea that signed networks tend over time to become more balanced has been widely used in several application areas, such as international relations. However investigation of this hypothesis has been complicated by the lack of a standard measure of partial balance, since complete balance is almost never achieved in practice. We formalize the concept of a measure of partial balance, discuss various measures, compare the measures on real-world and synthetic datasets and investigate their axiomatic properties. We use both well-known datasets from the sociology literature, such as Read's New Guinean tribes, and much more recent ones involving senate bill co-sponsorship. The synthetic data involves Barabási-Albert and specially structured random graphs. We show that some measures behave better than others in terms of axioms, computational tractability and ability to differentiate between graphs. We also find that under all our measures, real-world networks are more balanced than what is expected by chance. We make some recommendations for measures to be used in future work.

Keywords: *structural analysis, signed networks, balance theory, axiom, frustration index, algebraic conflict*

1 Introduction

Transitivity of relationships has a pivotal role in analyzing social interactions. Is the enemy of an enemy a friend? What about friend of an enemy or enemy of a friend? Network science is a key instrument in quantitative analysis for such questions. Researchers in the field are interested in knowing the extent of transitivity of ties and its impact on the global structure and dynamics in communities with positive and negative relationships. Whether the application involves international relationships among states, friendships and enmities between people, or ties of trust and distrust formed among shareholders, relationship to a third entity tends to be influenced by immediate ties.

There is a growing body of literature that aims to connect theories of social structure with network science tools and techniques to study local behaviors and global structures in signed graphs that come up naturally in many unrelated areas. The building block of structural balance is a work by Heider [Heider, 1944] that was expanded into a set of graph-theoretic concepts by Cartwright and Harary [Cartwright and Harary, 1956] to handle a social psychology problem a decade later. The relationship under study has an antonym or dual to be expressed by the opposite sign [Harary, 1957]. In a setting where the opposite of a negative relationship is a positive relationship, a tie to a distant neighbor can be expressed by the product of signs reaching him. Cycles containing an odd number of negative edges are considered unbalanced, guaranteeing total balance only in networks containing no cycles with an odd number of negative edges. This strict condition makes it quite unlikely for a signed network to be totally balanced. The literature on signed networks suggest many different formulae to measure balance. These measures are useful for detecting total balance and imbalance, but for intermediate cases their performance is not clear and has not been systematically studied.

Our contribution

The main focus of this paper is to provide insight into measuring partial balance, as much uncertainty still exists on this. The dynamics leading to specific global structures in signed networks remain speculative even after studies with fine-grained approaches. The central thesis of this paper is that measures of partial balance should relate to the application, as a considerable extent of balance is flexible in interpretation in different areas. Thus we need to consider a variety of measures. However, not all measures are equally useful. We provide a numerical comparison of several measures of partial balance on a variety of undirected signed networks, both randomly generated and inferred from well-known datasets. Using theoretical results for simple classes of graphs, we suggest an axiomatic framework to evaluate these measures and shed light on the context-dependency involved in using such measures.

This paper begins by laying out the theoretical dimensions of the research in Section 2 and looks at basic definitions and terminology. In Section 3 different means of checking for total balance are outlined. Section 4 discusses some approaches to measure partial balance in Eq. (3) – (12) summarized in Table 1. Basic numerical results on random networks are provided in Figures 1 – 2 in Section 5. Section 6 is concerned with performance of measures on specially structured graphs in Figure 3 – 5. A set of axioms is suggested in Section 7 to evaluate the measures systematically. Section 8 concerns recommendations for choosing a measure of balance. The numerical results on real signed networks are presented in Section 9. Section 10 presents the findings of the research regarding the interpretation and methodology. Finally, Section 11 sums up the research highlights and provides direction for future research. Throughout this paper, the terms signed graph and signed network will be used interchangeably to refer to a graph with positive and negative edges. While several definitions of the concept of balance have been suggested, this paper will only use the original definition for undirected signed graphs unless explicitly stated.

2 Problem statement and notation

We consider undirected signed network $G = (V, E, \sigma)$ where σ is the sign function $\sigma : E \rightarrow \{-1, +1\}$. The set of nodes is denoted by V, with $|V| = n$. E stands for the set of edges including m^- negative edges and m^+ positive edges adding up to a total of $m = m^+ + m^-$ edges. The *signed adjacency matrix* is defined in (1). We denote *unsigned adjacency matrix* by $|\mathbf{A}|$ which is the entrywise absolute value of $\mathbf{A}$.

$$\mathbf{A}_{uv} = \begin{cases} \sigma_{u,v} & \text{if } u,v \in E \\ 0 & \text{if } u,v \notin E \end{cases} \tag{1}$$

Positive and negative degree of nodes are denoted by d^+ and d^- and calculated based on $\mathbf{A}$. d_u represents the *degree of node u* and is calculated based on $|\mathbf{A}|$. A *walk* of length k in G is a sequence of nodes $v_0, v_1, ..., v_{k-1}, v_k$ such that for each $i = 1, 2, ..., k$ there is an edge from v_{i-1} to v_i. If $v_0 = v_k$, the sequence is a *closed walk* of length k. If the nodes in a closed walk are distinct except endpoints, it is a *cycle* of length k. The *sign of a cycle* is the product of the signs of its edges. A cycle is *balanced* if its sign is positive and is *unbalanced* otherwise. The total number of balanced cycles (closed walks) is denoted by O_k^+ (Q_k^+). Similarly, O_k^- (Q_k^-) denotes the total number of unbalanced cycles (closed walks). The total number of cycles (closed walks) is represented by $O_k = O_k^+ + O_k^-$ ($Q_k = Q_k^+ + Q_k^-$) .

3 Checking for balance

It is essential to have an algorithmic means of checking for balance. The characterization of *bi-polarity*, that a signed graph is balanced if and only if its vertex set can be partitioned into two subsets such that each negative edge joins vertices belonging to different subsets [Cartwright and Harary, 1956], leads to an obvious breadth-first search procedure similar to the usual algorithm for determining whether a graph is bipartite. As acyclic graphs are always bipartite, acyclic signed graphs are always balanced. Moreover, the eigenvalues of the signed and unsigned adjacency matrices are equal if and only if the signed network is balanced [Acharya, 1980]. For our purposes the following additional method is also important. We define the *switching function* $g(X)$ operating over a set of vertices $X \subseteq V$ as follows.

$$\sigma_{(u,v)}^{g(X)} = \begin{cases} \sigma_{u,v} & \text{if } u,v \in X \text{ or } u,v \notin X \\ -\sigma_{u,v} & \text{if } u \in X \text{ and } v \notin X \text{ or } u \notin X \text{ and } v \in X \end{cases} \tag{2}$$

As the sign of cycles remains the same when g is applied, any balanced graph can switch to an all-positive signature [Zaslavsky, 2010]. Accordingly, a balance detection algorithm can be developed by constructing a switching rule on a spanning tree and a root vertex, as suggested in [Zaslavsky, 2010]. Another method of checking for balance in connected signed networks makes use of the signed Laplacian matrix defined by $\mathbf{L} = \mathbf{D} - \mathbf{A}$ where $\mathbf{D}_{ii} = \sum_j |\mathbf{A}|_{ij}$. $\mathbf{L}$ is positive-semidefinite i.e. all of its eigenvalues are nonnegative [Kunegis, 2014]. The smallest eigenvalue of $\mathbf{L}$ equals 0 if and only if the graph is balanced [Hou, 2004].

4 Measures of partial balance

Several ways of measuring the extent to which a graph is balanced have been introduced by researchers. The simplest of such measures is the *degree of balance* suggested by Cartwright and Harary [Cartwright and Harary, 1956], which is the fraction of cycles that are balanced:

$$D(G) = \frac{\sum_{k=3}^{n} O_k^+}{\sum_{k=3}^{n} O_k} \tag{3}$$

There are two measures closely related to $D(G)$. The first is *relative k-balance*, denoted by $D_k(G)$ and formulated in (4). The special case $k = 3$ is called the *triangle index*, denoted by $T(G)$. Relative k-balance is proved by El Maftouhi, Manoussakis and Megalakaki [El Maftouhi et al., 2012] to tend to 0.5 for Erdős-Rényi graphs such that the probability of an edge being negative is equal to $1/2$.

$$D_k(G) = \frac{O_k^+}{O_k} \tag{4}$$

The second measure is *weighted degree of balance* and is obtained by weighting cycle based on length as in (5), in which $f(k) \geq 0$ is a monotonically decreasing function of the length of cycle. The selection of an appropriate function is briefly discussed by Norman and Roberts [Norman and Roberts, 1972], suggesting functions such as $1/k, 1/k^2, 1/2^k$, but no objective criterion for choosing such a weighting function is known.

$$C(G) = \frac{\sum_{k=3}^{n} f(k) O_k^+}{\sum_{k=3}^{n} f(k) O_k} \tag{5}$$

Although fast algorithms are developed for counting and listing cycles of undirected graphs [Birmelé et al., 2013], the number of cycles grows exponentially with network size. To tackle the computational complexity, Terzi and Winkler [Terzi and Winkler, 2011] suggested disregarding all cycles longer than three. Replacing the remaining triangles by closed walks of length 3, triangle index can be calculated efficiently in (6) where $\mathrm{Tr}(A)$ denotes the trace of A.

$$T(G) = D_3(G) = \frac{O_3^+}{O_3} = \frac{\mathrm{Tr}(A^3) + \mathrm{Tr}(|A|^3)}{2 \times \mathrm{Tr}(|A|^3)} \tag{6}$$

Relative signed clustering coefficient is suggested as a measure of balance by Kunegis [Kunegis, 2014] taking insight from the classic clustering coefficient. Being normalized, this measure is equal to triangle index. Having access to an easy-to-compute walk-based formula [Terzi and Winkler, 2011] for $T(G)$ obviates the need for a clustering-based calculation by iterating over all triads in the graph.

Bonacich argues that dissonance and tension are unclear in cycles of length greater than three [Bonacich, 2012], justifying the use of the triangle index to analyze structural balance. However the neglected interactions may represent potential tension and dissonance, though not as strong as that represented by unbalanced triads. Many prefer having a smaller weight for longer cycles, thereby reducing their impact rather than totally disregarding them. Note that $C(G)$ is a generalization of both $D(G)$ and $T(G)$.

Investigating a very basic random process for generating signed graphs, the expected value of cycle-based measures can be calculated. We consider generating signed graphs by a random process of negating edges independently with probability q in a given unsigned graph. Under this process, the probability of a cycle of length k being balanced equals the expected value of relative k-balance, denoted by $E(D_k(G))$ and calculated in (7) for a given q (details of calculations are given in Appendix B). Accordingly, the expected value of $D(G)$ is calculated in (8).

$$E(D_k(G)) = E(\frac{O_k^+}{O_k}) = \sum_{i\,\text{even}} \binom{k}{i} q^i (1-q)^{k-i} = (1+(1-2q)^k)/2 \tag{7}$$

$$E(D(G)) = \frac{\sum_{k=3}^{n} E(O_k^+)}{\sum_{k=3}^{n} O_k} = \frac{\sum_{k=3}^{n} (1+(1-2q)^k)(O_k)/2}{\sum_{k=3}^{n} O_k} \tag{8}$$

Based on (7), we expect $E(D_k(G)) \to 1$ when $q \to 0$ and $E(D_k(G)) \to 0.5$ when $q \to 0.5$. In case of $q \to 1$, $E(D_k(G))$ oscillates based on parity of k. It can be concluded from (8) that $E(D(G)) \to 0.5$ in random signed graphs with non-trivial negative edges. A similar conclusion can be made for $C(G)$. For the triangle index, $E(D_3(G)) = (1+(1-2q)^3)/2$ shows that the expected value of the measure merely depends on q.

Beside checking cycles, there are computationally easier approaches to structural balance such as the walk-based approach. *Walk-based measure of balance* is suggested by Pelino and Maimone [Pelino and Maimone, 2012] with more weight placed on shorter closed walks than the longer ones. Let $\text{Tr}(e^{\mathbf{A}})$ and $\text{Tr}(e^{|\mathbf{A}|})$ denote the trace of matrix exponential for $\mathbf{A}$ and $|\mathbf{A}|$ respectively. In this formula, closed walks are weighted by a function with a relatively fast rate of decay compared to functions suggested by [Norman and Roberts, 1972]. The weighted ratio of balanced to total closed walks is formulated in (9). Regarding the calculation of $\text{Tr}(e^{A})$, one may use the standard fact that $\mathbf{A}$ is a symmetric matrix for undirected graphs. It follows that $\text{Tr}(e^{\mathbf{A}}) = \sum_i e^{\lambda_i}$ in which λ_i ranges over eigenvalues of $\mathbf{A}$.

$$W(G) = \frac{K(G)+1}{2}, \quad K(G) = \frac{\sum_k \frac{Q_k^+ - Q_k^-}{k!}}{\sum_k \frac{Q_k^+ + Q_k^-}{k!}} = \frac{\text{Tr}(e^{\mathbf{A}})}{\text{Tr}(e^{|\mathbf{A}|})} \tag{9}$$

The idea of a walk-based measure was then used by Estrada and Benzi [Estrada and Benzi, 2014]. They have tested their measure on five signed networks resulting in values inclined towards imbalance which were in conflict with some previous observations [Facchetti et al., 2011].

The smallest eigenvalue of signed Laplacian matrix provides a measure of balance called *algebraic conflict* [Kunegis et al., 2010]. Algebraic conflict, denoted by $\lambda(G)$, equals zero if and only if the graph is balanced. Positive-semidefiniteness of $\mathbf{L}$ results in $\lambda(G)$ representing the amount of imbalance in a signed network. Algebraic conflict is used in [Kunegis, 2014] to compare the level of balance in online signed networks of different sizes. Moreover, Pelino and Maimone analyzed signed network dynamics based on $\lambda(G)$ [Pelino and Maimone, 2012]. Bounds for $\lambda(G)$ are investigated by [Hou, 2004] leading to recent applicable results in [Belardo, 2014, Belardo and Zhou, 2016]. Belardo and Zhou prove that $\lambda(G)$ for a fixed n is maximal in the complete all-negative graph of order n. $\overline{d}_{\max}$ represents the maximum average degree of endpoints over graph edges. $\lambda(G)$ is bounded by $\lambda_{\max}(G) = \overline{d}_{\max} - 1$ [Belardo, 2014]. We use this upper bound to normalize algebraic conflict. *Normalized algebraic conflict*, denoted by $A(G)$, is expressed in (10). Clearly $\lambda_{\max}(G)$ is maximized by complete graphs [Belardo and Zhou, 2016].

$$A(G) = 1 - \frac{\lambda(G)}{\overline{d}_{\max} - 1}, \quad \overline{d}_{\max} = \max_{(u,v)\in E} (d_u + d_v)/2 \tag{10}$$

A quite different measure is the *frustration index*. Originally proposed for applications on ferromagnetic molecules, it is also referred to as the *line index for balance* by [Harary, 1959]. A set E^* of edges is called *deletion-minimal* if deleting all edges in E^* results in a balanced graph, but no proper subset of E^* has this property. Each edge in E^* lies on an unbalanced cycle and every unbalanced cycle of the network contains an odd number of edges in E^*. The graph resulted from deleting all edges in E^* is called *balanced transformation* of a signed graph. Frustration index equals the minimum cardinality among deletion-minimal sets as in (12).

$L(G)$ is hard to compute as the problem can be reduced to graph maximum cut problem, in a special case of all negative edges, which is known to be NP-hard. However, upper bounds can be readily provided such as $L(G) \leq m^-$, which states the obvious result of removing all negative edges.

Facchetti, Iacono, and Altafini have used Ising spin glass computational methods to estimate frustration index in relatively large online social networks [Facchetti et al., 2011]. Using frustration index estimated by a heuristic algorithm, they concluded that the online signed networks are extremely close to total balance; an observation that contradicts some other research studies like [Estrada and Benzi, 2014].

The number of frustrated edges in Erdős-Rényi graphs is analyzed by El Maftouhi, Manoussakis and Megalakaki [El Maftouhi et al., 2012]. It follows a binomial distribution with parameters $n(n-1)/2$ and $p/2$ in which p represents equal probabilities for positive and negative edges in Erdős-Rényi graph. Therefore, the expected number for frustrated edges is $n(n-1)p/4$. They also prove that such a network is almost always not balanced when $p \geq \log 2/n$. It is straightforward to prove that frustration index is equal to the minimum number of negative edges over all switching functions [Zaslavsky, 2010]. Moreover, if $m^-(G^{g(X)}) = L(G)$ then every vertex under switching $g(X)$ satisfies $d^-(v^{g(X)}) \leq d^+(v^{g(X)})$. Tomescu [Tomescu, 1973] proves that this measure is bounded by $\left\lfloor (n-1)^2/4 \right\rfloor$. Bounds for the largest frustration index over all signings of vertices are provided by [Akiyama et al., 1981]:

$$\frac{m}{2} - \sqrt{mn} \leq \max L(G) \leq \frac{m}{2}. \tag{11}$$

In order to compare with the other indices which take values in the unit interval and give the value 1 for balanced graphs, we suggest *normalized frustration index*, denoted by $F(G)$ and formulated in (12).

$$F(G) = 1 - \frac{2L(G)}{m}, \quad L(G) = \min_{E^*} |E^*| \tag{12}$$

Balance can also be analyzed by blockmodeling based on iteratively calculating Pearson moment correlations from the columns of $\mathbf{A}$ [Doreian, 2005]. Blockmodeling reveals increasingly homogeneous sets of vertices. Doreian and Mrvar discuss this approach in partitioning signed networks [Doreian and Mrvar, 2009]. Applying the method to Correlates of War data on positive and negative international relationships, they refute the hypothesis that signed networks gradually move towards balance using blockmodeling alongside some variations of $D(G)$ and $L(G)$ [Doreian and Mrvar, 2015]. Some researchers suggest that studying the structural dynamics of signed networks is more important than measuring balance [Cai et al., 2015, Ma et al., 2015]. This approach is usually associated with considering an energy function to be minimized by local graph operations decreasing the energy. However, the energy function is somehow a measure of network imbalance which requires a proper definition and investigation of axiomatic properties. Six measures of partial balance investigated in this paper are outlined in Table 1.

Table 1: Measures of partial balance summarized

Measure	Name, Reference, and Description
D(G)	*Degree of balance* [Cartwright and Harary, 1956] [Harary, 1959] A cycle-based measure representing the ratio of balanced cycles
C(G)	*Weighted degree of balance* [Norman and Roberts, 1972] An extension of $D(G)$ using cycles weighted by a non-increasing function of length
W(G)	*Walk-based measure of balance* [Pelino and Maimone, 2012] [Estrada and Benzi, 2014] A simplified extension of $D(G)$ replacing cycles by closed walks
T(G)	*Triangle index* [Terzi and Winkler, 2011] [Kunegis, 2014] A triangle-based measure representing the ratio of balanced triangles
A(G)	*Normalized algebraic conflict* [Kunegis et al., 2010] [Kunegis, 2014] A normalized measure using least eigenvalue of the Laplacian matrix
F(G)	*Normalized frustration index* [Harary, 1959] [Facchetti et al., 2011] Normalized number of edges whose removal results balance

5 Basic results on random networks

In this section, we start with a brief discussion on the relationship between negative edges and imbalance in networks. According to the definition of structural balance, all-positive signed graphs (merely containing positive edges) are totally balanced. Intuitively, one may expect that all-negative signed graphs are very unbalanced. Perhaps another intuition derived by assuming symmetry is that increasing the number of negative edges in a network reduces partial balance proportionally. We analyze partial balance in randomly generated graphs to show neither of the intuitions is correct.

5.1 Barabási-Albert random network with various m^-

We calculate measures of partial balance, denoted by $\mu(G)$, for a Barabási-Albert network. A Barabási-Albert preferential attachment random network with 15 nodes and 50 edges is generated by attaching 5 edges from a new node to existing nodes. Figure 1 demonstrates the partial balance measured by different methods.

For each data point, we report the average of 50 runs, each assigning negative edges at random to the fixed underlying graph. The bottom subfigures of Figure 1 show the mean along with ± 1 standard deviation. It is worth mentioning that we observed similar decrease of the measures in other types of random graphs with various negative edges including Erdős-Rényi, small world, scale-free, and random regular graphs.

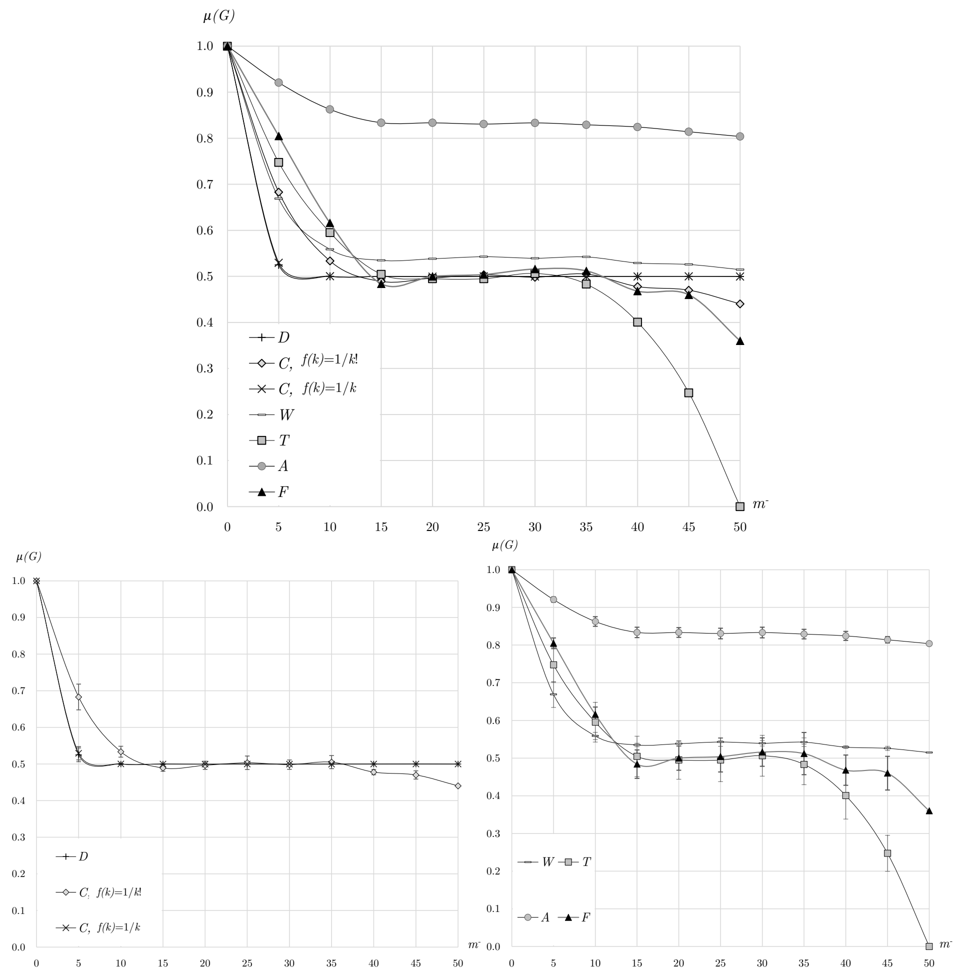


Fig. 1: Partial balance measured by different methods in Barabási-Albert network with various number of negative edges

Measures $D(G)$ and $C(G)$ (with $f(k) = 1/k$) are observed to tend to 0.5 where $m^- > 5$, not differentiating partial balance in graphs with a non-trivial number of negative edges. $C(G)$ weighted by $f(k) = 1/k!$ decreases slower than the former two and then provides values close to 0.5 for $m^- \geq 15$. $W(G)$ drops below 0.6 for $m^- = 10$ and then clusters around 0.55 for $m^- > 10$. $T(G)$ is the measure with the widest range of values, symmetric to m^- and confirming previous calculations (7). The single most striking observation to emerge is that $A(G)$ seems to have a completely different range of values, which we discuss further in 6.4. A steady linear decrease is observed from $F(G)$ for $m^- \leq 10$ providing smaller values for $m^- \geq 35$ compared to the other measures except for $T(G)$.

5.2 4-regular random networks of different orders

To investigate the impacts of graph order (number of nodes) and density on balance, we computed the measures for randomly generated 4-regular graphs with 50 percent negative edges. Intuitively we expect values to have low variation and no trends for similarly structured graphs of different orders. Figure 2 demonstrates the analysis in a setting where the degree of all the nodes remains constant, but the density $(4/n-1)$ is decreasing in larger graphs. For each data point the average and standard deviation of 100 runs (each generating a new 4-regular graph and randomly making half of the edges negative) are reported.

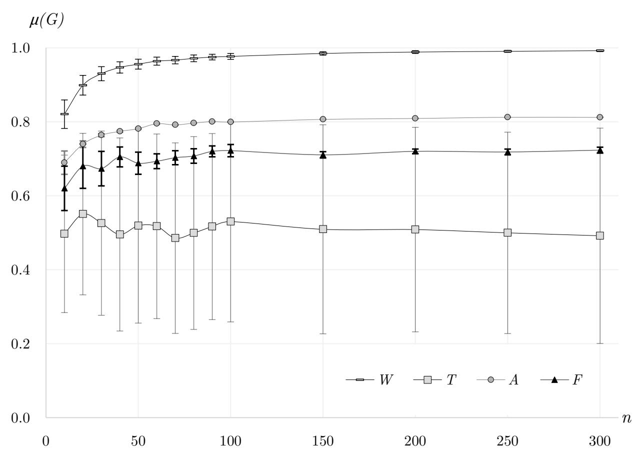


Fig. 2: Partial balance measured by different methods in 50% negative 4-regular graphs of different orders n and decreasing densities $4/n-1$

According to Figure 2, the four measures differ not only in the range of values, but also in their sensitivity to the graph order and density. First, $W(G) \to 1$ when $n \to \infty$ for larger graphs although the graphs are structurally similar, which goes against intuition. Clustered around 0.5 is $T(G)$ which features a substantial standard deviation for 4-regular random graphs. Values of $A(G)$ are around 0.8 and do not seem to change substantially when n increases. $F(G)$ provides stationary values around 0.7 when n increases. While $\lambda(G)$ and $L(G)$ depend on the graph order and size, the relative constancy of $A(G)$ and $F(G)$ values suggest the normalized measures $A(G)$ and $F(G)$ are largely independent of the graph size and order, as our intuition expects.

6 Balance in specially structured signed networks

In this section, we analyze the capability of measuring partial balance in some families of specially structured graphs. Closed-form formulae for the measures in two families of specially structured graphs can be found in Table 2. The three families of complete signed graphs that we investigate are as follows in 6.1–6.3.

6.1 Minimally unbalanced complete graphs with one negative edge

The first family includes complete graphs with one negative edge, denoted by K_n^a. Such graphs are only one edge away from a state of total balance. It is straight-forward to provide closed-form formulae for $\mu(K_n^a)$ as expressed in (B 1) – (B 7) in the Appendix B.

In K_n^a, intuitively we expect $\mu(K_n^a) \uparrow n$ and $\mu(K_n^a) \to 1$ as $n \to \infty$. We also expect the measure to detect the imbalance in K_3^a (a triangle with one negative edge). Figure 3 demonstrates the behavior of different indices for K_n^a. $W(K_n^a)$ gives unreasonably large values for $n < 5$. Except for $W(K_n^a)$, the measures are totally ordered over the given range of n.

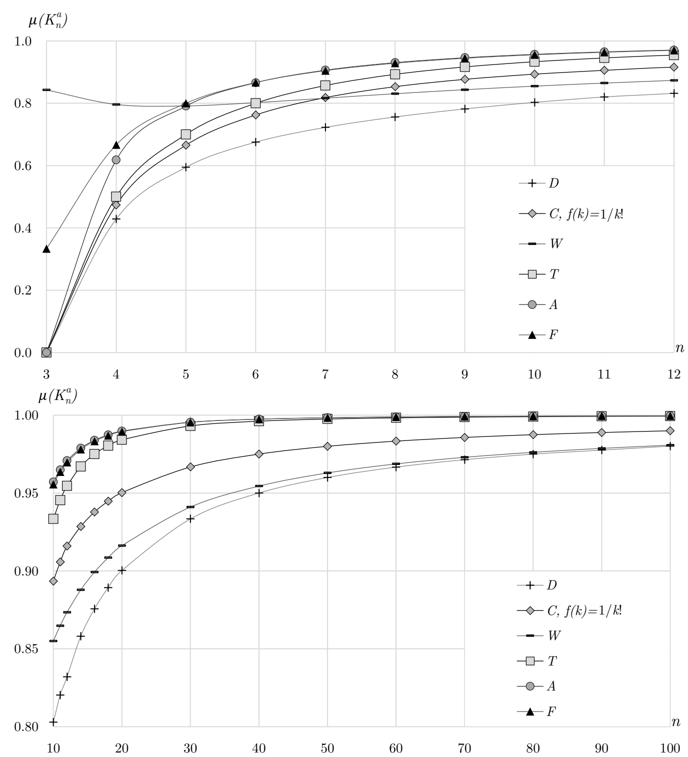


Fig. 3: Partial balance measured by different methods for K_n^a (6.1)

Table 2: Balance in minimally and maximally unbalanced graphs K_n^a (6.1) and K_n^c (6.3)

$\mu(G)$	D	C	W	T	A	F
K_n^a	$1-\frac{2\sum_{k=3}^{n}\frac{n!}{(n-k)!}}{n(n-1)\sum_{k=3}^{n}\frac{n!}{(n-k)!k}}$	$1-\frac{2\sum_{k=3}^{n}\frac{n!}{k!(n-k)!}}{n(n-1)\sum_{k=3}^{n}\frac{n!}{k!(n-k)!k}}$	$\sim 1-\frac{2}{n}$	$1-\frac{6}{n(n-1)}$	$\sim 1-\frac{4}{n^2}$	$1-\frac{4}{n(n-1)}$
K_n^c	$\frac{\sum_{\text{even}}^{n}\frac{n!}{2k(n-k)!}}{\sum_{k=3}^{n}\frac{n!}{2k(n-k)!}}$	$\frac{\sum_{\text{even}}^{n}\frac{n!}{2k(n-k)!k!}}{\sum_{k=3}^{n}\frac{n!}{2k(n-k)!k!}}$	$\sim\frac{1+e^{2-2n}}{2}$	0	0	$\frac{1}{n}, \frac{1}{n-1}$

6.2 Negative complete graphs with one maximally sized cycle of positive edges

The second family of specially structured graphs is referred to as maximally unbalanced graphs by [Estrada and Benzi, 2014]. These graphs, denoted by K_n^b, are comprised of one cycle of n positive edges with the remaining pairs of nodes connected by negative edges forming a complete graph. The adjacency matrix can be defined as $\mathbf{A}(K_n^b) = 2\mathbf{C}_n - \mathbf{K}_n$ in which $\mathbf{C}_n$ and $\mathbf{K}_n$ are the unsigned adjacency matrices of a cycle graph of order n and a complete graph of the same order.

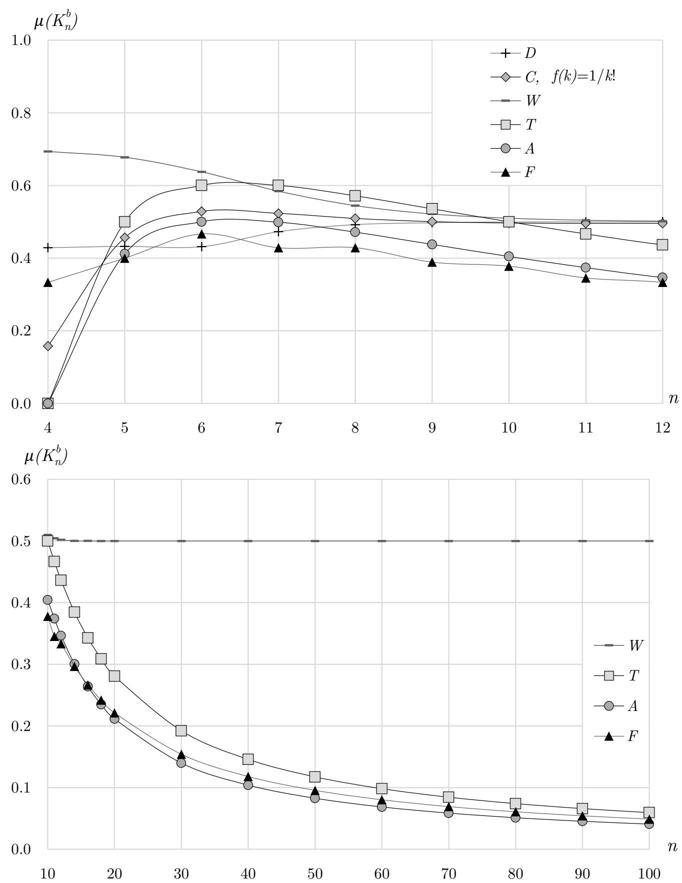


Fig. 4: Partial balance measured by different methods for K_n^b (6.2)

Assuming K_n^b to be highly unbalanced, intuitively we expect $\mu(K_n^b)$ to decrease with n and $\mu(K_n^b) \to 0$ as $n \to \infty$. Figure 4 demonstrates the behavior of different indices for K_n^b. Note that $D(K_n^b), C(K_n^b), W(K_n^b) \to 0.5$ as n increases which suggests their incapability of measuring low balance in K_n^b. However, $T(K_n^b), A(K_n^b), F(K_n^b) \to 0$ as $n \to \infty$ which supports their performance. For this family, we may calculate an exact closed-form formula for $L(K_n^b)$ as shown in (B 8) which reveals a gap between $L(K_n^b)$ and its upper bound to be discussed in 6.4.

6.3 Maximally unbalanced complete graphs with all-negative edges

The third family of specially structured graphs to analyze includes all-negative complete graphs denoted by K_n^c. All 3-cycles in K_n^c are unbalanced leading to $T(K_n^c) = 0$. Based on maximality of $\lambda(G)$ in K_n^c, $A(K_n^c) = 0$. The other indices are calculated in (B 10) – (B 15) in the Appendix B.

Intuitively, we expect $\mu(K_n^c) \to 0$ as $n \to \infty$. Figure 5 illustrates $D(K_n^c)$ oscillating around 0.5 while $W(K_n^c), C(K_n^c) \to 0.5$ as n increases. However $F(K_n^c) \to 0$ as $n \to \infty$.

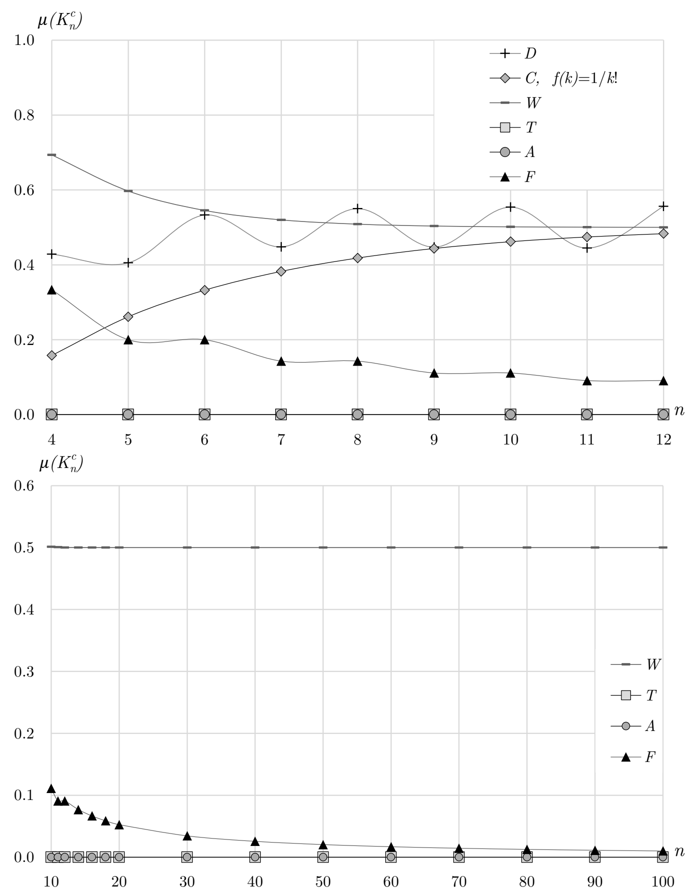


Fig. 5: Partial balance measured by different methods for K_n^c (6.3)

6.4 Maximally unbalanced graphs and tightness of upper bounds

Having discussed 3 families of complete graphs, it is worth mentioning that measures of partial balance may lead to different maximally unbalanced complete graphs. Based on $\lambda(G)$ and $L(G)$, K_n^c represents the family of maximally unbalanced graphs, while it is merely one family among the maximally unbalanced graphs according to $T(G)$. Estrada and Benzi have found K_n^b to be a family of maximally unbalanced graphs based on $W(G)$

[Estrada and Benzi, 2014], while this argument is not supported by any other measures. As the signs of cycles in a graph are not independent, the structure of maximally unbalanced graphs under the cycle-based measures, $D(G)$ and $C(G)$, is not known.

A simple comparison of $L(K_n^b), L(K_n^c)$ and the proposed upper bound $m/2 = (n^2 - n)/4$ reveals substantial gaps. These gaps equal $(5n-16)/4$ and $n/4$ (for even n) respectively for K_n^b and K_n^c. This supports the previous discussions on looseness of $m/2$ as an upper bound for frustration index. Assuming K_n^c to be "the maximally unbalanced graphs" under $L(G)$, $m/2 - n/4$ would be a tight upper bound for the frustration index. This allows a modified version of normalized frustration index, denoted by $F'(G)$ and defined in (13), to take the value zero for K_n^c. Similarly, the upper bound, $\lambda_{\max}(G)$, used to normalize algebraic conflict, is not tight for many graphs. For instance, in the Barabási-Albert graph studied in Section 5, the existence of an edge with $\overline{d}_{\max} = 13$ makes $\lambda_{\max}(G) = 12$, while $\lambda(G) = 2.36$.

$$F'(G) = 1 - L(G)/(m/2 - n/4) \tag{13}$$

The two observations mentioned above suggest that tighter upper bounds can be used for normalization. However, the statistical analysis we use in Section 9 is independent of the normalization method, so we do not pursue this question further now. Having discussed major differences of the measures and their capabilities, the next section of this paper addresses an axiomatic framework for evaluating the measures of partial balance.

7 Axiomatic framework of evaluation

The results in Section 5 and Section 6 indicate that the choice of measure substantially affects the values of partial balance. Besides that, the lack of a standard measure calls for a framework of comparing different methods. Two different sets of axioms are suggested in [Norman and Roberts, 1972], which characterize the measure $C(G)$ (up to the choice of $f(k)$). Moreover, the theory of structural balance itself is axiomatized in [Schwartz, 2010]. However, to our knowledge, axioms for general measures of balance have never been developed. Here we provide the first set of axioms for measures of partial balance, in order to shed light on their characteristics and performance.

7.1 Axioms for measures of partial balance

We define a *measure of partial balance* to be a function μ taking each signed graph to an element of $[0,1]$. Worthy of mention is that some of these measures were originally defined as a measure of imbalance (algebraic conflict, frustration index and the original walk-based measure) calibrated at 0 for completely balanced structures, so that some normalization was required, and perhaps our normalization choices can be improved on. As the choice of $m/2$ as the upper bound for normalizing the line index of balance was somewhat arbitrary, another version of normalized frustration index is defined in (14).

$$X(G) = 1 - L(G)/m^- \tag{14}$$

Before listing the axioms, we justify the need for an axiomatic evaluation of balance measures. As an attempt to understand the need for axiomatizing measures of balance, we introduce two unsophisticated and trivial measures that comes to mind for measuring balance. The fraction of positive edges, denoted by $Y(G)$, is defined in (15) on the basis that all-positive signed graphs are balanced. Moreover, a binary measure of balance, denoted by $Z(G)$, is defined in (16).

$$Y(G) = m^+/m \tag{15}$$

$$Z(G) = \begin{cases} 1 & \text{if G is totally balanced} \\ 0 & \text{if G is not balanced} \end{cases} \tag{16}$$

While $Y(G)$ and $Z(G)$ appear to be irrelevant, there is currently no reason to avoid using such measures. We consider the following notation for referring to basic operations on signed graphs:
$G^{g(X)}$ denotes signed graph G switched by $g(X)$ (switched graph).
$G \oplus H$ denotes the disjoint union of two signed graphs G and H (disjoint union).
$G \ominus e$ denotes G with e deleted (removing an edge).
$G \ominus E^*$ denotes G with deletion-minimal edges deleted (balanced transformation).
C^+ denotes a positive cycle.
$e \in E^*$ denotes an edge in the deletion-minimal set.
$G \ominus E^* \oplus e$ denotes the balanced transformation of a graph with an edge e added to it.

We list the following axioms:

A1 $0 \leq \mu(G) \leq 1$.
A2 $\mu(G) = 1$ if and only if G is balanced.
A3 If $\mu(G) \leq \mu(H)$, then $\mu(G) \leq \mu(G \oplus H) \leq \mu(H)$.
A4 If $\mu(G) \neq 1$, then $\mu(G \oplus C^+) > \mu(G)$.
A5 $\mu(G^{g(X)}) = \mu(G)$.
A6 If $e \in E^*$, then $\mu(G \ominus e) \geq \mu(G)$.
A7 If $\mu(G) \neq 0$ and $\mu(G \ominus E^* \oplus e) \neq 1$, then $\mu(G \oplus e) \leq \mu(G)$.

Table 3 shows how some measures fail on particular axioms. It is worth mentioning that the axiomatic evaluation of the measures are somewhat independent of parametrization: for each strictly increasing function h such that $h(0) = 0$ and $h(1) = 1$, the results in Table 3 hold for $h(\mu(G))$. The results provide important insights into suitability of $F(G)$ as a measure of partial balance. A more detailed discussion on the proof ideas and counter examples related to Table 3 is provided in the Appendix C.

Table 3: Different measures satisfying(✓) or failing(✗) Axioms

	$D(G)$	$C(G)$	$W(G)$	$T(G)$	$A(G)$	$F(G)$	$X(G)$	$Y(G)$	$Z(G)$
A1	✓	✓	✓	✓	✓	✓	✓	✓	✓
A2	✓	✓	✓	✗	✗	✓	✓	✗	✓
A3	✓	✓	✓	✓	✗	✓	✓	✓	✓
A4	✓	✓	✓	✗	✓	✓	✗	✗	✗
A5	✓	✓	✓	✓	✓	✓	✗	✗	✓
A6	✗	✗	✗	✗	✗	✓	✓	✗	✗
A7	✗	✗	✗	✗	✗	✓	✓	✗	✓

7.2 Some other desirable properties

Another desirable property, which we have not formulated as a formal requirement owing to its vagueness, is that the measure takes on a wide range of values. For example, $D(G)$ and $C(G)$ tend rapidly to 0.5 as n increases which makes their interpretation and possibly comparison with other measures difficult. A possible way to formalize it would be expecting $\mu(G)$ to give 0 and 1 on each complete graph of order at least 3, for some assignment of signs of edges . This condition would be satisfied by $T(G)$ and $A(G)$, as well as $F'(G)$. However, $D(G),C(G)$ and $W(G)$ would not satisfy this condition due to the existence of balanced cycles and closed walks in complete signed graphs of general orders. Moreover, the very small standard deviation of $D(G)$, $C(G)$, and $W(G)$ makes statistical testing against random balance of reshuffled networks complicated. $D(G)$, $C(G)$, and $W(G)$ are also shown to have some unexpected behaviors for complete graphs discussed in Section 6.

8 Recommendations for choosing a measure of partial balance

Taken together, the findings above give strong reason not to use cycle-based measures, whether weighted or not. The major issues with cycle-based measures $D(G)$ and $C(G)$ include the very small variance in randomly generated and reshuffled graphs, computational complexity of counting/listing cycles, clustering of values around 0.5 for graphs with a non-trivial number of negative edges, and the numerical values which are difficult to interpret.

The triangle index, however, seems to behave better, and is easy to compute based on closed walks of length 3. However, $T(G)$ fails 4 out of 7 axioms and cannot be used for graphs that do not have triangles, such as square grids. Accepting these shortcomings, networks can be differentiated by the wider range of values that $T(G)$ provides.

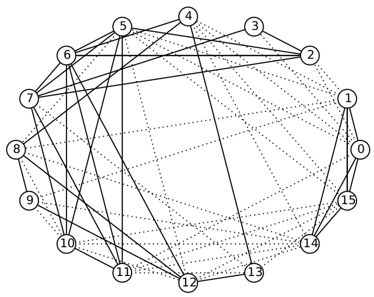
Walk-based measures like $W(G)$ can perhaps be improved by a more systematic way of weighting closed walks to avoid double-counting of closed walks with repeated edges. However the clustering of values near 0.5 may present problems. Moreover, the model behind $W(G)$ is strange as signs of closed walks do not represent balance or imbalance.

Satisfying all the axioms, normalized frustration index seems to measure something different from cycle balance, and be worth pursuing in future. We recommend using $F(G)$ for graphs up to 1000 edges. For larger graphs, computing $F(G)$ would be time consuming and $A(G)$ and $T(G)$ seem to be the other options. Depending on the type of the graph, triangles might not necessarily capture global structural properties which would make $T(G)$ an improper choice for some specific graphs like sparse 4-regular graphs and square grids.

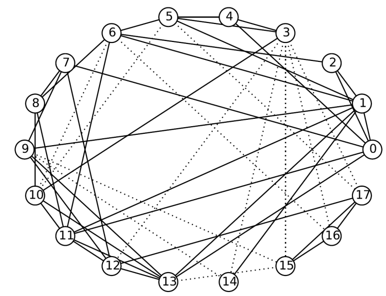
9 Results on real signed networks

In this section, we analyze partial balance for a range of signed networks inferred from datasets on small communities with positive and negative interactions and preferences.

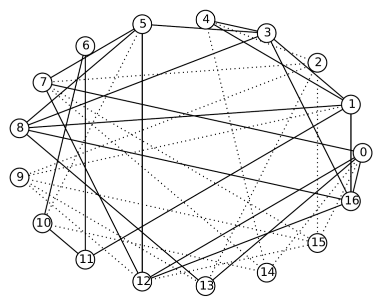
Read's dataset for New Guinean highland tribes [Read, 1954] is demonstrated as a signed graph (G1) in Figure 6(a), where dotted lines represent negative edges and solid



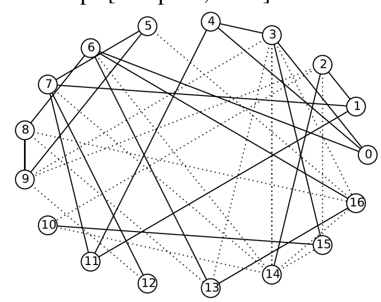
(a) Highland tribes network (G1), a signed network of 16 tribes of the Eastern Central Highlands of New Guinea [Read, 1954]



(b) Monastery interactions network (G2) of 18 New England novitiates inferred from the integration of all positive and negative relationships [Sampson, 1968]



(c) Fraternity preferences network (G3) of 17 boys living in a pseudo-dormitory inferred from ranking data of the last week in [Newcomb, 1961]



(d) College preferences network (G4) of 17 girls at an Eastern college inferred from ranking data of house B in [Lemann and Solomon, 1952]

Fig. 6: Four well-studied signed datasets illustrated as signed graphs in which dotted lines represent negative edges and solid lines represent positive edges

lines represent positive ones. Sampson's dataset for monastery interactions [Sampson, 1968] (G2) is drawn in Figure 6(b). There are also datasets of students' choice and rejection (G3 and G4) [Newcomb, 1961, Lemann and Solomon, 1952] as demonstrated in Figure 6(c) and Figure 6(d). The last three are converted to undirected signed graphs by considering mutually agreed relations. A further explanation on the details of inferring signed graphs from the choice and rejection data is provided in Appendix A. Moreover, a larger signed network (G5) is inferred by [Neal, 2014] through implementing a stochastic degree sequence model on Fowler's data on Senate bill co-sponsorship [Fowler, 2006].

The results are shown in Table 4 where the average and standard deviation of measures for the reshuffled graphs (G_r), denoted by mean$(\mu(G_r))$ and SD$(\mu(G_r))$, are also provided for comparison. $T(G)$ and $F(G)$ give reasonable values to distinguish partial balance in real networks and their corresponding reshuffled graphs. Although neither of the networks is completely balanced, the small values of $L(G)$ suggest that removal of some edges makes the networks completely balanced. Table 4 also provides a comparison of partial balance between different datasets of similar sizes. In this regard, it is essential to know that the choice of measure can make a substantial difference. For instance among G1–G4, under

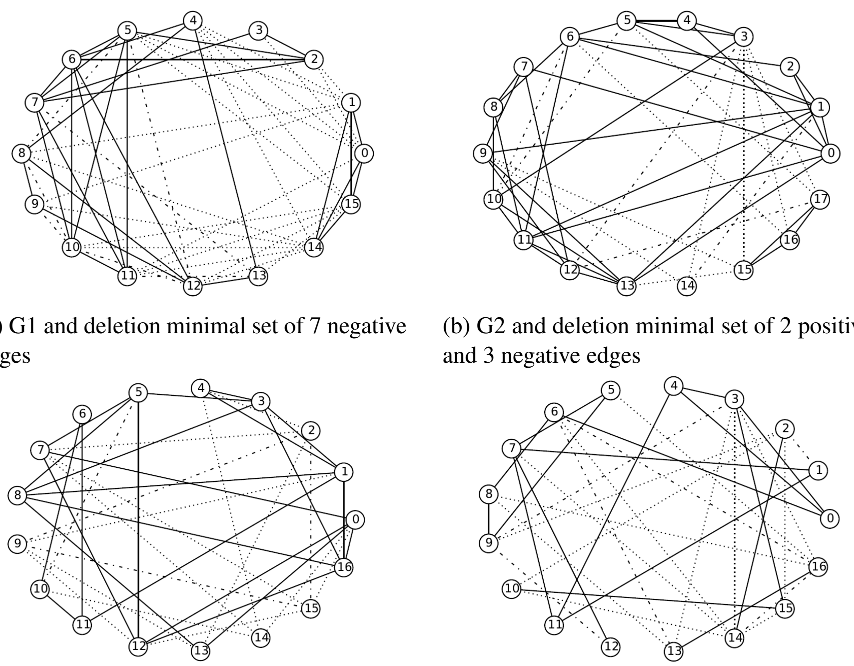
$T(G)$, G1 and G3 are respectively the most and the least partially balanced networks. However, if we choose $A(G)$ as the measure, G1 and G3 would be the least and the most partially balanced networks respectively.

Table 4: Partial balance compared between signed graphs (G1–5) and reshuffled graphs

Graph	n	m	m^-		T	A	F	λ	L
G1	16	58	29	$\mu(G)$	0.87	0.88	0.76	1.04	7.00
				mean($\mu(G_r)$)	0.50	0.75	0.49	2.12	14.80
				SD($\mu(G_r)$)	0.07	0.02	0.04	0.20	1.25
				Z-score	5.36	5.52	6.24	-5.52	-6.24
G2	18	49	12	$\mu(G)$	0.86	0.88	0.80	0.75	5.00
				mean($\mu(G_r)$)	0.54	0.79	0.59	1.39	10.02
				SD($\mu(G_r)$)	0.10	0.03	0.05	0.18	1.22
				Z-score	3.13	3.57	4.10	-3.57	-4.10
G3	17	40	17	$\mu(G)$	0.78	0.90	0.80	0.50	4.00
				mean($\mu(G_r)$)	0.51	0.83	0.60	0.87	8.02
				SD($\mu(G_r)$)	0.12	0.03	0.04	0.15	0.88
				Z-score	2.32	2.44	4.55	-2.44	-4.55
G4	17	36	16	$\mu(G)$	0.79	0.88	0.67	0.71	6.00
				mean($\mu(G_r)$)	0.48	0.87	0.61	0.78	7.04
				SD($\mu(G_r)$)	0.14	0.03	0.06	0.17	1.00
				Z-score	2.28	0.46	1.04	-0.46	-1.04
G5	100	2461	1047	$\mu(G)$	0.86	0.87	0.73	8.92	331.00
				mean($\mu(G_r)$)	0.50	0.75	0.21	17.46	973.83
				SD($\mu(G_r)$)	0.00	0.00	0.01	0.02	9.30
				Z-score	112.69	395.19	69.13	-395.19	-69.13

We have implemented a very basic statistical analysis using mean$(\mu(G_r))$ and SD$(\mu(G_r))$. Reshuffling the signs on the edges 100 times, we obtain two parameters of balance distribution for the fixed underlying structure. For measures of balance, Z-scores are calculated based on $Z = \frac{\mu(G)-\text{mean}(\mu(G_r))}{\text{SD}(\mu(G_r))}$ to show how far the balance is with regards to balance distribution of the underlying structure. Positive values of Z-score for $T(G)$, $A(G)$, and $F(G)$ can be interpreted as existence of more partial balance than the average random level of balance. Z-score values also represent the significance when compared to the standard range of $(-3,3)$.

From the Z-scores, we can see that all the 5 graphs have a level of partial balance more than the average expected. The significance and level of partial balance more than expected by chance is very high for G5, high for G1, G2, and G3, and low for G4. It indicates that ties the real signed networks investigated are more transitive than what we expect when signs are allocated by chance. It is worth pointing out that the statistical analysis we have implemented is independent of the normalization method used in $A(G)$ and $F(G)$. The two right columns of 4 provide $\lambda(G)$ and $L(G)$ alongside their associated Z-scores. Representing more balance by smaller values, Z-scores obtained for $\lambda(G)$ and $L(G)$ equal the opposite of the Z-scores obtained for $A(G)$ and $F(G)$.



(a) G1 and deletion minimal set of 7 negative edges

(b) G2 and deletion minimal set of 2 positive and 3 negative edges

(c) G3 and deletion minimal set of removing 4 negative edges

(d) G4 and deletion minimal set of 3 positive and 3 negative edges

Fig. 7: The deletion minimal set illustrated by dotdash lines for four signed networks

Figure 7 shows the four small signed networks, with a deletion minimal set of edges indicated by dotdash lines whose removal makes the network balanced. It shows how such networks are close to balance when measured by frustration index. The numerical results support previous observations of networks closeness to balance [Facchetti et al., 2011] and contradicts some other arguments provided by using other measures [Estrada and Benzi, 2014]. A possible reason for the conflicts in the literature may be the method of measuring balance itself, which we discuss further in the following section.

10 Discussion

One criticism of much of the literature on balance theory is that it is widely used on directed signed graphs. It seems that this approach is questionable in two ways. First, it neglects the fact that many edges in signed digraphs are not reciprocated. Bearing that in mind, investigating balance theory in signed digraphs deals with conflict avoidance when one actor in such a relationship may not necessarily be aware of good will or ill will on the part of other actors. This would make studying balance in directed networks analogous to studying how people avoid potential conflict resulting from potentially unknown ties. Secondly, balance theory does not make use of the directionality of ties and the concepts of sending and receiving positive and negative links.

In a parallel line of research on network structural analysis, researchers differentiate between classical balance theory and structural balance specifically in the way that the latter is directional [Bonacich, 2012]. They consider another setting for defining balance where absence of ties implies negative relationships. This assumption makes the theory limited to complete signed digraphs. Accordingly, 64 possible structural configurations emerge for three nodes. These configurations can be reduced to 16 classes of triads, referred to as 16 MAN triad census, based on the number of Mutual, Asymmetric, and Null relationships they contain. There are only 2 out of 16 classes that are considered balanced. New definitions are suggested by researchers in order to make balance theory work in a directional context. According to Prell [Prell, 2012], there is a second, a third, and a fourth definition of permissible triads allowing for 3, 7, and 9 classes of all 16 MAN triads. However, there have been many instances of findings in conflict with expectations [Prell, 2012].

Leskovec, Huttenlocher and Kleinberg compare the reliability of predictions made by competing theories of social structure: balance theory and status theory (a theory that explicitly includes direction and gives quite different predictions) [Leskovec et al., 2010]. The consistency of these theories with observations is investigated through large signed directed networks such as Epinions, Slashdot, and Wikipedia. The results suggest that status theory predicts accurately most of the time while predictions made by balance theory are incorrect half of the time. This supports the inefficacy of balance theory for structural analysis of signed digraphs. For another comparison of the two theories, one may refer to a study of 8 theories to explain signed tie formation between students [Yap and Harrigan, 2015].

Apart from directionality, the interpretation of balance measures is very important. Numerous studies have compared balance measures with their extremal values and found that signed networks are far from balanced [Estrada and Benzi, 2014]. However, with such a strict criterion, caution must be applied not to look for properties that are almost impossible to satisfy. A much more systematic approach is to compare values of partial balance in the signed graphs in question to reshuffled graphs. According to this approach, we suggest comparing signed network balance with analogous reshuffled graphs having the same structure (as in Section 9). Table 4 provides this comparison showing that the extent of transitivity of ties in signed networks is substantial. The real signed networks analyzed are more balanced than we expect by chance. An alternative approach would identify how measures of balance interact with other network parameters that are linked to transitivity of ties [Szell et al., 2010, Szell and Thurner, 2010].

11 Conclusion and future research

Taking axiomatic properties of the measures into account, using $D(G)$, $C(G)$, and $W(G)$ is recommended. $T(G)$ and $A(G)$ may introduce some minor problems as discussed, but overall using them seem to be more appropriate compared to cycle based and walk based measures. The numerical results taken together with the axiomatic properties, recommend

$F(G)$ as the best overall measure of partial balance. However, considering the difficulty of computing $F(G)$ for large graphs, one may prefer to use $T(G)$ or $A(G)$ instead while accepting their potential shortcomings.

Returning to the questions posed at the beginning of this paper, it is now possible to state that many signed networks exhibit a level of partial (but not total) balance beyond that expected by chance. One of the more significant findings to emerge from this study is that methods suggested for measuring balance have their context and interpretation. The present study confirms previous findings that some measures of partial balance cannot be taken as a reliable static measure to be used for analyzing network dynamics. It contributes additional evidence that suggests a gray-scale for transitivity of positive and negative relationships. Although the major numerical part of the current study is based on signed networks with less than a few thousand edges, the findings suggest the inefficacy of some methods for analyzing larger networks as well. One gap in this study is that we avoid using structural balance theory for analyzing directed networks, making a significant part of the literature (including Epinions, Slashdot, and Wikipedia datasets) untested by our approach for now. However, see our discussion in Section 10.

The findings of this study have a number of important implications for future investigation. Although this study focuses on partial balance, the findings may well have a bearing on link prediction and clustering in signed networks [Gallier, 2016]. Some other theoretical topics of interest in signed networks are network dynamics [Tan and Lü, 2016] and opinion dynamics [Li et al., 2015]. Effective methods of signed network structural analysis can contribute to these topics as well. From a practical viewpoint, international relationships is a crucial area to implement signed network structural analysis. Having an efficient measure of partial balance in hand, international relations can be investigated in terms of partial balance in networks of states.

A Inferring undirected signed graphs

Sampson collected different sociometric rankings from a group of 18 monks at different times [Sampson, 1968]. The data provided includes rankings on like, dislike, esteem, disesteem, positive influence, negative influence, praise, and blame. We have considered all the positive rankings as well as all the negative ones. Then only the reciprocated relations with similar signs are considered to infer an undirected signed edge between two monks (see [Doreian and Mrvar, 2009] and how the authors inferred a directed signed graph in their Table 5 by summing the influence, esteem and respect relations.).

Newcomb reported rankings made by 17 men living in a pseudo-dormitory [Newcomb, 1961]. We used the ranking data of the last week which includes complete ranks from 1 to 17 gathered from each men. As the gathered data is related to complete ranking, we considered ranks 1-5 as one-directional positive relations and 12-17 as one-directional negative relations. Then only the reciprocated relations with similar signs are considered to infer an undirected signed edge between two men (See [Doreian and Mrvar, 2009] and how the authors converted the top three and bottom three ranks to a directed signed edges in their Fig. 4.).

Lemann and Solomon collected ranking data based on multiple criteria from female students living in off campus dormitories [Lemann and Solomon, 1952]. We used the data for house B which is resulted by integrating top and bottom three one-directional rankings each for multiple criteria. As the gathered data itself is related to top and bottom rankings, we considered all the ranks as one-directional signed relations. Then only the reciprocated relations with similar signs are considered to infer an undirected signed edge between two women (See [Doreian, 2008] and how the author inferred a directed signed graph in their Fig. 5 from the data for house B.)

B Details of calculations

In order to simplify the sum $E(D_k(G)) = \sum_{i \text{ even}} \binom{k}{i} q^i (1-q)^{k-i}$, one may add the two following equations and divide the result by 2:

$$\sum_i \binom{k}{i} q^i (1-q)^{k-i} = (q+(1-q))^k$$
$$\sum_i \binom{k}{i} (-q)^i (1-q)^{k-i} = (-q+(1-q))^k$$

In K_n^a, a k-cycle is specified by choosing k vertices in some order, then correcting for the overcounting by dividing by 2 (the possible directions) and k (the number of starting points, namely the length of the cycle). If the unique negative edge is required to belong to the cycle, we need choose only $k-2$ further elements and no overcounting occurs. The number of negative cycles and total cycles are as follows.

$$\sum_{k=3}^{n} O_k^- = \sum_{k=3}^{n} \frac{(n-2)!}{(n-k)!}, \quad \sum_{k=3}^{n} O_k = \sum_{k=3}^{n} \frac{n!}{2k(n-k)!}$$

The unsigned adjacency matrix $|\mathbf{A}|$ of the complete graph has the form $\mathbf{E}-\mathbf{I}$ where $\mathbf{E}$ is the matrix of all 1's. The latter matrix has rank 1 and nonzero eigenvalue n. Thus $|\mathbf{A}|_{(K_n^a)}$ has eigenvalues $n-1$ (with multiplicity 1) and -1 (with multiplicity $n-1$). The matrix $\mathbf{A}_{(K_n^a)}$ has a similar form and we can guess eigenvectors of the form $(-1,1,0,\dots,0)$ and $(a,a,1,1,\dots,1)$. Then a satisfies a quadratic $2a^2+(n-3)a-(n-2)=0$. Solving for a and the corresponding eigenvalues, we obtain eigenvalues $(n-4 \pm \sqrt{(n-2)(n+6)})/2, 1, -1$ (with multiplicity $n-3$)).

This yields $K(K_n^a) = \frac{(n-3)e^{-1}+e+e^{\frac{n-4-\sqrt{(n-2)(n+6)}}{2}}+e^{\frac{n-4+\sqrt{(n-2)(n+6)}}{2}}}{(n-1)e^{-1}+e^{n-1}}$ which results in $W(K_n^a) \sim \frac{1+e^{-4/n}}{2}$.

Furthermore, since every node of K_n has degree $n-1$, the eigenvalues of $\mathbf{L} := (n-1)\mathbf{I} - \mathbf{A}$ are precisely of the form $n-1-\lambda$ where λ is an eigenvalue of $\mathbf{A}$.

Measures of partial balance for K_n^a can be expressed exactly according to the closed-form formulae as stated in (B 1) – (B 7):

$$D(K_n^a) = 1 - \frac{2}{n(n-1)} \frac{\sum_{k=3}^{n} \frac{n!}{(n-k)!}}{\sum_{k=3}^{n} \frac{n!}{(n-k)!k}} \tag{B 1}$$

$$C(K_n^a) = 1 - \frac{2}{n(n-1)} \frac{\sum_{k=3}^{n} \frac{n!}{k!(n-k)!}}{\sum_{k=3}^{n} \frac{n!}{k!(n-k)!k}} \tag{B 2}$$

$$W(K_n^a) \sim \frac{1+e^{-4/n}}{2} \sim 1 - \frac{2}{n} \tag{B 3}$$

$$T(K_n^a) = 1 - \frac{\frac{(n-2)!}{(n-3)!}}{\frac{n!}{2\times 3(n-3)!}} = 1 - \frac{6}{n(n-1)} \tag{B 4}$$

$$\lambda(K_n^a) = n-1-(n-4+\sqrt{(n-2)(n+6)})/2 = (n+2-\sqrt{(n-2)(n+6)})/2 \tag{B 5}$$

$$A(K_n^a) = 1 - \frac{n+2-\sqrt{(n-2)(n+6)}}{2n-4} \sim 1 - \frac{4}{n^2} \tag{B 6}$$

$$F(K_n^a) = 1 - \frac{2}{n(n-1)/2} = 1 - \frac{4}{n(n-1)} \tag{B 7}$$

In K_n^b, the deletion-minimal set has a specific structure. This can be used to calculate $L(K_n^b)$ and $F(K_n^b)$ exactly as in (B 8) – (B 9) :

$$L(K_n^b) = \begin{cases} (n^2-6n+16)/4 & \text{if } n \text{ is even} \\ (n^2-6n+17)/4 & \text{if } n \text{ is odd} \end{cases} \tag{B 8}$$

$$F(K_n^b) = \begin{cases} (5n-16)/(n^2-n) & \text{if } n \text{ is even} \\ (5n-17)/(n^2-n) & \text{if } n \text{ is odd} \end{cases} \tag{B 9}$$

In K_n^c, all cycles of odd length are unbalanced and all cycles of even length are balanced. Therefore:

$$\sum_{k=3}^{n} O_k^+ = \sum_{\text{even}}^{n} \frac{n!}{2k(n-k)!}$$

$|\mathbf{A}|_{(K_n^c)}$ has eigenvalues $n-1$ (with multiplicity 1) and -1 (with multiplicity $n-1$). The matrix $\mathbf{A}_{(K_n^c)}$ has a similar form and the corresponding eigenvalues would be $1-n$ (with multiplicity 1) and 1 (with multiplicity $n-1$). This yields $K(K_n^c) = \frac{(n-1)e^1+e^{1-n}}{(n-1)e^{-1}+e^{n-1}}$ which results in $W(K_n^c) \sim \frac{1+e^{2-2n}}{2}$.

Moreover, a closed-form formula for $L(K_n^c)$ can be expressed based on a maximum cut. Measures of partial balance for K_n^c can be expressed exactly according to the closed-form formulae as stated in (B 10) – (B 15):

$$D(K_n^c) = \frac{\sum_{\text{even}}^{n} \frac{n!}{2k(n-k)!}}{\sum_{k=3}^{n} \frac{n!}{2k(n-k)!}} \tag{B 10}$$

$$C(K_n^c) = \frac{\sum_{\text{even}}^{n} \frac{n!}{2k(n-k)!k!}}{\sum_{k=3}^{n} \frac{n!}{2k(n-k)!k!}} \tag{B 11}$$

$$W(K_n^c) \sim \frac{1+e^{2-2n}}{2} \tag{B 12}$$

$$\lambda(K_n^c) = \lambda_{\max} = \overline{d}_{\max} - 1 = n-2 \tag{B 13}$$

$$L(K_n^c) = \begin{cases} (n^2-2n)/4 & \text{if } n \text{ is even} \\ (n^2-2n+1)/4 & \text{if } n \text{ is odd} \end{cases} \tag{B 14}$$

$$F(K_n^c) = \begin{cases} 1-\frac{n(n-2)/4}{n(n-1)/4} = \frac{1}{n-1} & \text{if } n \text{ is even} \\ 1-\frac{(n-1)(n-1)/4}{n(n-1)/4} = \frac{1}{n} & \text{if } n \text{ is odd} \end{cases} \tag{B 15}$$

C Counter examples and proof ideas for the axioms

Axiom 1 holds in all the measures introduced due to the systematic normalization implemented.

$T(G), A(G)$, and $Y(G)$ do not satisfy Axiom 2. All 3-cycles being balanced, $T(G)$ fails to detect the imbalance in graphs with unbalanced cycles of longer than three. $A(G \oplus C^+) = 1$ for unbalanced graphs which makes $A(G)$ fail Axiom 2. $Y(G)$ fails on detecting balance in completely bi-polar signed graphs that are indeed balanced.

As long as $\mu(G \oplus H)$ can be written in the form of $(a+c)/(b+d)$ where $\mu(G) = a/b$ and $\mu(H) = c/d$, μ satisfies Axiom 3. So all the measures considered satisfy Axiom 3, except for $A(G)$. In case of $\lambda(G) < \lambda(H)$ and $\lambda_{\max}(G) < \lambda_{\max}(H)$, $A(G \oplus H) = 1 - \frac{\lambda(G)}{\lambda_{\max}(H)} > A(H)$ which shows that $A(G)$ fails Axiom 3.

Clearly in Axiom 4, C^+ contributes positively to $D(G)$ and $C(G)$ while $T(G \oplus C^+) = T(G)$ when C^+ is longer than 3. As $W(C)$ equals 1 for C^+, $\mathrm{Tr}(e^A)/\mathrm{Tr}(e^{|A|})$ would be added by equal terms in both the numerator and denominator leading to $W(G)$ satisfying Axiom 4. $A(G)$ satisfies Axiom 4 because $A(G \oplus C^+) = 1$. As m increases by the length C^+, $F(G)$ satisfies Axiom 4. The dependency of $X(G)$ and $Y(G)$ on m^- and incapability of the binary measure, $Z(G)$, in providing values between 0 and 1 make them fail Axiom 4.

The sign of cycles (closed walks), the Laplacian eigenvalues [Belardo and Zhou, 2016], and the frustration index [Zaslavsky, 2010] will not change by applying the switching function introduced in (2). Therefore, Axiom 5 holds for all the measures discussed except for $X(G)$ and $Y(G)$ because they depend on m^-, which changes in switching. This observation supports the normalization used for $F(G)$.

All the cycle-based measures, namely $D(G), C(G)$, and $T(G)$ fail Axiom 6 (for example, take $G = K_4$ with two symmetrically located negative edges). $W(G)$ is also observed to fail Axiom 6 (for instance, take G as the disjoint union of a 3-cycle and a 5-cycle each having 1 negative edge). It is known that $\lambda(G \ominus e) \leq \lambda(G)$ [Belardo and Zhou, 2016]. However in some cases where $\lambda_{\max}(G \ominus e) < \lambda_{\max}(G)$ counter examples are found showing $A(G)$ fails on Axiom 6 (consider a graph with $n = 8, m^+ = 10, m^- = 3, \min|E^*| = 3$ in which $\lambda_{\max}(G) = 6$ and $\lambda_{\max}(G \ominus e) = 3$). $Y(G)$ and $Z(G)$ fail Axiom 6. Moreover, $F(G)$ satisfies Axiom 6 because $L(G \ominus e) = L(G) - 1$.

The cycle-based measures and $W(G)$ do not satisfy Axiom 7. For $T(G)$, we tested a graph with $n = 7, m = 15, |E^*| = 3$ and we observed $T(G \oplus e) > T(G)$. According to Belardo and Zhou, $\lambda(G \oplus e) \geq \lambda(G)$ [Belardo and Zhou, 2016]. However in some cases where $\lambda_{\max}(G \oplus e) > \lambda_{\max}(G)$ counter examples are found showing $A(G)$ fails on Axiom 7. Counter examples showing $D(G), C(G), W(G)$, and $A(G)$ fail Axiom 7, are similar to that of Axiom 6. Moreover, $F(G)$ satisfies Axiom 7 as do $X(G)$ and $Z(G)$, while $Y(G)$ fails Axiom 7 when e is positive.

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