CITR

The University of Auckland
NEW ZEALAND

PhD Thesis Topic:
**"Non-Mean Curvature Estimators"**

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ELECTRICAL & COMPUTER ENGINEERING

MANUKAU INSTITUTE OF TECHNOLOGY
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**Background:**

Mostly electronics and computer engineering, with mathematics as well.

Introduced 3D computer graphics into the computer engineering curriculum at MIT in 2004.

Summer 2004-5 meetings with Vladimir Kovalevsky.

April 2005 Reinhard Klette suggests topic.

1) **Definition of curvature?**

2) **"Non-mean" curvature?**

3) **Curvature "estimators"?**

Differential Geometry

Computational Geometry

Digital Geometry

3D Computer Graphics

**Definition of curvature:**

It's a property associated with points of planar curves, space curves and surfaces.

It is a real-valued scalar, sometimes restricted to being non-negative.

**Non-mean curvature?**

What are the different curvatures?

1) Only one type for **plane curves**.

2) *Curvature* and *torsion* for **space curves**.

3) For **surfaces**

*normal curvature*,

*principle curvature*,

*mean curvature*,

*Gaussian curvature.*

**Estimators?**

Digitization of 3D *analogue* objects.

e.g. range scan, density scan

Digitization generally results in loss of information.

Given data from digitization, the properties of the original object can only be *estimated*.

## Planar curves.

Tangent & curvature.

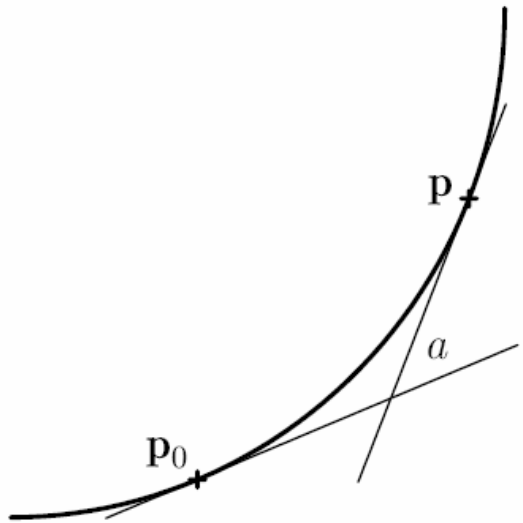


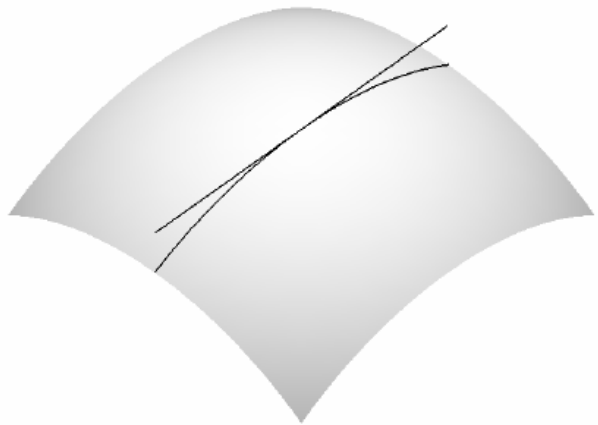

FIGURE 1. Curvature.

Consider a planar curve with arc-length $l$ from the point $\mathbf{p}_0$ to the point $\mathbf{p}$, and the counter-clockwise angular advance $a$ between the tangents at $\mathbf{p}_0$ and $\mathbf{p}$ as illustrated, for example, in figure (1). Then the *curvature* of the curve at the point $\mathbf{p}_0$ is defined to be

$$k = \lim_{\mathbf{p} \to \mathbf{p}_0} \frac{a}{l} = \frac{da}{dl}$$

**Surfaces.**

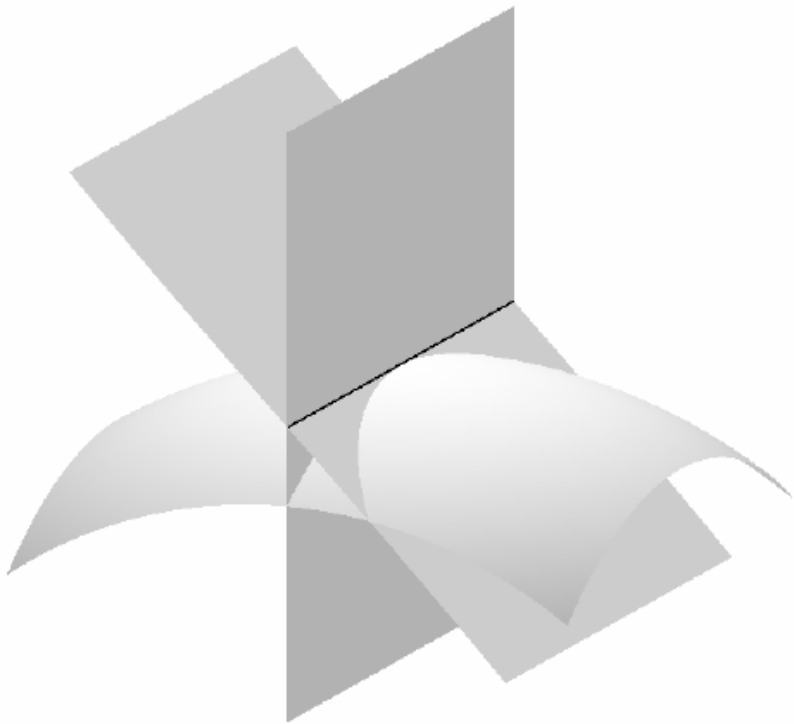
The intersection of surface and plane is a plane curve in the surface.



Tangent & curvature.

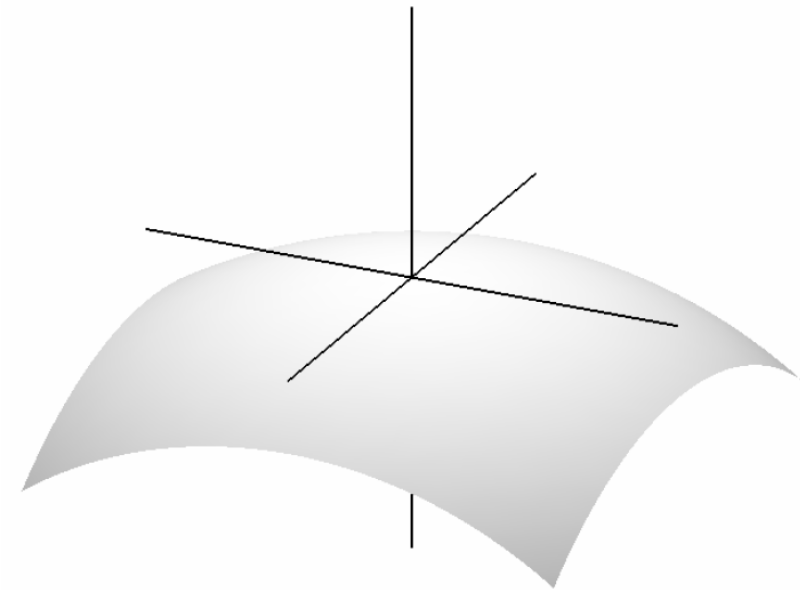
**Surfaces.**

Many plane curves share the same tangent but give different curvature at the same surface point!



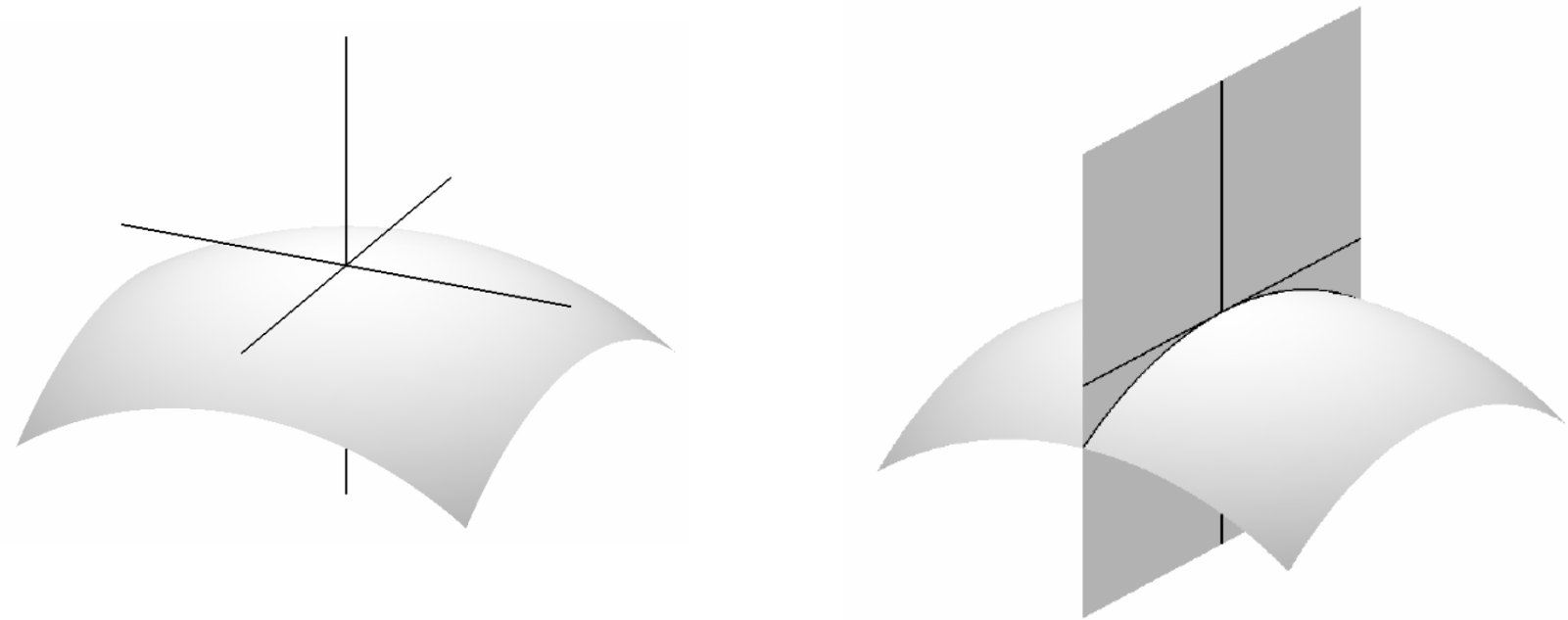
**Surfaces.**

Two tangents & the normal.



**Surfaces.**

*Normal curvature* is unique for each tangent direction.



But there are still many normal curvatures at each point of the surface!

**Surfaces.**

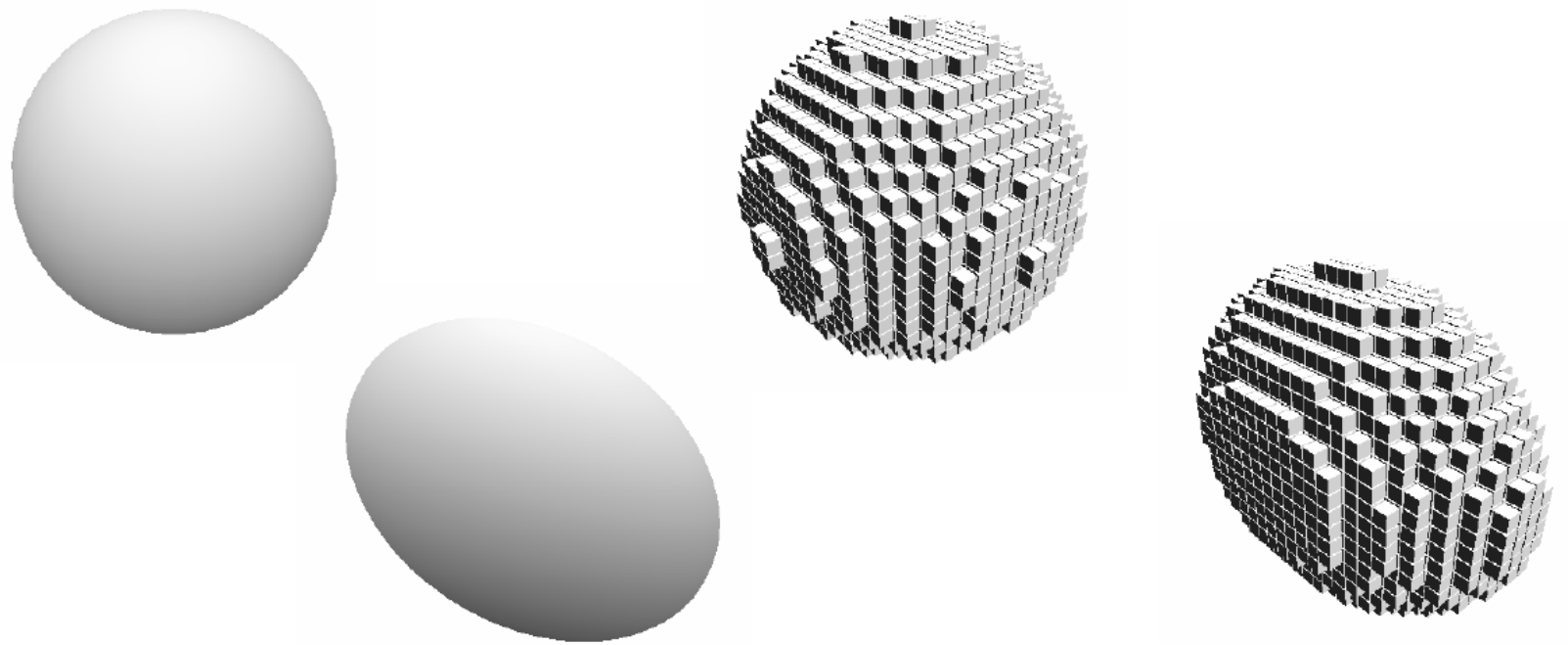
The two *principle curvatures* $k_1$ and $k_2$ are defined as the minimum and maximum normal curvatures.

The *mean curvature* is defined as: $\dfrac{k_1 + k_2}{2}$

The *Gaussian curvature* is defined as: $k_1 k_2$

**Surfaces from 3D digitization.**

Gaussian digitization.



Curvature estimates from digitized objects can be checked for accuracy when synthetic objects are used.

**Surfaces.**
**Synthetic objects.**

**Spheres.**

Surface curvature estimation. [HK03][KR04]

Start with grid based "cuts".

1) Tangents?
   Digital Straight Segments.
2) Normal curvature?
   Meusnier's Theorem.
3) Mean curvature?
   Average of normal curvature from any two orthogonal cuts.

Easy to check because sphere has constant curvature!

**Surfaces.**
**Synthetic objects.**

**Ellipsoids.**

Note that surface area calculation requires numerical evaluation of integral! [Tee04]

What about curvature calculations?

**Planar curves.**

Parametric specification.

A planar curve $P \subset \mathbb{R}^2$ can be specified parametrically as the set of points

$$P = \left\{ \mathbf{p}(u) = \begin{bmatrix} p_1(u) \\ p_2(u) \end{bmatrix} : u_{min} < u < u_{max} \right\}$$

**Planar curves.**

Speed.

The *speed* of a plane curve parameterization at the point $\mathbf{p}$ is

$$s(u) = ||\mathbf{p}_u|| = \sqrt{p_{1u}^2 + p_{2u}^2}$$

Note that with this notational convention $p_{1u}^2$ has the same meaning as $(dp_1/du)^2$

**Planar curves.**

Arc-length.

The *arc-length* of a curve $\mathbf{p}$ is given by

$$\int_{u_{min}}^{u_{max}} s(u)\,du$$

Arc-length can more generally be considered as a function of $u$ where the length is that from some starting point in the curve, parameterized by $u_0$, to an *arbitrary point* in the curve parameterized by $u$.

$$l(u) = \int_{u_0}^{u} s(u)\,du$$

**Planar curves.**

Tangent.

The unit length *tangent* to a curve at the point $\mathbf{r}$ is given by

$$\mathbf{t} = \frac{\mathbf{p}_u}{||\mathbf{p}_u||} = \frac{\mathbf{p}_u}{s}$$

**Planar curves.**
Curvature calculation.

Curvature for plane curves can be computed directly from the curve parameterization as

$$k = \frac{p_{1u}p_{2uu} - p_{1uu}p_{2u}}{||\mathbf{p}_u||^3}$$

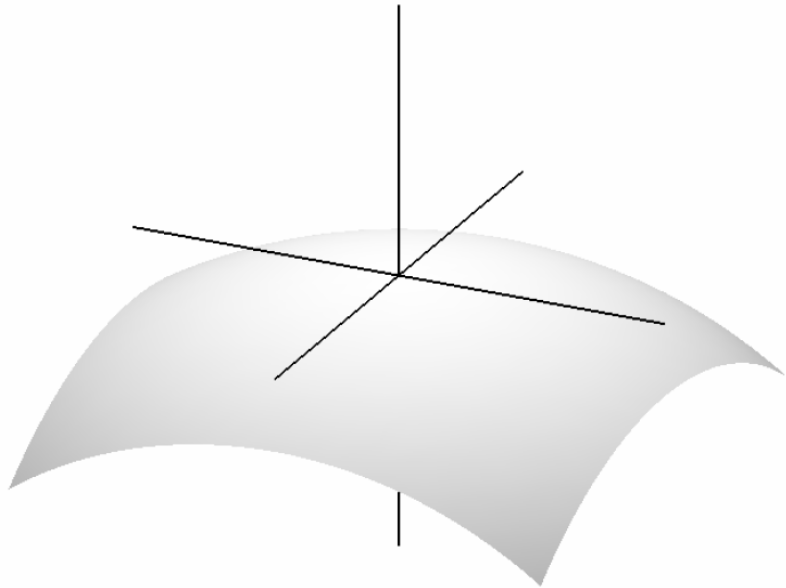
Note that curvature is dependant on both the first and second derivative of the curve parameterization.

**Surfaces.**
Parametric specification.

A surface $P \subset \mathbb{R}^3$can be specified parametrically as the set of points

$$P = \left\{ \mathbf{p}(u,v) = \begin{bmatrix} p_1(u,v) \\ p_2(u,v) \\ p_3(u,v) \end{bmatrix} : \begin{array}{c} u_{min} < u < u_{max} \\ v_{min} < v < v_{max} \end{array} \right\}$$

**Surfaces.**
Normal.



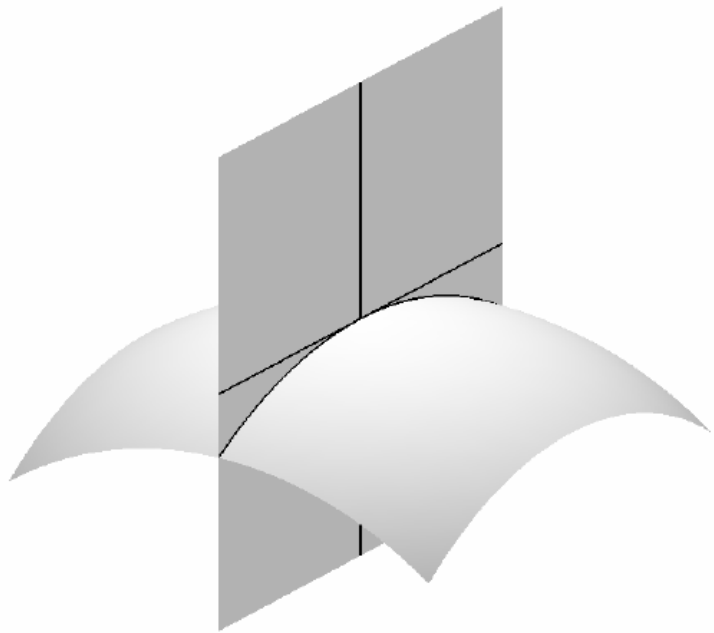
The unit length *surface normal* at the surface point $\mathbf{p}$ is

$$\mathbf{n} = \frac{\mathbf{p}_u \times \mathbf{p}_v}{||\mathbf{p}_u \times \mathbf{p}_v||}$$

Note that with this notational convention $\mathbf{p}_u$ means $\frac{\partial \mathbf{p}}{\partial u}$ and $\mathbf{p}_v$ means $\frac{\partial \mathbf{p}}{\partial v}$.

**Surfaces.**

How can the minimum and maximum normal curvatures be found when their directions are not known first?



**Surfaces.**
The Weingarten matrix!

The *Weingarten matrix* for a given point $\mathbf{p}$ in the surface $P$ is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} \mathbf{n}_u & \mathbf{n}_v \end{bmatrix} = \begin{bmatrix} \mathbf{p}_u & \mathbf{p}_v \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The two eigenvalues of the Weingarten matrix at a surface point are negations of the principle curvatures $k_{min}$ and $k_{max}$ at that point, and the two eigenvectors are the directions of those curvatures.

**Surfaces.**
Curvature calculations.

*First fundamental (quadratic) form* variables:

$$E = \mathbf{p}_u \cdot \mathbf{p}_u \qquad F = \mathbf{p}_u \cdot \mathbf{p}_v \qquad G = \mathbf{p}_v \cdot \mathbf{p}_v$$

*Second fundamental form* variables:

$$l = \mathbf{n} \cdot \mathbf{p}_{uu} \qquad m = \mathbf{n} \cdot \mathbf{p}_{uv} \qquad n = \mathbf{n} \cdot \mathbf{p}_{vv}$$

The Weingarten matrix can be calculated as follows

$$\mathbf{L} = \frac{1}{EG - F^2} \begin{bmatrix} Fm - Gl & Fn - Gm \\ Fl - Em & Fm - En \end{bmatrix}$$

**Surfaces.**
Curvature calculations.

*First fundamental (quadratic) form* variables:

$$E = \mathbf{p}_u \cdot \mathbf{p}_u \qquad F = \mathbf{p}_u \cdot \mathbf{p}_v \qquad G = \mathbf{p}_v \cdot \mathbf{p}_v$$

*Second fundamental form* variables:

$$l = \mathbf{n} \cdot \mathbf{p}_{uu} \qquad m = \mathbf{n} \cdot \mathbf{p}_{uv} \qquad n = \mathbf{n} \cdot \mathbf{p}_{vv}$$

The mean curvature can be calculated as follows

$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)}$$

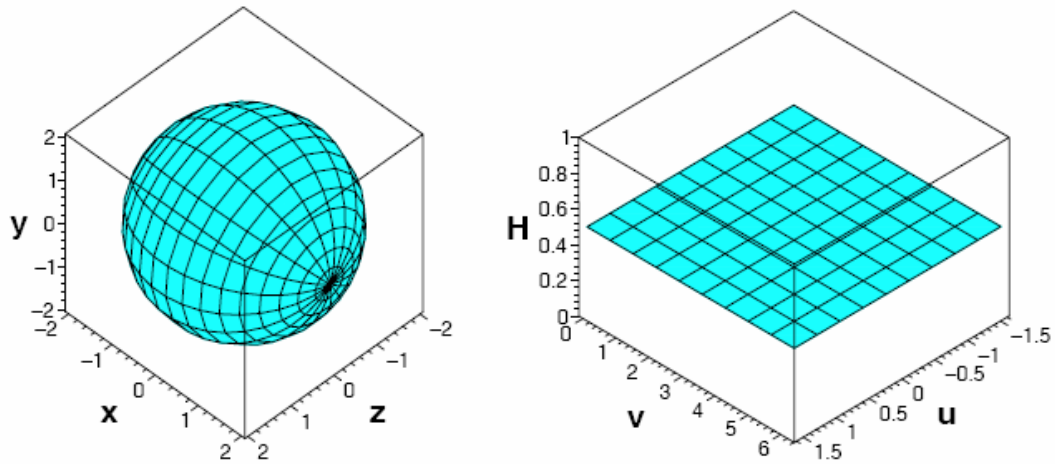
The Gaussian curvature can be calculated as follows

$$K = \frac{ln - m^2}{EG - F^2}$$

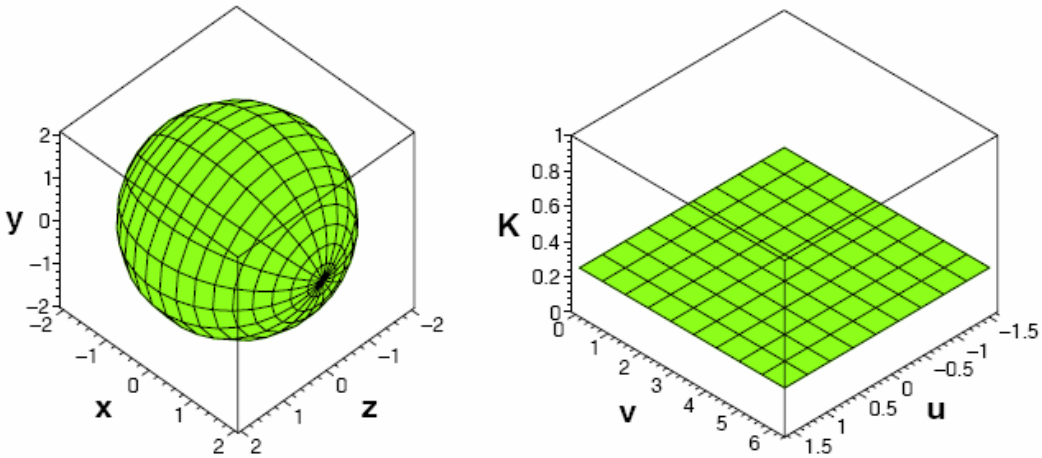
**Surfaces.**

Sphere curvature.

Mean



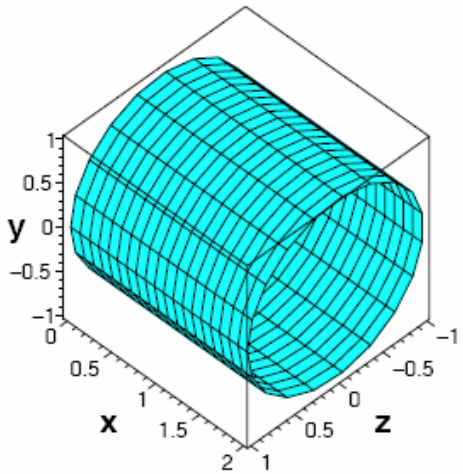

Gaussian

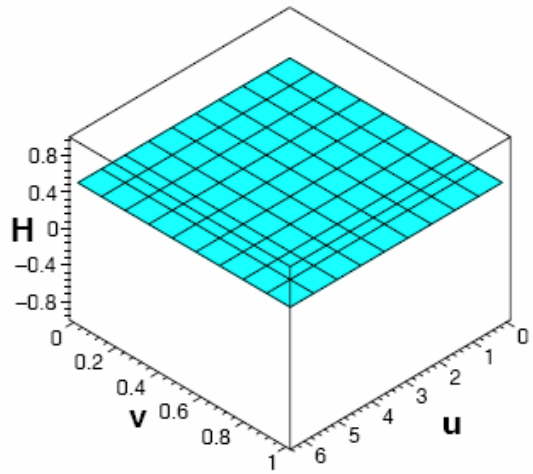


**Surfaces.**

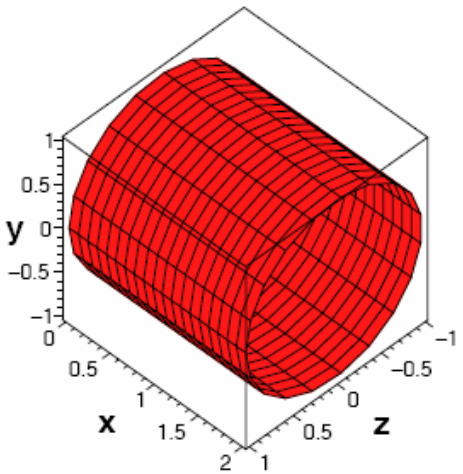
Cylinder.

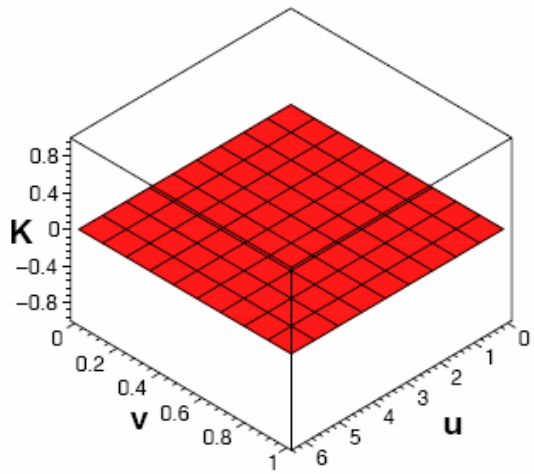
Mean




Gaussian

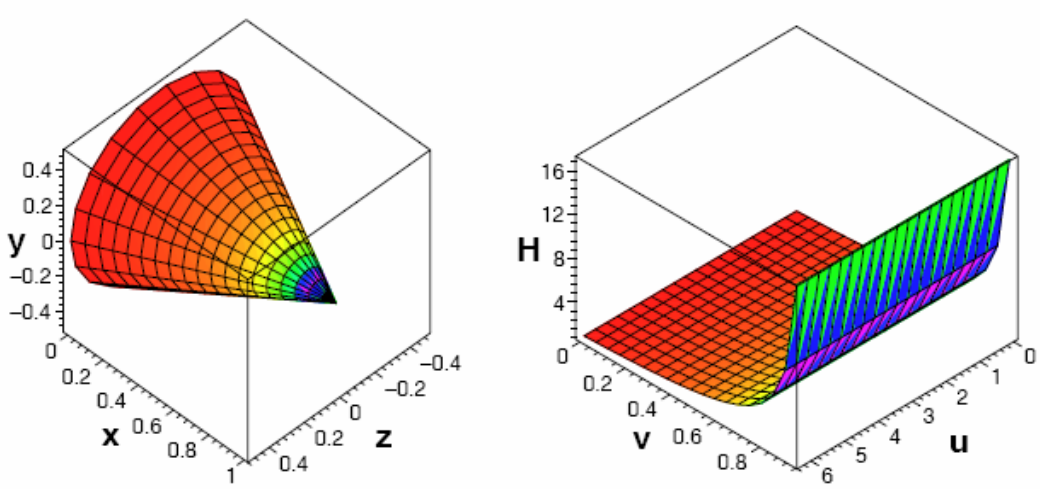


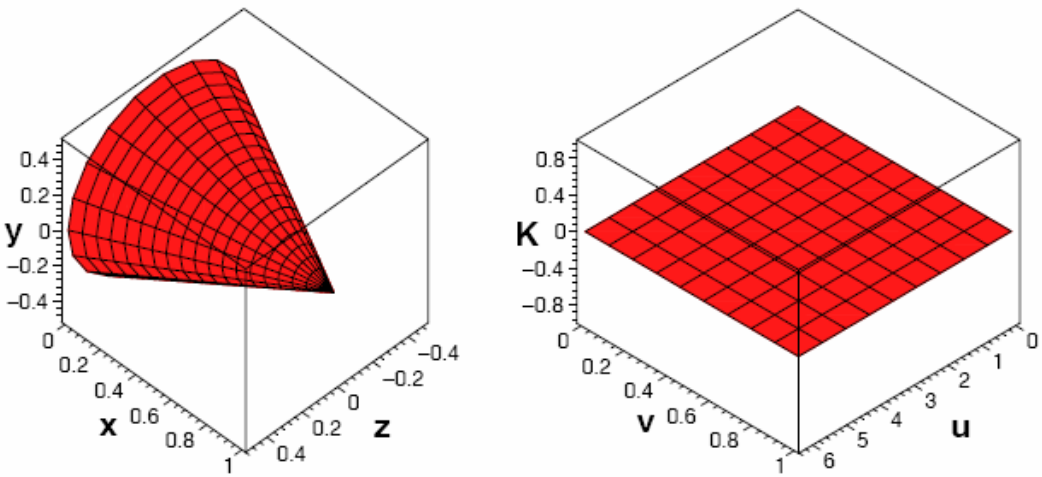
**Surfaces.**

Cone.

Mean



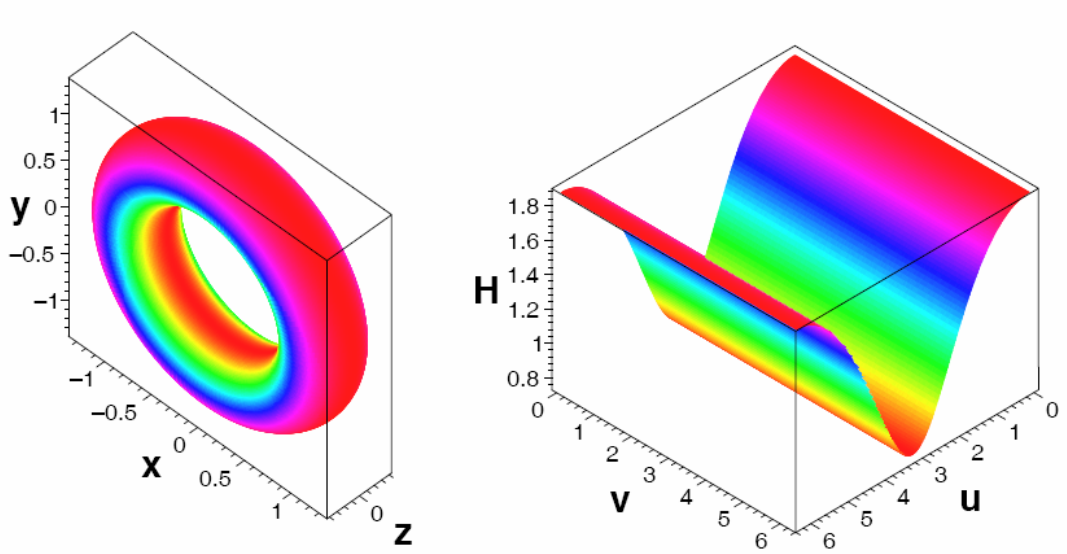

Gaussian



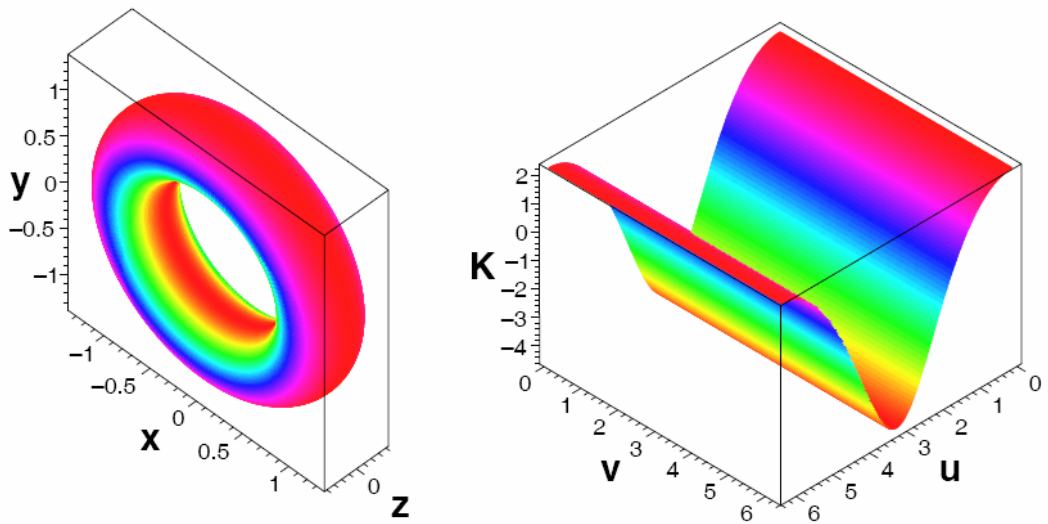
**Surfaces.**

Torus curvature.

Mean




Gaussian



**Surfaces.**

Ellipsoid curvature.

An ellipsoid parametization:

$$P = \left\{ \mathbf{p}(u,v) = \begin{bmatrix} a\sin u \\ b\cos u\cos v \\ c\cos u\sin v \end{bmatrix} : \begin{matrix} -\pi/2 < u < \pi/2 \\ 0 < v < 2\pi \end{matrix} \right\}$$

Mean curvature of ellipsoid:

$$H = \left(\frac{abc}{2}\right) \frac{b^2 + c^2 + (a^2 - c^2 + (c^2 - b^2)\cos^2 v)\cos^2 u}{(b^2c^2 - ((b^2 - c^2)a^2\cos^2 v + b^2(c^2 - a^2))\cos^2 u)^{3/2}}$$

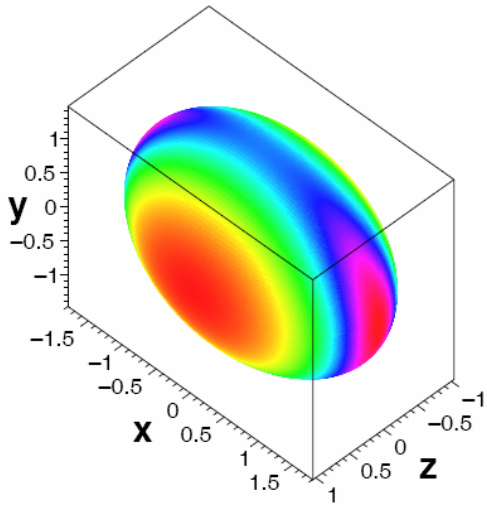
Gaussian curvature of ellipsoid:

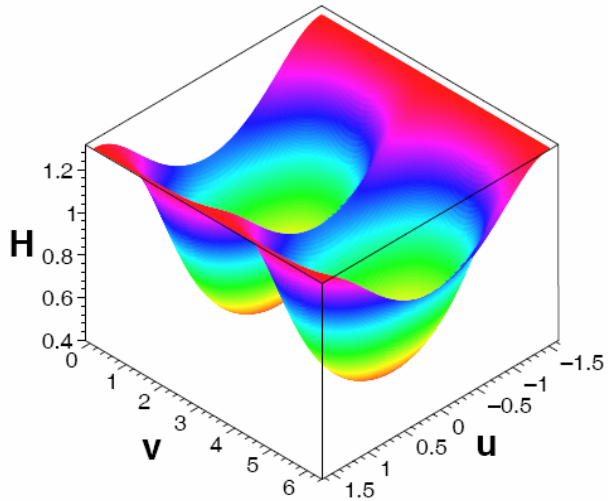
$$K = \frac{a^2b^2c^2}{(b^2c^2 - ((b^2 - c^2)a^2\cos^2 v + b^2(c^2 - a^2))\cos^2 u)^2}$$

**Surfaces.**

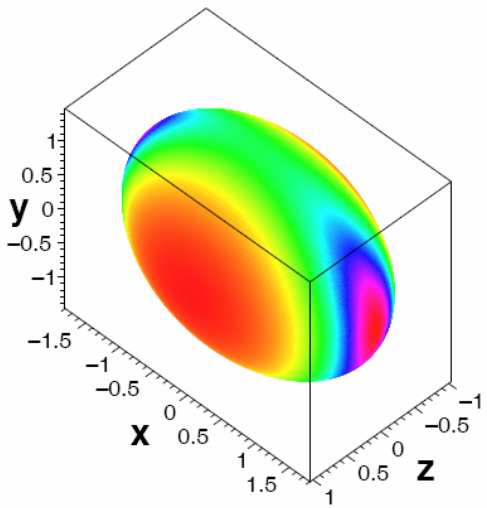
Ellipsoid curvature.

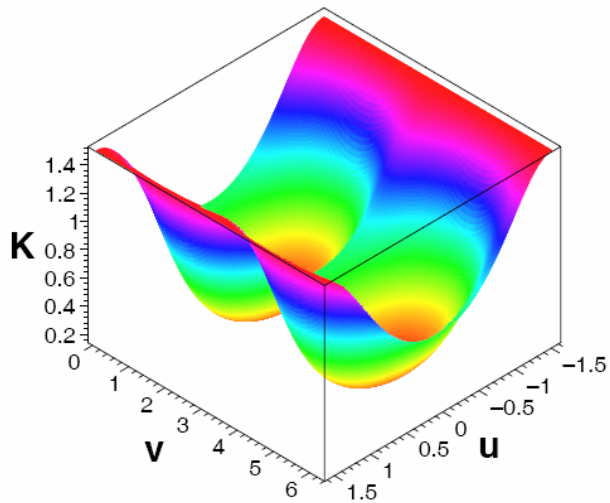
Mean




Gaussian



**Ellipsoid curvature estimation.**

1) Use the [HK03] method?
2) At which point in the real surface is the curvature estimated?
3) Use the geometric interpretation of Gaussian curvature?
4) Other methods? [TT05]?
5) Comparison of Gaussian & mean?
6) ...

## References


[AF97] Ehoud Ahronovitz and Christophe Fiorio, editors. *Discrete Geometry for Computer Imagery, 7th International Workshop, DCGI'97, Montpellier, France, Secember 3-5, 1997, Proceedings*, volume 1347 of *Lecture Notes in Computer Science*. Springer, 1997.

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[Tee04] Garry J. Tee. Surface area and capcity of ellipsoids in n dimensions. Technical Report CITR-TR-139, Centre for Image Technology and Robotics (CITR), University of Auckland, 2004.

[TT05] Wai-Shun Tong and Chi-Keung Tang. Robust estimation of adaptive tensors of curvature by tensor voting. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 27(3):434–449, March 2005.