Closure Properties of Real Number Classes under Limits and Computable Operators

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Abstract. In effective analysis, various classes of real numbers are discussed. For example, the classes of computable, semi-computable, weakly computable, recursively approximable real numbers, etc. All these classes correspond to some kind of (weak) computability of the real numbers. In this paper we discuss mathematical closure properties of these classes under the limit, effective limit and computable function. Among others, we show that the class of weakly computable real numbers is not closed under effective limit and partial computable functions while the class of recursively approximable real numbers is closed under effective limit and partial computable functions.

1 Introduction

In computable analysis, a real number x is called *computable* if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x effectively. That is, the sequence satisfies the condition that $|x_n - x| < 2^{-n}$, for any $n \in \mathbb{N}$. In this case, the real number x is not only approximable by some effective procedure, there is also an effective error-estimation in this approximation. In practice, it happens very often that some real values can be effectively approximated, but an effective error-estimation is not always available. To characterize this kind of real numbers, the concept of recursively approximable real numbers is introduced. Namely, a real number x is *recursively approximable* (r.a., in short) if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x. It is first noted by Ernst Specker in [15] that there is a recursively approximable real number which is not computable by encoding the halting problem into the binary expansion of a recursively approximable real numbers.

The class $\mathbf{C}_e$ of computable real numbers and the class $\mathbf{C}_{ra}$ of recursively approximable real numbers shares a lot of mathematical properties. For example, both $\mathbf{C}_e$ and $\mathbf{C}_{ra}$ are closed under the arithmetical operations and hence they are algebraic fields. Furthermore, these two classes are closed under the computable real functions, namely, if x is computable (r.a.) real number and f is a computable real function in the sense of, say, Grzegorczyk [6], then $f(x)$ is also computable (resp. r.a.).

The classes of real numbers between $\mathbf{C}_e$ and $\mathbf{C}_{ra}$ are also widely discussed (see e.g. [12,13,4,2,18]). Among others, the class of, so called, recursively enumerable real numbers might be the first widely discussed such class. A real

D.-Z. Du et al. (Eds.): COCOON 2000, LNCS 1858, pp. 170–179, 2000.

number x is called *recursive enumerable* if its left Dedekind cut is an r.e. set of rational numbers, or equivalently, there is an increasing computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x. We prefer to call such real numbers *left computable* because it is very naturally related to the left topology $\tau_< := \{(a;\infty) : a \in \mathbb{R}\}$ of the real numbers by the admissible representation of Weihrauch [16]. Similarly, a real number x is called *right computable* if it is a limit of some decreasing computable sequence of rational numbers. Left and right computable real numbers are called *semi-computable*. Robert Soare [12,13] discusses widely the recursion-theoretical properties of the left Dedekind cuts of the left computable real numbers. G. S. Ceĭtin [4] shows that there is an r.a. real number which is not semi-computable. Another very interesting result, shown by a series works of Chaitin [5], Solovay [14], Calude et al. [2] and Slaman [10], says that a real number x is r.e. random if and only if it is an Ω-number of Chaitin which is the halting probability of an universal self-delimiting Turing machine. We omit the details about these notions here and refer the interested readers to a nice survey paper of Calude [3].

Although the class of left computable real numbers has a lot of nice properties, it is not symmetrical in the sense that the real number $-x$ is right computable but usually not left computable for a left computable real number x. Furthermore, even the class of semi-computable real numbers is also not closed under the arithmetical operations as shown by Weihrauch and Zheng [18]. Namely, there are left computable real numbers y and z such that $y-z$ is neither left nor right computable. As the arithmetical closure of semi-computable real numbers, Weihrauch and Zheng [18] introduces the class of weakly computable real numbers. That is, a real number x is *weakly computable* if there are two left computable real numbers y and z such that $x = y-z$. It is shown in [18] that x is weakly computable if and only if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ which converges to x weakly effectively, i.e. $\lim_{n\to\infty} x_n = x$ and $\sum_{n=0}^{\infty} |x_n - x_{n+1}|$ is finite. By this characterization, it is also shown in [18] that the class of weakly computable real numbers is an algebraic field and is strictly between the classes of semi-computable and r.a. real numbers. In this paper we will discuss other closure properties of weakly computable real numbers for limits, effective limits and computable real functions. We show that weakly computable real numbers are not closed under the effective limits and partial computable real functions. For other classes mentioned above, we carry out also a similar discussion.

At the end of this section, let us explain some notions at first. For any set $A \subseteq \mathbb{N}$, denote by $x_A := \sum_{n\in A} 2^{-n}$ the real number whose binary expansion corresponds to set A. For any $k \in \mathbb{N}$, we define $kA := \{kn : n \in A\}$. For any function $f : \mathbb{N} \to \mathbb{N}$, a set A is called f-r.e. if there is a computable sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ such that $A = \cup_{i\in\mathbb{N}} \cap_{j\geq i} A_j$ and $|\{s : n \in A_s \Delta A_{s+1}\}| < f(n)$ for all $n \in \mathbb{N}$, where $A\Delta B := (A\backslash B) \cup (B\backslash A)$. If $f(n) := k$ is a constant function, then f-r.e. sets are also called k-r.e. A is called ω-r.e. iff there is a recursive function f such that A is f-r.e.

2 Computability of Real Numbers

In this section we give at first the formal definition of various versions of computability of real numbers and then recall some important properties about these notions. We assume that the reader familiar the computability about subsets of the natural nmbers $\mathbb{N}$ and number-theoretical functions. A sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers is computable iff there are recursive functions $a, b, c : \mathbb{N} \to \mathbb{N}$ such that $x_n = (a(n)-b(n))/(c(n)+1)$. We summarize the computability notions for real numbers as follows.

Definition 1. For any real number $x \in \mathbb{R}$,

1. x is *computable* iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$ and $\forall n\,(|x_n - x_{n+1}| < 2^{-n})$. In this case, the sequence $(x_n)_{n\in\mathbb{N}}$ is called *fast convergent* and it converges to x *effectively.*

2. x is *left (right) computable* iff there is an increasing (decreasing) computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$. Left and right computable real numbers are all called *semi-computable.*

3. x is *weakly computable* (w.c. in short) iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$ and $\sum_{n=0}^{\infty} |x_n - x_{n+1}|$ is finite. $(x_n)_{n\in\mathbb{N}}$ is called converging to x *weakly effectively.*

4. x is *recursively approximable* (r.a., in short) iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$.

The class of computable, left computable, right computable, semi-computable, w.c., r.a. real numbers is denoted by $\mathbf{C}_e$, $\mathbf{C}_{lc}$, $\mathbf{C}_{rc}$, $\mathbf{C}_{sc}$, $\mathbf{C}_{wc}$, $\mathbf{C}_{ra}$, respectively.

As shown in [18], the relationship among these classes looks like the following

$$\mathbf{C}_e = \mathbf{C}_{lc} \cap \mathbf{C}_{rc} \subsetneqq \begin{matrix}\mathbf{C}_{lc}\\ \mathbf{C}_{rc}\end{matrix} \subsetneqq \mathbf{C}_{sc} = \mathbf{C}_{lc} \cup \mathbf{C}_{rc} \subsetneqq \mathbf{C}_{wc} \subsetneqq \mathbf{C}_{ra}.$$

Note that in above definition, we define various versions of computability of real numbers in a similar way. Namely, a real number x is of some version of computability iff there is a computable sequence of rational numbers which satisfies some special property and converges to x. For example, if $P_{lc}[(x_n)]$ means that $(x_n)_{n\in\mathbb{N}}$ is increasing, then $x \in \mathbf{C}_{lc}$ iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $P_{lc}[(x_n)]$ and $\lim_{n\to\infty} x_n = x$. In general, for any reasonable property on sequences, we can define a corresponding class of real numbers which have some kind of (weaker) computability. This can even be extended to the case of sequences of real numbers as in the following definition.

Definition 2. Let P be any property about the sequences of real numbers. Then

1. A real number x is called *P-computable* if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbres which satisfies property P and converges to x. The class of all P-computable real numbers is denoted by $\mathbf{C}_P$

2. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is called *P-computable*, or it is a *computable sequence of* $\mathbf{C}_P$ iff there is a computable double sequence $(r_{nm})_{nm\in\mathbb{N}}$ of rational numbers such that $(r_{nm})_{m\in\mathbb{N}}$ satisfies P and $\lim_{m\to\infty} r_{nm} = x_n$ for all $n \in \mathbb{N}$.

3. The class $\mathbf{C}_P$ is called "*closed under limits*", iff for any computable sequences $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{C}_P$, the limits $x := \lim_{n\to\infty} x_n$ is also in $\mathbf{C}_P$ whenever $(x_n)_{n\in\mathbb{N}}$ satisfies P and converges.

4. The class $\mathbf{C}_P$ defined in 2. is called "*closed under effective limits*", iff for any fast convergent computable sequences $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{C}_P$, the limits $x := \lim_{n\to\infty} x_n$ is also in $\mathbf{C}_P$.

Now we remind the notion of computable real function. There are a lot of approaches to define the computability of real functions. Here we use Grzegorczyk-Ko-Weihrauch's approach and define computable real function in terms of "Type-two Turing Machine" (TTM, in short) of Weihrauch.

Let Σ be any alphabet. Σ^* and Σ^∞ are sets of all finite strings and infinite sequences on Σ, respectively. Roughly, TTM M extends the classical Turing machine in such a way that it can be inputed and also can output infinite sequences as well as finite strings. For any $p \in \Sigma^* \cup \Sigma^\infty$, $M(p)$ outputs a finite string q, if $M(p)$ writes q in output tape and halt in finite steps similar to the case of classical Turing machine. $M(p)$ outputs an infinite sequence q means that $M(p)$ will compute forever and keep writing q on the output tape. We omit the formal details about TTM here and refer the interested readers to [16,17]. We will omit also the details about the encoding of rational numbers by Σ^* and take directly the sequences of rational numbers as inputs and outputs to TTM's.

Definition 3. A real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is *computable* if there is a TTM M such that, for any $x \in \mathrm{dom}(f)$ and any sequence $(u_n)_{n\in\mathbb{N}}$ of rational numbers which converges effectively to x, $M((u_n)_{n\in\mathbb{N}})$ outputs a sequence $(v_n)_{n\in\mathbb{N}}$ of rational numbers which converges to $f(x)$ effectively.

Note that, in this definition we do not add any restriction on the domain of computable real function. Hence a computable real function can have any type of domain, because $f \restriction A$ is always computable whenever f is computable and $A \subseteq \mathrm{dom}(f)$. Furthermore, for a total function $f : [0,1] \to \mathbb{R}$, f is computable iff f is sequentially computable and effectively uniformly continuous (see [9]).

Definition 4. For any subset $\mathbf{C} \subseteq \mathbb{R}$,

1. $\mathbf{C}$ is *closed under computable operators*, iff $f(x) \in \mathbf{C}$ for any $x \in \mathbf{C}$ and any total computable real function $f : \mathbb{R} \to \mathbb{R}$.

2. $\mathbf{C}$ is *closed under partial computable operators*, iff $f(x) \in \mathbf{C}$, for any $x \in \mathbf{C}$ and any partial computable real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ with $x \in \mathrm{dom}(f)$.

Following proposition follows immediately from the definition. Remember that, $A \subseteq \mathbb{N}$ is Δ^0_2 iff A is Turing reducible to the halting problem $\emptyset'$.

Proposition 1. *1. $x_A \in \mathbf{C}_e \iff A$ is rcursive.*
2. $x_A \in \mathbf{C}_{ra} \iff A$ is a Δ^0_2-set, or equivalently, $A \leq_T \emptyset'$.
3. $\mathbf{C}_e$ and $\mathbf{C}_{ra}$ are closed under arithmetical operations $+, -, \times$ and $\div$, hence they are algebraic fields.

4. $\mathbf{C}_e$ *are closed under limits and computable real functions.*
5. $\mathbf{C}_{lc}$ *and* $\mathbf{C}_{rc}$ *are closed under addition.*

Some other non-trivial closure properties are shown in [18] and [19].

Theorem 1 (Weihrauch and Zheng). *1.* $\mathbf{C}_{sc}$ *is not closed under addition, i.e. there are left computable* y *and right computable* z *such that* $y+z$ *is neither left nor right computable.*

2. $\mathbf{C}_{wc}$ *is closed under arithmetical operations. In fact* $\mathbf{C}_{wc}$ *is just the closure of* $\mathbf{C}_{sc}$ *under the arithmetical operations.*

It is not very surprising that the classes $\mathbf{C}_{lc}$ and $\mathbf{C}_{rc}$ are not closed under "subtraction" and, in general, under computable real functions, because they are not symmetrical. On the other hand, the class $\mathbf{C}_{wc}$ is symmetrical and closed under arithmetical operations. So it is quite natural to ask whether it is also closed under limits and computable real functions. In the following we will give the negative answers to both questions. To this end we need the following observations about weakly computable real numbers.

Theorem 2 (Ambos-Spies, [1]). *1. If* $A, B \subseteq \mathbb{N}$ *are incomparable under Turing reduction, then* $x_{A\oplus\overline{C}}$ *is not semi-computable.*

2. For any set $A \subseteq \mathbb{N}$*, if* x_{2A} *is weakly computable, then* A *is* f*-r.e. for* $f(n) := 2^{3n}$*, hence* A *is* ω*-r.e.*

Theorem 3 (Zheng [20]). *There is a non-*ω*-r.e.* Δ^0_2*-set* A *such that* x_A *is weakly computable.*

3 Closure Property under Limits

In this section, we will discuss the closure properties of several classes of real numbers under limits. We first consider the classes of left and right computable real numbers. The following result is quite straightforward.

Theorem 4. *The classes of left and right computable real numbers are closed under limits, respectively.*

For semi-computable real numbers, the situation is different.

Theorem 5. *The class* $\mathbf{C}_{sc}$ *is not closed under limits.*

Proof. Define, for any $n, s \in \mathbb{N}$, at first the following sets:

$$A := \{e \in \mathbb{N} : \varphi_e \text{ is total}\}$$
$$A_n := \{e \in \mathbb{N} : (\forall x \le n)\varphi_e(x)\downarrow\}$$
$$A_{n,s} := \{e \in \mathbb{N} : (\forall x \le n)\varphi_{e,s}(x)\downarrow\}$$

Since $A_{n,s} \subseteq A_{e,s+1}$, $(x_{A_{n,s}})_{n,s\in\mathbb{N}}$ is obviously a computable sequence of rational numbers such that, for and $n \in \mathbb{N}$, $(x_{A_{n,s}})_{s\in\mathbb{N}}$ is nondecreasing and converges to x_{A_n}. That is, $(x_{A_n})_{n\in\mathbb{N}}$ is a computable sequence of $\mathbf{C}_{lc}$, hence it is a computable sequence of $\mathbf{C}_{sc}$. But its limit x_A is not semi-computable. In fact x_A is even not r.a. by Proposition 1, since A is not a Δ^0_2-set. □

Note that in above proof, as a computable sequence of $\mathbf{C}_{lc}$, $(x_{A_n})_{n\in\mathbb{N}}$ is also a computable sequence of $\mathbf{C}_{wc}$ and $\mathbf{C}_{ra}$. Then the folowing corollary follows immediately.

Corollary 1. *The classes $\mathbf{C}_{wc}$ and $\mathbf{C}_{ra}$ are not closed under the limit.*

Now we discuss the closure property under the effective limits. We will show that the class of semi-computable real numbers is closed under effective limits and the class of weakly computable real numbers, hence also the class of r.a. real numbers, is not closed under effective limits.

Theorem 6. *The class $\mathbf{C}_{sc}$ is closed under the effective limits.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a computable sequence of $\mathbf{C}_{sc}$ which satisfies the condition that $\forall n(|x_n - x_{n+1}| < 2^{-(n+1)})$ and converges to x. We shall show that $x \in \mathbf{C}_{sc}$.

By Definition 2, there is a computable sequence $(r_{ij})_{i,j\in\mathbb{N}}$ of rational numbers such that, for any $n \in \mathbb{N}$, $(r_{nj})_{n\in\mathbb{N}}$ is monotonic and converges to x_n. For any n, we can effectively determine whether x_n is left or right computable by comparing, say, r_{n0} and r_{n1}. Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ can be split into two computable subsequences $(x_{n_i})_{i\in\mathbb{N}}$ and $(x_{m_i})_{i\in\mathbb{N}}$ of left and right computable real numbers, respectively. At least one of them is infinite. Suppose w.l.o.g. that $(x_{n_i})_{i\in\mathbb{N}}$ is an infinite sequence. Obviously it is also a fast convergent computable sequence, i.e., $|x_{n_i} - x_{n_{i+1}}| < 2^{-i}$, since $(x_n)_{n\in\mathbb{N}}$ converges fast. Define a new sequence $(y_n)_{n\in\mathbb{N}}$ by $y_i := x_{n_i} - 2^{-(i-1)}$. Since $y_{i+1} = x_{n_{i+1}} - 2^{-i} = (x_{n_{i+1}} - x_{n_i} + 2^{-i} + (x_{n_i} - 2^{-(i-1)}) \geq x_{n_i} - 2^{-(i-1)} = y_i$ $(y_i)_{i\in\mathbb{N}}$ is an increasing sequence. Let $r'_{ij} := r_{ij} - 2^{-(i-1)}$. Then $(r'_{ij})_{i,j\in\mathbb{N}}$ is a computable sequence of rational numbers such that, for any i, $(r'_{ij})_{j\in\mathbb{N}}$ is increasing and converges to y_i. Namely, $(y_i)_{i\in\mathbb{N}}$ is an increasing computable sequence of $\mathbf{C}_{lc}$. By Theorem 4, its limit $\lim_{i\to\infty} y_i = \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} x_i = x$ is also left computable, i.e., $x \in \mathbf{C}_{lc} \subseteq \mathbf{C}_{sc}$. □

Theorem 7. *The class $\mathbf{C}_{wc}$ is not closed under effective limits.*

Proof. Suppose by Theorem 3 that A is a non-ω-r.e. Δ_2^0-set such that x_A is weakly computable. Then x_{2A} is not weakly computable by Theorem 2. Let $(A_s)_{s\in\mathbb{N}}$ be a recursive approximation of A such that $(x_{A_s})_{s\in\mathbb{N}}$ converges to x_A weakly effectively, i.e. $\sum_{s=0}^{\infty} |x_{A_s} - x_{A_{s+1}}| \leq C$ for some $C \in \mathbb{N}$. Define, for $n, s \in \mathbb{N}$,

$$B_{n,s} := 2(A_s \upharpoonright (n+1)) \cup (A_s \downharpoonright 2n)$$
$$B_n := 2(A \upharpoonright (n+1)) \cup (A \downharpoonright 2n)$$

It is easy to see that $(B_{n,s})_{n,s\in\mathbb{N}}$ is a computable sequence of finite subsets of $\mathbb{N}$, hence $(x_{B_{n,s}})_{n,s\in\mathbb{N}}$ is a computable sequence of rational numbers.

Since $\lim_{s\to\infty} A_s = A$, there is an $N(n)$, for any $n \in \mathbb{N}$ such that, for any $s \geq N(n)$, $A_s \upharpoonright (n+1) = A \upharpoonright (n+1)$. Let $C_1 = \sum_{s=0}^{N(n)} |x_{B_{n,s}} - x_{B_{n,s+1}}|$. Then

$\sum_{s=0}^{\infty} |x_{B_{n,s}} - x_{B_{n,s+1}}| = \sum_{s=0}^{N(n)} |x_{B_{n,s}} - x_{B_{n,s+1}}| + \sum_{s>N(n)}^{\infty} |x_{B_{n,s}} - x_{B_{n,s+1}}| = C_1 + \sum_{s>N(n)}^{\infty} |x_{A_{n,s}} - x_{A_{n,s+1}}| < C_1 + C$. On the other hand, it is easy to see that $\lim_{s\to\infty} x_{B_{n,s}} = x_{B_n}$. Therefore, the sequence $(x_{B_{n,s}})_{n,s\in\mathbb{N}}$ converges to x_{B_n} weakly effectively. Hence $(x_{B_n})_{n\in\mathbb{N}}$ is a weakly computable sequence of real numbers. By the definition of B_n, $B_n \Delta 2A \subseteq \{2n+1, 2n+2, \ldots\}$. It follows that $|x_{B_n} - x_{2A}| \leq 2^{-2n} \leq 2^{-n}$ This means that $(x_{B_n})_{n\in\mathbb{N}}$ converges to x_{2A} effectively and this ends the proof of the theorem. □

Theorem 8. *The class $\mathbf{C}_{ra}$ is closed under effective limits.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be any computable sequence of $\mathbf{C}_{ra}$ which converges effectively to x. Assume w.l.o.g. that it satisfies, for all $n \in \mathbb{N}$, the condition $|x_n - x_{n+1}| < 2^{-(n+1)}$. By Definition 2, there is a computable sequence $(r_{ij})_{i,j\in\mathbb{N}}$ of rational numbers such that, for any $n \in \mathbb{N}$, $\lim_{s\to\infty} r_{ns} = x_n$. We shall show that $x \in \mathbf{C}_{ra}$.

It suffices to construct a computable sequence $(u_s)_{s\in\mathbb{N}}$ of rational numbers such that $\lim_{s\to\infty} u_s = x$. This sequence will be constructed from $(r_{ij})_{i,j\in\mathbb{N}}$ in following stages:

The construction of sequence $(u_s)_{s\in\mathbb{N}}$:

Stage $s = 0$: Define $u_0 := r_{00}$, $t(0,0) := 0$ and $i(0) := 0$.

Stage $s+1$: Given $i(s)$, $u_0, \ldots, u_{i(s)}$ and $t(j,s)$ for all $j \leq s$. If there is $j \leq s$ satisfying $|u_{t(j-1,s)} - r_{js}| < 2^{-(j-1)}$ such that either $t(j,s) \neq -1$ & $|u_{t(j,s)} - r_{js}| \geq 2^{-(j+1)}$ or $t(j-1,s) \neq -1$ & $t(j,s) = -1$, then choose j_0 as minimal such j and define

$$(*) \quad \begin{cases} i(s+1) & := i(s) + 1 \\ u_{i(s+1)} & := r_{j_0,s} \\ t(j, s+1) & := \begin{cases} t(j,s) & \text{if } 0 \leq j < j_0 \\ i(s)+1 & \text{if } j = j_0 \\ -1 & \text{if } j_0 < j \leq s+1. \end{cases} \end{cases}$$

Otherwise, if no such j exists, then define, $i(s+1) := i(s)$, $t(s+1, s+1) := -1$ and $t(j, s+1) := t(j,s)$ for all $j \leq s$.

To show this construction succeeds, we need only to prove the following claims.

1. For any $j \in \mathbb{N}$, the limit $t(j) := \lim_{s\to\infty} t(j,s)$ exists and satisfies the conditions that $t(j) \neq -1$ and $|u_{t(j)} - x_j| \leq 2^{-(j+1)}$.
2. $\lim_{s\to\infty} i(s) = +\infty$.
3. For any $j \in \mathbb{N}$, $(\forall s \leq t(j))(|u_{t(j)} - u_s| \leq 2^{-j})$.

Now it is clear that the sequence $(u_s)_{s\in\mathbb{N}}$ constructed above is a computable infinite sequence of rational numbers. Furthermore, this sequence converges to x. This completes the proof of Theorem. □

4 Closure Property under Computable Operators

In this section we will discuss the closure property under computable operators. The following result about left and right computable real numbers is immediate by the fact that the real function f defined by $f(x) = -x$ is computable.

Proposition 2. *The classes* $\mathbf{C}_{lc}$ *and* $\mathbf{C}_{rc}$ *are not closed under the computable operators, hence is also not closed under partial computable operators.*

To discuss the closure property under partial computable operators for other classes, we will apply the following observation of Ko [7].

Theorem 9 (Ker-I Ko [7]). *For any sets* $A, B \subseteq \mathbb{N}$, $A \leq_T B$ *iff there is a (partial) computable real function* $f :\subseteq \mathbb{R} \to \mathbb{R}$ *such that* $f(x_B) = x_A$.

¿From this result, it is easy to show that a lot of classes of real numbers are not closed under the partial computable operators.

Theorem 10. *The classes* $\mathbf{C}_{sc}$ *and* $\mathbf{C}_{wc}$ *are not closed under the partial computable operators. The class* $\mathbf{C}_{ra}$ *is closed under partial computable operators.*

Proof. 1. For class $\mathbf{C}_{sc}$. By Muchnik-Friedberg Theorem (see [11]), there are two r.e. sets A and B such that they are incomparable under Turing reduction. Then $x_{A\oplus\overline{B}}$ is not semi-computable by Theorem 2. On the other hand, $x_{A\oplus B}$ is left computable since $A \oplus B$ is r.e. Obviously, we have the reduction that $A \oplus \overline{B} \leq_T A \oplus B$. By Theorem 9, there is a computable real function f such that $f(x_{A\oplus B}) = x_{A\oplus\overline{B}}$. Therefore, $\mathbf{C}_{sc}$ is not closed under partial computable operators.

2. For class $\mathbf{C}_{wc}$. By Theorem 3, there is a non-ω-r.e. set A such that x_A is weakly computable. On the other hand, x_{2A} is not weakly computable by Theorem 2 since $2A$ is obviously not ω-r.e. Because $2A \leq_T A$, by Theorem 9, there is a computable real function f such that $f(x_A) = x_{2A}$. That is, $\mathbf{C}_{wc}$ is not closed under the partial computable operators.

3. For class $\mathbf{C}_{ra}$, it follows immediately from the fact that a real number x_A is r.a iff A is a Δ^0_2-set and the class of all Δ^0_2-sets is closed under the Turing reduction, i.e. if $A \leq_T B$ and B is Δ^0_2-set, then A is also Δ^0_2-set. □

It is shown in Theorem 1 that the class $\mathbf{C}_{sc}$ is not closed under addition. Hence it is not closed under the polynomial functions with several arguments. Namely, if $p(x, \cdots, x_n)$, $n \geq 2$, is a polynomial with rational coefficients and $a_1, \cdots, a_n$ are semi-computable real numbers, then $p(a_1, \cdots, a_n)$ is not necessary semi-computable. But for the case of $n = 1$, it is not clear. Furthermore it is also not known whether the class of semi-computable real numbers is closed under the total computable real functions.

Similarly, it remains still open whether the class $\mathbf{C}_{wc}$ is closed under (total) computable operators. We guess it is not. One possible approach is to define a computable real function which maps some weakly computable x_A for a non-ω-r.e. set A to a not weakly computable real number x_{2A}. Using the idea in the proof of Theorem 9, it is not difficult to show that there is a computable real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ such that $f(x_A) = x_{2A}$ for any irrational x_A. Unfortunately, such function cannot be extended to a total computable real function as shown by next result.

Theorem 11. *1. Let $f :\subseteq \mathbb{N} \to \mathbb{N}$ be a function such that $f(x_A) = x_{2A}$ for any irrational x_A. If x_A is a rational number, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of irrational numbers such that $\lim_{n\to\infty} x_n = x_{2A}$ and $\lim_{n\to\infty} f(x_n) = x_{2A}$.*
2. The function $f : [0;1] \to \mathbb{R}$ defined by $f(x_A) := x_{2A}$ for any $A \subseteq \mathbb{N}$ is not continuous at any rational points, hence it is not computable.

Proof. 1. Suppose that function $f :\subseteq \mathbb{N} \to \mathbb{N}$ satisfies $f(x_A) = x_{2A}$ for any irrational x_A. Let x_A be rational, hence A is a finite set. We define a sequence $(x_n)_{n\in\mathbb{N}}$ of irrational numbers by $x_n := x_A + \sqrt{2}\cdot 2^{-(n+1)}$. Let n_0 be the maximal element of A. Define a set A_n by $x_{A_n} = \sqrt{2} \cdot 2^{-(n+1)}$ for any $n \in \mathbb{N}$. Then for any $n > n_0$, $x_{A_n} < 2^{-n} \leq 2^{-n_0}$. This implies that A_n contains only the elements which are bigger than n_0. Therefore, for any $n \geq n_0$, $A \cap A_n = \emptyset$ and $f(x_n) = f(x_A + \sqrt{2} \cdot 2^{-(n+1)}) = f(x_A + x_{A_n}) = f(x_{A\cup A_n}) = x_{2(A\cup A_n)} = x_{(2A)\cup(2A_n)} = x_{2A} + x_{2A_n}$. Since $\lim_{n\to\infty} x_{A_n} = 0$, it is easy to see that $\lim_{n\to\infty} x_{2A_n} = 0$ too. So we conclude that $\lim_{n\to\infty} f(x_n) = x_{2A}$.

2. Suppose that $f : [0;2] \to \mathbb{R}$ satisfies $f(x_A) = x_{2A}$ for any $A \subseteq \mathbb{N}$. For any rational x_A, A is finite. Let n_0 be the maximal element of A and $A' := A\backslash\{n_0\}$ and define, for all $n \in \mathbb{N}$, a finite set A_n by $A_n := A' \cup \{n_0+1, n_0+2, \cdots, n_0+n\}$. Then it is easy to see that $\lim_{n\to\infty} x_{A_n} = x_A$. On the other hand we have:

$$\begin{aligned} f(x_{A_n}) = x_{2A_n} &= x_{2A'} + \sum_{i=1}^{n} 2^{-2(n_0+i)} \\ &= x_{2A} - 2^{-2n_0} + 2^{-2n_0} \cdot (1 - 2^{-2n})/3 \end{aligned}$$

This implies that $\lim_{n\to\infty} f(x_{A_n}) = x_{2A} - 2^{-2n_0+1}/3 \neq x_{2A}$. □

In summary, the closure properties of several classes of real numbers under arithmetic operations (+, −, × and ÷), limits, effective limits, partial computable operators and computable operators are listed in the following table:

	arithmetic operations	limits	effective limits	computable operators	partial computable operators
$\mathbf{C}_e$	Yes	Yes	Yes	Yes	Yes
$\mathbf{C}_{lc}$	No	Yes	Yes	No	No
$\mathbf{C}_{rc}$	No	Yes	Yes	No	No
$\mathbf{C}_{sc}$	No	No	Yes	?	No
$\mathbf{C}_{wc}$	Yes	No	No	?	No
$\mathbf{C}_{ra}$	Yes	No	Yes	Yes	Yes

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