

# Incompleteness, Uncertainty and Complexity

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# Outline

- 1 Axiomatics
- 2 Incompleteness
- 3 Information
- 4 Complexity invariance
- 5 Impact revisited

## The axiomatic method and Hilbert's credo

Any mathematical theory has to be formulated in the form of an axiomatic theory where all substantial assumptions appear as explicit axioms.

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- Is the method 'universal'?
- Is the method 'secure'?

## Hilbert radio address

On 8 September 1930 in Königsberg, David Hilbert gave a speech entitled **Natureerkennen und Logik**, which ended with the famous words:

For us there is no *ignorabimus*, and in my opinion none whatever in natural science. In opposition to the foolish *ignorabimus* I offer our answer:

We must know,  
We will know.

A four minute excerpt was broadcast by radio:

▶ Wir müssen wissen, Wir werden wissen

## The incompleteness theorem

The Incompleteness Theorem as announced on 7 October 1930 in Königsberg at the *First International Conference on the Philosophy of Mathematics*:

### Theorem

In any formal system adequate for number theory there exists an undecidable formula—that is, a formula that is not provable and whose negation is not provable.

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**Caution:** Gödel's incompleteness does not show that there exist true, but absolutely unprovable formulas/propositions!

## Gödel's own interpretation



*“Either mathematics is too big for the human mind or the human mind is more than a machine”.*

K. Gödel

▶ GMath

▶ GPhil

▶ GVaria

▶ GObit

## Incompleteness impact

## 1

With the exception of von Neumann, who immediately grasped Gödel's line of thought and its importance, incompleteness passed un-noticed in Königsberg.

Hans Reichenbach, who summarised the discussion for *Die Naturwissenschaften*, **made no reference to Gödel's contributions at all.**

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Still, it's interesting to compare Gödel's with Wittgenstein's incompleteness (we cannot speak the unspeakable truths, but they exist).

Martin Davis (1999), in a comment on a review I published in the *London Review of Books*, wrote:

*Calude complains about the assertion that Gödel's theorems have had "very little impact on the bulk of mathematical research". **Alas, this is unquestionably true.***

Still, for many, incompleteness is

“the third leg, together with Heisenberg’s uncertainty principle and Einstein’s relativity, of that tripod of theoretical cataclysms that have been felt to force disturbances deep down in the foundations of the ‘exact sciences.’ ”

## Chaitin's complexity-theoretic proof

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His proof is based on program-size complexity  $H$ : the complexity  $H(s)$  of a binary string  $s$  is the size, in bits, of the shortest program for a universal self-delimiting Turing machine to calculate  $s$ .

### Theorem [Chaitin]

For every finitely-specified, sound, consistent theory strong enough to formalize arithmetic, there exists a positive constant  $M$  such that no sentence of the form " $H(x) > m$ " is provable in the theory unless  $m$  is less than  $M$ .

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## Theorem [Chaitin]

For every finitely-specified, sound, consistent theory strong enough to formalize arithmetic, there exists a positive constant  $M$  such that no sentence of the form " $H(x) > m$ " is provable in the theory unless  $m$  is less than  $M$ .

There are infinitely many true sentences of the form " $H(x) > m$ " with  $m > M$ , and each of them is unprovable in the theory.

## In high complexity a 'general cause' of incompleteness?

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For example, ZFC or Peano Arithmetic trivially prove all sentences of the form " $n + 1 = 1 + n$ ". The  $H$ -complexity of the sentences " $n + 1 = 1 + n$ " grows unbounded with  $n$ .

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### Main Open Problem

Is the Principle valid?

## Related Open Problems

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Which kinds of sentences can be expected to be unprovable?

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- a sentence with a large  $\delta$ -complexity has also a large  $H$ -complexity, but the converse is not true;
- there are only *finitely* many strings with bounded  $H$ -complexity, but *infinitely* many strings with bounded  $\delta$ -complexity.

## Relative complexity of a theorem

The  $\delta$ -complexity of a theorem  $u \in \mathcal{T}$  induced by the Gödel numbering  $g$  (a computable one-to-one function from the set of wff's to binary strings) is defined by:

$$\delta_g(u) = H_2(g(u)) - \lceil \log_2 Q \rceil \cdot |u|_Q.$$

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### Theorem

Let  $\mathcal{T} \subset X_Q^*$  be c.e. and  $g, g'$  be two Gödel numberings. Then, there effectively exists a constant  $c$  (depending upon  $U_2, g$  and  $g'$ ) such that for all  $u \in \mathcal{T}$  we have:

$$|\delta_g(u) - \delta_{g'}(u)| \leq c.$$

## Complexity and incompleteness revisited

### Theorem [Calude-Jürgensen]

Consider a finitely-specified, arithmetically sound (i.e. each arithmetical proven sentence is true), consistent theory strong enough to formalize arithmetic, and denote by  $\mathcal{T}$  its set of theorems written in the alphabet  $X_Q$ . Let  $g$  be a Gödel numbering for  $\mathcal{T}$ . Then, there exists a constant  $N$ , which depends upon  $U_Q$ ,  $U_2$  and  $\mathcal{T}$ , such that  $\mathcal{T}$  contains no  $x$  with  $\delta_g(x) > N$ .

## Is Incompleteness Wide-Spread?

### Theorem

Consider a consistent, sound, finitely-specified theory strong enough to formalize arithmetic. The probability that a true sentence of length  $n$  is provable in the theory tends to zero when  $n$  tends to infinity, while the probability that a sentence of length  $n$  is true is strictly positive.

## Incompleteness and uncertainty

1

In 1927, four years before Gödel's incompleteness, Heisenberg discovered that the “more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa”.

Are there any connections between uncertainty and incompleteness?

## Incompleteness and uncertainty

## 2

Gödel's hostility to any suggestion regarding possible connections between his incompleteness theorem and physics, particularly, Heisenberg's uncertainty relation, is well-known: J. Wheeler was thrown out of Gödel's office for asking the question "Professor Gödel, what connection do you see between your incompleteness theorem and Heisenberg's uncertainty principle?"

Hawking's view (in 2002) was that "a physical theory is self-referencing, like in Gödel's theorem . . . Theories we have so far are both inconsistent and incomplete" .

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Finally let  $\Delta_s = 2^{-s}$ .

## Incompleteness and uncertainty

## 4

Chaitin's halting probability  $\Omega = 0.\omega_1\omega_2\dots\omega_n\dots$  is **algorithmically random** and this property can be presented as an **uncertainty relation**:

$$\Delta_s \cdot \Delta(\omega_1 \dots \omega_s) \geq 1. \quad (1)$$

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**Theorem [Calude-Stay]**

Heisenberg uncertainty relation implies Gödel's incompleteness.

## Gödel's mathematics

Perhaps it's a surprise for many [mathematicians], that Gödel—a logician—has contributed enormously to the solution of three of the famous Hilbert's unsolved 23 (or 24) problems: the first two and the tenth: Cantor's continuum hypothesis, the consistency of arithmetic and the decision problem for Diophantine equations. His contributions went against Hilbert's expectations. [▶ Gödel](#)

## Gödel's philosophy

- Einstein was as little committed to the “relativity of truth” as his good friend Gödel was committed to the view that mathematics is the result of “the human activity that mathematicians carry on.” They are both as far from seconding Sophist’s “man is measure of all things” as it possible to be. [▶ Gödel](#)

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- Gödel’s “interesting axiom”, Die Welt ist vernünftig, the world is reasonable/rational/sensible/intelligible. [▶ Gödel](#)

## Gödel: varia

- A paranoid person is irrationally rational; not illogic, but by a misguided logic, by logic run wild. [▶ Gödel](#)

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- A paranoid person is irrationally rational; not illogic, but by a misguided logic, by logic run wild. [▶ Gödel](#)
- I don't believe in natural science. [Said to physicist John Bahcall.] [▶ Gödel](#)

## Gödel: obituary

Times obituary

http://www.times.co.uk/Link/Story

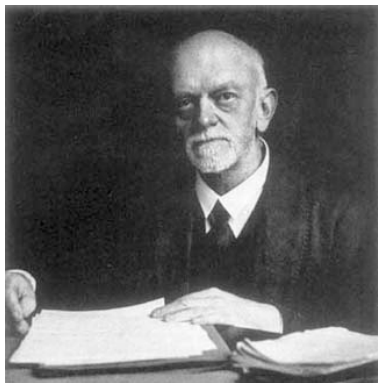
**PROFESSOR KURT GÖDEL**  
**Influential work in**  
**mathematical logic**

Professor Kurt Gödel, the foundations of mathematics was regarded by many as the most important of his mathematical discoveries. In 1931 he proved that any consistent system of arithmetic is incomplete, meaning that there are true statements which cannot be proved within the system. This result, known as Gödel's incompleteness theorem, has had a profound influence on the philosophy of mathematics and on the foundations of computer science. Gödel's work also led to the development of the theory of recursive functions, which is a branch of mathematical logic that deals with the properties of computable functions. Gödel's contributions to logic and mathematics have been widely recognized and he is considered one of the most important mathematicians of the 20th century. He was born in Brno, Czechoslovakia, in 1905 and died in Princeton, New Jersey, in 1982. He was a member of the National Academy of Sciences and the American Academy of Arts and Sciences. Gödel's work has inspired many other mathematicians and logicians, and his ideas continue to influence the field of mathematics today.

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## Hilbert radio address 8 September 1930



Wir müssen wissen, Wir werden wissen!

▶ Hilbert