COMPLEXITY, PROVABILITY, AND INCOMPLETENESS

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Outline

"Either mathematics is too big for the human mind or the human mind is more than a machine".
K. Gödel

GMath

GPhil

GVaria

GObit

Hilbert radio address

On 8 September 1930 in Königsberg, David Hilbert gave a speech entitled Naturerkennen und Logik, which ended with the famous words:

For us there is no *ignorabimus*, and in my opinion none whatever in natural science. In opposition to the foolish *ignorabimus* I offer our answer:

We must know,
We will know.

A four minute excerpt was broadcast by radio:

▸ Wir müssen wissen, Wir werden wissen

The incompleteness theorem

The Incompleteness Theorem as announced on 7 October 1930 in Königsberg at the *First International Conference on the Philosophy of Mathematics*:

Theorem

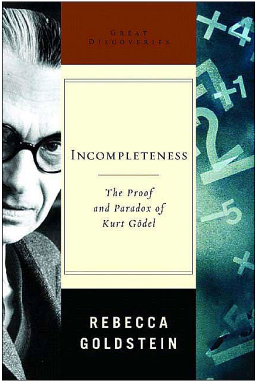
In any formal system adequate for number theory there exists an undecidable formula—that is, a formula that is not provable and whose negation is not provable.

Incompleteness impact

According to Hintikka, with the exception of von Neumann, who immediately grasped Gödel's line of thought and its importance, incompleteness passed un-noticed in Königsberg: even the speaker who summarized the discussion omitted Gödel's result.

Today, incompleteness is considered as "the third leg, together with Heisenberg's uncertainty principle and Einstein's relativity, of that tripod of theoretical cataclysms that have been felt to force disturbances deep down in the foundations of the 'exact sciences.' " ▸ WG

R. Goldstein



There are many books about incompleteness. An important recent book is

R. Goldstein. *Incompleteness. The Proof and Paradox of Kurt Gödel*, Atlas, NY, 2005.

Chaitin's complexity-theoretic proof

Chaitin presented a complexity-theoretic proof of incompleteness which shows that high complexity is a reason of the unprovability of infinitely many (true) sentences.

His proof is based on program-size complexity H: the complexity $H(s)$ of a binary string s is the size, in bits, of the shortest program for a universal self-delimiting Turing machine to calculate s.

Chaitin's complexity-theoretic proof

Chaitin proved the following:

Theorem

For every finitely-specified, sound, consistent theory strong enough to formalize arithmetic, there exists a positive constant M such that no sentence of the form "$H(x) > m$" is provable in the theory unless m is less than$M$.

There are infinitely many true sentences of the form "$H(x) > m$" with $m > M$, and each of them is unprovable in the theory.

A problem

The high H-complexity of the sentences "$H(x) > m$" with $m > M$ is a source of their unprovability.

Is every true sentence s with $H(s) > M$ unprovable by the theory?

Unfortunately, the answer is negative: only finitely many sentences s have complexity $H(s) < M$, in contrast with the fact that the set of all theorems of the theory is infinite.

For example, ZFC or Peano Arithmetic trivially prove all sentences of the form "$n + 1 = 1 + n$". The H-complexity of the sentences "$n + 1 = 1 + n$" grows unbounded with n.

Chaitin'heuristic principle'

This result suggested him the following "heuristic principle", an information-preservation principle:

Principle

The theorems of a finitely specified theory cannot be significantly more complex than the theory itself.

Main Open Problem

Is the Principle valid?

Related Open Problems

Open Problem

Is independence is a widespread phenomenon?

Open Problem

Which kinds of sentences can be expected to be unprovable.

A new complexity measure

The δ-complexity is defined by

$$\delta_Q(x) = H_Q(x) - |x|_Q.$$

The H-complexity of the sentences "$n + 1 = 1 + n$" grows unbounded with n, but the "intuitive complexity" of the sentences "$n + 1 = 1 + n$" remains bounded; this intuition is confirmed by δ-complexity. In fact:

- a sentence with a large δ-complexity has also a large H-complexity, but the converse is not true;
- there are only *finitely* many strings with bounded H-complexity, but *infinitely* many strings with bounded δ-complexity.

Relative complexity of a theorem

The δ-complexity of a theorem $u \in \mathcal{T}$ induced by the Gödel numbering g is defined by:

$$\delta_g(u) = H_2(g(u)) - \lceil \log_2 Q \rceil \cdot |u|_Q .$$

Theorem

Let $\mathcal{T} \subset X_Q^*$ be c.e. and g, g' be two Gödel numberings. Then, there effectively exists a constant c (depending upon U_2, g and g') such that for all $u \in \mathcal{T}$ we have:

$$|\delta_g(u) - \delta_{g'}(u)| \leq c .$$

Complexity and incompleteness revisited

Theorem [Calude-Jürgensen]

Consider a finitely-specified, arithmetically sound (i.e. each arithmetical proven sentence is true), consistent theory strong enough to formalize arithmetic, and denote by $\mathcal{T}$ its set of theorems written in the alphabet X_Q. Let g be a Gödel numbering for $\mathcal{T}$. Then, there exists a constant N, which depends upon U_Q, U_2 and $\mathcal{T}$, such that $\mathcal{T}$ contains no x with $\delta_g(x) > N$.

Complexity and incompleteness revisited

Corollary [Chaitin]

Every finitely-specified, arithmetically sound, consistent theory strong enough to formalize arithmetic can prove only, for finitely many of its theorems, that they have high δ-complexity.

Is Incompleteness Wide-Spread?

Theorem

Consider a consistent, sound, finitely-specified theory strong enough to formalize arithmetic. The probability that a true sentence of length n is provable in the theory tends to zero when n tends to infinity, while the probability that a sentence of length n is true is strictly positive.

▸ incomsimul

Open Problems

- Calculate the δ-complexity of some concrete independent sentences, like the sentence expressing the consistency of the theory itself.
- Find other (more interesting?) measures of complexity satisfying Chaitin's "heuristic principle".
- Sentences expressed by strings with large δ-complexity are unprovable. Is the converse implication true? In other words, given a theory as above, are there independent sentences x with low δ-complexity? Even if such sentences do exist, the probability that a true sentence of length n with δ-complexity less than or equal to N is unprovable in the theory tends to zero when n tends to infinity.

Open Problems

- Check the validity of the following stronger result: under the same conditions, the probability that a sentence of length n, expressible in the language of the theory, is provable in the theory tends to zero when n tends to infinity.

Summary

- We proved Chaitin's heuristic principle, 'the theorems of a finitely- specified theory cannot be significantly more complex than the theory itself' for an appropriate measure of complexity.
- Previous results showing that incompleteness is not accidental, but ubiquitous are here reinforced in probabilistic terms: the probability that a true sentence of length n is provable in the theory tends to zero when n tends to infinity, while the probability that a sentence of length n is true is strictly positive.

Further Reading

- C S. Calude. *Information and Randomness*, 2nd Edition, Revised and Extended, Springer-Verlag, Berlin, 2002.
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- J. Hintikka. *On Gödel*, Wadsworth, 2000.
- D. Hofstadter. *Gödel, Escher, Bach*, Basic Books, 1979.
- E. Nagel, J. Newman. *Gödel's Proof*, New York University Press, 1958.
- H. Wang. *A Logical Journey: From Gödel to Philosophy*, MIT Press, 1996.

Appendix Outline

Gödel's mathematics

Perhaps it's a surprise for many [mathematicians], that Gödel—a logician—has contributed enormously to the solution of three of the famous Hilbert's unsolved 23 (or 24) problems: the first two and the tenth: Cantor's continuum hypothesis, the consistency of arithmetic and the decision problem for Diophantine equations. His contributions went against Hilbert's expectations. Gödel

Gödel's philosophy

- Einstein was as little committed to the "relativity of truth" as his good friend Gödel was committed to the view that mathematics is the result of "the human activity that mathematicians carry on." They are both as far from seconding Sophist's "man is measure of all things" as it possible to be. Gödel
- Gödel's "interesting axiom", Die Welt ist vernünftig, the world is reasonable/rational/sensible/intelligible. Gödel

Wittgenstein on Gödel

Wittgenstein dismissed Gödel's theorems as "logische Kunststücken," logical conjuring tricks. Proposition 7, "Where we cannot speak we must remain silent" was once commented (by an exasperated mathematician) as "accomplishing the difficult task of being at once portentous and vacuous".
Still, it's interesting to compare Gödel's with Wittgenstein's incompleteness (according to which our linguistic systems cannot exhaust all that there is to non-mathematical reality, we cannot speak the unspeakable truths, but they exist). impact

Gödel: varia

- A paranoid person is irrationally rational; not illogic, but by a misguided logic, by logic run wild. ▸ Gödel
- I don't believe in natural science. [Said to physicist John Bahcall.] ▸ Gödel

Gödel: obituary

PROFESSOR KURT GÖDEL

Influential work in mathematical logic

Professor Kurt Gödel, the man regarded by common consent as the most influential mathematical logician of the century died at the age of 71 in Princeton, New Jersey, on January 14. Born in Brünn, Czechoslovakia, on April 28, 1906, Kurt Gödel was Privat-Dozent at the University of Vienna from 1933 to 1938. Having visited the Institute for Advanced Study several times, he took up residence in Princeton in 1938 becoming a Permanent Member of the Institute in 1946 and Professor in 1953. Besides honorary doctorates from several universities, he received the Einstein Award in 1951 and the National Medal of Science in 1974. Professor Gödel was a Member of the United States National Academy of Sciences, Fellow of the American Academy of Arts and Sciences, Foreign Member of the Royal Society, Corresponding Member of the Institut de France, a Corresponding Fellow of the British Academy, and Honorary Member of the London Mathematical Society.

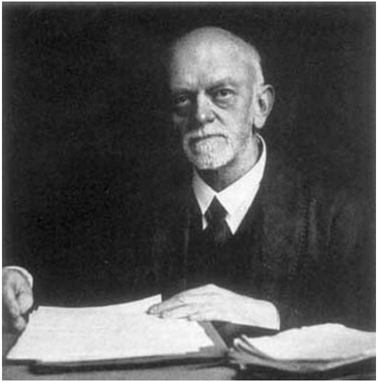
The fame of Gödel rests on three outstanding results obtained during the decade of the 1930's. The first is his completeness proof for the first-order functional calculus; this was his doctoral dissertation and was published in 1930. Scarcely a year later he published his most celebrated result: the incompleteness thorem for various axiomatic systems. He showed in particular that the vast structure of the *Principia Mathematica* of Whitehead and Russell was inadequate for deciding all mathematical questions; indeed, the system could not even prove its own formal consistency. This inadequacy is inherent in any reasonably strong system which is effectively axiomatized, and so Gödel's Theorem changed the whole philosophical view of the foundations of mathematics; the repercussions of this unexpected discovery are felt and debated today.

On the positive side his techniques led directly to a new concept of effectively calculable function which had major influence on the development of computers and is still central in theoretical studies in computer science. However, Gödel himself held a very Platonist view of mathematical objects and higher infinities, and his next main achievement in 1938 was a reassuring one for his beliefs. He showed that if the system of *Principia Mathematica* (or even a certain stronger system) is consistent, then it remains so upon the addition of the Axiom of Choice and the Generalized Continuum Hypothesis, principles of prime importance in the arithmetic of infinite cardinal numbers. It was not until 1963 that Paul J. Cohen finally proved the independence of those axioms by a new idea but building on Gödel's work. Gödel's original methods for the consistency problem have recently had new applications reflecting on apparently quite unrelated mathematical problems, so it seems clear that the fruitfulness of his ideas will continue to stimulate new work. Few mathematicians are granted this kind of immortality.

A slight person and very fastidious, Gödel was generally worried about his health and did not travel or lecture widely in later years. He had no doctoral students, but through correspondence and personal contact with the constant succession of visitors to Princeton, many people benefited from his extremely quick and incisive mind. Friend to Einstein, von Neumann and Morgenstern he particularly enjoyed philosophical discussion. His widow, Adele; whom he married in 1938, survives him; there were no children.

Gödel

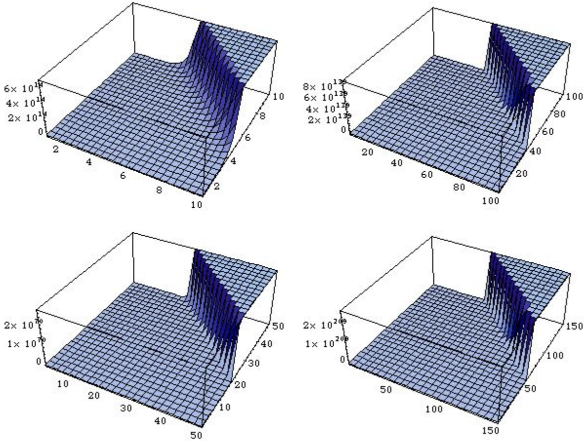
Hilbert radio address 8 September 1930



Wir müssen wissen,Wir werden wissen!

Hilbert

Graphical illustration



WS

Abstract

In this talk we report some results, jointly obtained with H. Jürgensen, regarding Chaitin's heuristic principle: 'the theorems of a finitely- specified theory cannot be significantly more complex than the theory itself'. We show that this principle is valid for an appropriate measure of complexity. We show that the measure is invariant under the change of the Gödel numbering. For this measure, the theorems of a finitely-specified, sound, consistent theory strong enough to formalize arithmetic which is arithmetically sound (like Zermelo-Fraenkel set theory with choice or Peano Arithmetic) have bounded complexity, hence every sentence of the theory which is significantly more complex than the theory is unprovable. Previous results showing that incompleteness is not accidental, but ubiquitous are here reinforced in probabilistic terms: the probability that a true sentence of length n is provable in the theory tends to zero when n tends to infinity, while the probability that a sentence of length n is true is strictly positive. The talk will conclude with a few open problems.