Generalisations of disjunctiv sequences

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Disjunctive ω-words

An ω-word $\xi \in \{0, \ldots, r-1\}^{\omega}$ is called *disjunctive* o
and only if it has every word $w \in \{0, \ldots, r-1\}^{*}$ (inf
often) as an infix.

$$D_r = \bigcap_{w \in X_r^*} X_r^* \cdot w \cdot X_r^{\omega}$$

is the set of all disjunctive ω-words in X_r^{ω}.

Example. Champernowne's number ($r = 10$):

$$0.12345678910111213\ldots$$

Properties of ω-words $\xi \in \{0, \ldots, r-1\}^{\omega}$ and nu

ω-Words	Real Numbers	Remar
random	random	
$\underline{\kappa}(\xi) = 1$	$\underline{\kappa}(0.\xi) = 1$	
$\kappa(\xi) = 1$	$\kappa(0.\xi) = 1$	
Borel normal	r-Borel normal	base depende
		Cassels '59, Sc
disjunctive, rich	r-disjunctive	base depende
		Cassels '59, Sc
		Hertling '96

Properties of D_r

1. $D_r = \{\xi : \xi \in X^\omega \wedge |\mathbf{pref}(\xi) \cap W| = \aleph_0\}$

   for $W := \{wx : \exists n(\mathbf{infix}(wx) \supseteq X_r^n \wedge \mathbf{infix}(w) \not\supseteq X_r^{$

2. D_r is Π_2-definable and residual in X_r^ω.

3. $\mu(D_r) = 1$ for all non-atomic product measures

4. $X_r^\omega \setminus D_r = \bigcup_{w \in X_r^*} \left(X_r^\omega \setminus X_r^* \cdot w \cdot X_r^\omega\right) = \bigcup_{w \in X_r^*} (X_r^{|w|} \setminus \{$

5. $X_r^\omega \setminus D_r$ is σ-porous.

6. $X_r^\omega \setminus D_r$ is the union of all nowhere dense ω-lan definable by finite-automata

Cantor expansion

Let

$$f(1), f(2), \ldots, f(n), \ldots$$

be a fixed infinite sequence of positive integers gre
than 1. Using a point we form the finite or infinite

$$0.x_1 x_2 \ldots x_i \ldots$$

such that $x_n \in \mathbb{N},\ 0 \leq x_n < f(n)$, for all $n \geq 1$.

The real number

$$\alpha := \sum_{i=1}^{n} \frac{x_i}{f(1) \cdot f(2) \cdots f(i)}$$

has $0.x_1 x_2 \ldots$ as (one of) its *Cantor expansion(s)*.

Spaces $X^{(f)}$ as subspaces of the Baire sp

$$X^{(f)} := \prod_{i=1}^{\infty} X_{f(i)}$$

Metric in $X^{(f)}$

$$\begin{aligned} \rho_f(\xi,\eta) &:= \inf\Big\{\prod_{i=1}^{|w|} f(i)^{-1} : w \sqsubset \xi \wedge w \sqsubset \\ \mathrm{diam}_f(F) &:= \sup\{\rho_f(\xi,\eta) : \xi,\eta \in F\} \end{aligned}$$

Theorem 1 *$(X^{(f)}, \rho_f)$ is a compact metric space.*

ω-words disjunctive in a subset F of the Baire s

Requirement:

An ω-word $\xi \in F$ should be called *disjunctive* if an
every word $w \in \mathbb{N}^*$ which can appear infinitely oft
infix in F has to appear infinitely often as an infix

Question:

When can a word w appear infinitely often as an i

$$\mathbf{Infix}_\infty(F) := \{w : \exists^\infty n \exists u(|u| = n \wedge uw \in \mathbf{pref}($$

An ω-word ξ is called F-*disjunctive* provided $\xi \in F$

$$\mathbf{Infix}_\infty(F) = \mathbf{Infix}_\infty(\xi) .$$

A set $W \subseteq \mathbb{N}^*$ is referred to as *left prolongable* if ar
for every $w \in W$ there is a $x \in \mathbb{N}$ such that $x \cdot w \in W$

Proposition 2 *If* $\mathbf{Infix}_\infty(F)$ *is left prolongable,*
$\mathbf{infix}(\xi) \supseteq \mathbf{Infix}_\infty(F)$ *implies* $\mathbf{Infix}_\infty(\xi) \supseteq \mathbf{Infix}_\infty(F)$ *f*
$\xi \in F$.

Corollary 3 *If* $\mathbf{Infix}_\infty(F)$ *is left prolongable, then*

$$D_F = \bigcap_{w \in \mathbf{Infix}_\infty(F)} (F \cap \mathbb{N}^* \cdot w \cdot \mathbb{N}^\omega)$$

is the set of all F*-disjunctive* ω*-words.*

Corollary 4 *If* $\mathbf{Infix}_\infty(F)$ *is left prolongable then*

$$F \setminus D_F = \bigcup_{w \in \mathbf{Infix}_\infty(F)} (F \setminus \mathbb{N}^* \cdot w \cdot \mathbb{N}^\omega)$$

$$= \bigcup_{w \in \mathbf{Infix}_\infty(F)} \Big(F \cap \bigcap_{j=0}^{|w|-1} \mathbb{N}^j \cdot (\mathbb{N}^{|w|} \setminus \{w$$

Observation:

$\mathbf{Infix}_\infty(X^{(f)})$ is left prolongable, for every function $f : \mathbb{N} \to \mathbb{N} \setminus \{0,1\}$.

Properties of $\mathbf{Infix}_\infty(X^{(f)})$
general

Lemma 5 $\{0,1\}^* \cdot i \cdot \{0,1\}^* \subseteq \mathbf{Infix}_\infty(X^{(f)})$ *iff* $f(n) \geq$

Example. For $f(i) := \begin{cases} i+1, \text{ if } i = n^2 \text{ , and} \\ 2 \quad\quad \text{, otherwise.} \end{cases}$ we h

$\mathbf{Infix}_\infty(X^{(f)}) = \{0,1\}^* \cdot \mathbb{N} \cdot \{0,1\}^*$

Lemma 6 *If* $\lim_{n\to\infty} f(n) = \infty$ *then* $\mathbf{Infix}_\infty(X^{(f)}) = \mathbb{N}^*$.

Properties of $\mathbf{Infix}_\infty(X^{(f)})$

f recursive

Lemma 7 *If f is a recursive function then* $\mathbf{Infix}_\infty(X$

Example. $M \subseteq \mathbb{N}$, $M \in \Pi_2$ has a representation $M := \{n : \exists^\infty m((m,n) \in R)\}$ where $R \subseteq \mathbb{N} \times \mathbb{N}$ and $R \in \Sigma_1 \cap \Pi_1$.

Then

$$X^{(f)} := \prod_{(m,n)\in R} X_2^{m+1} \cdot X_4 \cdot X_3^n \cdot X_4$$

and

$$\mathbf{Infix}_\infty(X^{(f)}) \cap 3 \cdot 2^* \cdot 3 = \{32^n 3 : n \in M\} \ .$$

Properties of $X^{(f)} \setminus \mathbb{N}^* \cdot w \cdot \mathbb{N}^\omega$

Lemma 8 *The set* $X^{(f)} \setminus \mathbb{N}^* \cdot w \cdot \mathbb{N}^\omega$ *is nowhere de*
$(X^{(f)}, \rho_f)$ *whenever* $w \in \mathbf{Infix}_\infty(X^{(f)})$.

Let μ be the measure on $X^{(f)}$ induced by the Lebes
measure on $[0,1]$.

Example. Let $f(i) := (i+1)^2$.

Then in view of $\prod_{i=1}^{\infty}(1 - \frac{1}{f(i)}) = \prod_{j=2}^{\infty}(1 - \frac{1}{j^2}) > 0$ t

$E = X^{(f)} \setminus \mathbb{N}^* \cdot 0 \cdot \mathbb{N}^\omega$ is nowhere dense but has me
$\mu(E) > 0$, and contains a random sequence.

Porosity in $X^{(f)}$

$\lambda(E,u) := \sup\{\mathrm{diam}_f(w \cdot \mathbb{N}^\omega \cap X^{(f)}) : u \sqsubseteq w \wedge w \cdot \mathbb{N}^\omega$
is the diameter of a largest ball contained $u \cdot \mathbb{N}^\omega \cap$
disjoint to E, and

$$\mathbf{p}(E,\xi) := \limsup_{u \to \xi} \frac{\lambda(E,u)}{\mathrm{diam}_f(u \cdot \mathbb{N}^\omega \cap X^{(f)})}$$

is the *porosity* of E at the point ξ.

E is *porous* iff $\mathbf{p}(E,\xi) > 0$ for all $\xi \in E$.

Lemma 9 *If $f : \mathbb{N} \to \mathbb{N}$ is bounded then for every $w \in \mathbf{Infix}_\infty(X^{(f)})$ the set $X^{(f)} \setminus \mathbb{N}^* \cdot w \cdot \mathbb{N}^\omega$ is porous*

Porosity in $X^{(f)}$ where f is unboun

For unbounded functions $f : \mathbb{N} \to \mathbb{N}$ we introduce
following number

$$k_f := \begin{cases} -1 & \text{, if all } f^{-1}(k) \text{ are f} \\ \sup\{k : f^{-1}(k) \text{ is infinite}\}, \text{ otherwise.} \end{cases}$$

Theorem 10 *If $f : \mathbb{N} \to \mathbb{N}$ is unbounded then for e*
and $w \in \{0,1\}^*$ the set $X^{(f)} \setminus \mathbb{N}^* \cdot w \cdot i \cdot \mathbb{N}^\omega$ is not po
in $X^{(f)}$.

Corollary 11 *If $\lim_{n\to\infty} f(n) = \infty$ then $X^{(f)} \setminus \mathbb{N}^* \cdot w \cdot \mathbb{N}$*
porous in $X^{(f)}$ for any $w \in \mathbf{Infix}_\infty(X^{(f)})$.