

Paradoxes of Voting
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The Scenario

- A** A set of alternatives that you have to choose from (examples: political parties, candidates for a committee, social projects, etc.).
- V** A set of voters (each voter has an individual **ranking** of the alternatives, i.e. first best, second best, third best, etc.).

How will people in V choose from the alternatives in A ?

Answer: They vote.

Question: But how?

- When there are two alternatives they use the **majority rule**: the alternative with the majority of votes (i.e. $> 50\%$) wins.
- What about the case when there are more alternatives?

Voting Methods

- A Pairwise voting:** majority voting between pairs of alternatives in a given order
- **Condorcet method:** Condorcet winner beats everything else in majority voting
 - **Amendment procedure** (when there is a status-quo alternative): first, vote between a and b (a, b two new proposals), then vote between the winner and c (c status-quo)
- B Plurative methods:** voting on all alternatives at once.
- **Plurality rule:** the alternative with the most number of votes wins
 - **Borda count:** each voter ranks alternatives; points are assigned; add up points, highest wins
 - **Approval voting:** each voter chooses alternatives; the alternative with the highest approval votes wins

C **Mixed methods:** mixtures of the previous types

- **Majority runoff:** each voter chooses one alternative; if the alternative is the majority winner, it wins; otherwise, majority voting between the first and the second
- **Voting in rounds:** use a single vote or ranking in each round; at the end of each round, eliminate the worst-performing alternative
- **Proportional representation:** the chosen set must mirror the voters' votes
- **Single transferable vote:** voters declare ranking; bottom alternatives eliminated and their votes are transferred

We can choose any of these rules
for your society . . .

- the outcome will depend on the voting procedure used
- one can choose strategically
- one can manipulate
- can we guarantee “rationality”?

Denote by \succ the binary relation:

$a \succ b$ means a is ranked **higher** than b .

Voting Paradoxes

Condorcet Paradox

What is the social ranking between alternatives $G =$ generous, $A =$ average and $L =$ limited?

LEFT	CENTER	RIGHT
G	A	L
A	L	G
L	G	A

Table 1: Preferences over welfare policies

Note that:

1. $G \succ A \succ L \succ G$ (with the majority voting)
2. an intransitive ranking (each voter has transitive ranking)

Who is the winner?

Voting Paradoxes

Borda Paradox

Seven sportswriters trying to choose among soccer players A, B, C, D have expressed the following preferences:

1	2	3	4	5	6	7	ranking
D	B	D	A	B	B	A	4
A	D	A	C	D	D	C	3
C	A	C	B	A	A	B	2
B	C	B	D	C	C	D	1

Table 2: Seven preferences, four candidates

Applying the Borda rule we get:

A	20 points: winner
B	19 points
C	19 points
D	13 points

Table 3: Borda hierarchy

Unfortunately, they discovered that D cannot be a candidate ... Should this affect who wins?

1	2	3	4	5	6	7	ranking
D	B	D	A	B	B	A	3
A	D	A	B	D	D	B	2
B	A	B	D	A	A	D	1

Table 4: Seven preferences, three candidates

Applying the Borda rule we get:

B	15 points: new winner
D	14 points
A	13 points

Table 5: Borda hierarchy

Voting Paradoxes

Agenda Paradox

The chair decides the order of voting (sets the agenda).

Consider the preferences

LEFT	CENTER	RIGHT
G	A	L
A	L	G
L	G	A

Table 6: Preferences over welfare policies revisited

According to the majority voting, $G \succ A \succ L \succ G$.

However, if the order of voting is L, A, G, then

- from $A \succ L$, A wins
- from $G \succ A$, G wins

so G **wins**. Any outcome can be obtained ...

Changing the Voting Method

Assume that 100 voters vote:

- 40 voters prefer $\mathbf{A} \succ \mathbf{B} \succ \mathbf{C}$
- 25 voters prefer $\mathbf{B} \succ \mathbf{C} \succ \mathbf{A}$
- 35 voters prefer $\mathbf{C} \succ \mathbf{B} \succ \mathbf{A}$

so *strategically* choose the voting method:

- Plurality rule: \mathbf{A} wins
- Borda rule: \mathbf{B} wins (\mathbf{B} : 225 points, \mathbf{C} : 195 points, \mathbf{A} : 180 points)
- Majority runoff: \mathbf{C} wins (\mathbf{A} and \mathbf{C} move to second round)

Rational Aggregation?

Suppose we have a finite set V of voters who vote for a finite set of candidates C . A *voting system* is a function which takes as input the voting preferences of each voter in V and returns as output a ranking of the candidates.

Here are some natural conditions on the voting system:

Rationality: The output of the voting system is a total ordering.

Determinism: The output of the voting system is determined only by the ranking preferences of the voters and on no other factors (in particular, no randomization is allowed).

Consensus: If all voters prefer A to B , then the output of the voting system also prefers A to B .

Impartiality: All candidates are treated equally, i.e., the voting system is invariant under permutations of candidates.

Independence: The relative ranking of X and Y in the output of a voting system is independent of the voters preferences for any third candidate Z .

No dictators: If a single voter prefers X to Y , but all other voters disagree, then voting system should override the wishes of that single voter and rank Y higher than X .

Can we construct a voting system satisfying all the above conditions?

Arrow's Theorem (1950)

If there are at least three candidates, then the above six conditions are inconsistent.

If there are at least three candidates, then the any voting system satisfying the first five conditions above is dictatorial.

Ways Around?

- Sacrificing independence: third parties can “spoil” the results of an election even they have no realistic chance of winning. Borda violates independence.
- Restricting the freedom of individuals by structuring their preferences in some way; for example, allowing limited veto power.
- Adopting a *random dictatorship system* in which a single voter is chosen by lottery to rank the candidates (one of the fairest, but impossible to implement systems).

An Example

Assume that a group of five voters A, B, C, D, E vote for one of their members to be the chairperson. Their preferences are:

A prefers $A \succ B \succ C \succ D \succ E$,

B prefers $B \succ C \succ D \succ E \succ A$,

C prefers $C \succ D \succ E \succ A \succ B$,

D prefers $D \succ E \succ A \succ B \succ C$,

E prefers $E \succ A \succ B \succ C \succ D$.

Four out of five voters prefer A to B . Thus, to obey the no dictators axiom, the voting system output must rank A above B . But a similar argument forces the voting system to rank B above C , C above D , D above E and E above A !

A Metric Method

- a) The voters v_1, \dots, v_n grade the objects o_1, \dots, o_m .
- b) We construct the “ideal” object o_{ideal} , which has the “best” grades.
- c) The “distance” between each o_i and o_{ideal} induces the final hierarchy.

A possible metric is the following: assume (L, \cup, \cap, \neg) is a complemented lattice and grades are taken from a set $X \subset L$, which is closed under the operation of negation (\neg).

For example, take $X = L$ where $L = \{0, 1, \dots, n\}$,
 $L = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$, or $L = [0, 1]$.

Then, $d(x, y) = x \oplus y = (x \cap \neg y) \cup (\neg x \cap y)$, if $x \neq y$ and $d(x, x) = 0$ is such a metric.

Gibbard-Satterthwaite Theorem (1973)

We have a set of n voters V voting for a set of candidates C . Denote by $L(C)$ the set of all linear orderings of C . So, each voter v_i votes for some ordering l_i of C . The voting system S aggregates the orders (l_1, \dots, l_n) into an linear order $l = S(l_1, \dots, l_n)$.

- the voter v_i **votes tactically** in a profile (L_1, \dots, L_n) if $S(L_1, \dots, L_{i-1}, L_i, L_{i+1}, \dots, L_n) \succ_i S(L_1, \dots, L_{i-1}, l_i, L_{i+1}, \dots, L_n)$ (the voter v_i “improves” – in the sense of his true preferences, by tactically voting L_i rather than honestly voting l_i),
- A voting system is **strategy-proof** if for every profile L , no voter votes tactically; that is, there is no possible situation in which a voter, knowing the votes cast by everyone else, benefits by doing other than declaring his true preferences.

In an election with three or more outcomes, a voting rule is strategy-proof if and only if it is dictatorial.