

Natural Complexity, Partial Randomness and Zeta Functions

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Outline

Halting Probabilities

Natural complexity

Partial randomness

Asymptotic randomness

Open problems

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Natural halting probabilities

1

Chaitin's halting probability of a self-delimiting Turing machine C

$$\Omega_{\text{dom}(C)} = \sum_{p \in \text{dom}(C)} 2^{-|p|},$$

allows an apparent “ambiguity” as strings with the same length contribute equally towards the halting probability. This motivates us to introduce a slightly different “halting probability” in which different strings in the domain of the machine have different contributions to the “halting probability”.

Natural halting probabilities

2

Let $\mathbf{N} = \{1, 2, \dots\}$ and $\text{bin}(n)$ be the binary expansion of $n \geq 1$ without the leading 1. If $A \subset \Sigma^*$, then we define

$$\Upsilon[A] = \{n \in \mathbf{N} \mid \text{bin}(n) \in A\}.$$

Definition

The *natural halting probability* ζ_M of a Turing machine M is

$$\zeta_M = \sum_{n \in \Upsilon[\text{dom}(M)]} \frac{1}{n}.$$

Natural halting probabilities

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A classification of Turing machines

According to the *natural halting probability*, Turing machines can be classified into the following three classes:

- ▶ *natural divergent Turing machines*: those machines M for which $\zeta_M = \infty$,
- ▶ *natural convergent Turing machines*: those machines M for which $\zeta_M < \infty$,
- ▶ *tutara machines*: those machines M for which $\zeta_M \leq 1$.

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▶ Tuatara picture

▶ Universality

Natural halting probabilities

3

Fact

Every self-delimiting Turing machine is a tuatara machine. More precisely, for every self-delimiting Turing machine C , ζ_C is c.e. and

$$1 \geq \Omega_C \geq \zeta_C \geq \Omega_C/2 \geq 0.$$

Corollary

For every universal self-delimiting Turing machine U ,

$$1 > \Omega_U > \zeta_U > \Omega_U/2 > 0.$$

Theorem

The natural halting probability ζ_U of a universal self-delimiting Turing machine U is random.

Natural halting probabilities

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Natural halting probabilities

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The classification of Turing machines in terms of natural probabilities can be paralleled with a classification in terms of Omega: *Omega divergent Turing machines*, *Omega convergent Turing machines*, *Omega Turing machines*.

Fact

a) For every Turing machine M , $\zeta_M < \infty$ iff $\Omega_M < \infty$, hence the classes of natural divergent (convergent) Turing machines coincide.

b) If $\Omega_M \leq 1$, then $\zeta_M \leq 1$, but there exists a tuatara machine T such that $\Omega_T > 1$, hence the class of Omega Turing machines is strictly included in the class of tuatara machines.

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Natural halting probabilities

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Theorem

For each tuatara machine V there effectively exists a self-delimiting Turing machine C such that $\Omega_C = \zeta_V$. If V is tuatara universal, then C can be taken to be a universal self-delimiting Turing machine.

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For each self-delimiting Turing machine C there is a tuatara machine V such that $\zeta_V = \Omega_C$.

Open question

Assume U is a universal self-delimiting Turing machine. Does there exist a tuatara universal machine W such that $\zeta_W = \Omega_U$?

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Halting Probabilities

Natural complexity

Partial randomness

Asymptotic randomness

Open problems

Natural complexity

1

Definition

The *natural complexity* of the string $x \in \Sigma^*$ (with respect to the tuatara machine V) is $\nabla_V(x) = \min\{n \geq 1 \mid V(\text{bin}(n)) = x\}$.

Fact

- ▶ A tuatara machine W is universal iff for every tuatara machine V there exists a constant ε (depending upon W and V) such that $\nabla_W(x) \leq \varepsilon \cdot \nabla_V(x)$, for all strings $x \in \Sigma^*$.
- ▶ The real $\alpha \in (0, 1)$ is random iff there exists $\varepsilon > 0$ such that for all $n \geq 1$, $2^{-n} \cdot \nabla_W(\alpha[n]) \geq \varepsilon$.
- ▶ The natural halting probability ζ_W of a universal tuatara machine W is random.

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Partial randomness

1

Chaitin found that the number of characters required in a program to produce the first n bits of LISP's halting probability was asymptotic to n / \log_2 (number of characters).

A more general approach was followed by Staiger, Tadaki, Calude, Terwijn and Staiger, Reimann and Stephan; they studied the degree of randomness of sequences or reals by measuring their “degree of compression”.

Downey, Hirschfeldt, Miller and Nies studied relativisations of the halting probability.

Partial randomness

2

Following Tadaki, for every $s > 0$ and universal self-delimiting Turing machine U we define the “Omega function of U ”

$$\Omega_U(s) = \sum_{p \in \text{dom}(U)} 2^{-s|p|}.$$

Theorem (Tadaki)

If $0 < s < 1$, then $\Omega_U(s) = \infty$.

For every computable $s \geq 1$, there exists a constant $c > 0$ such that for all $m \geq 1$,

$$H_U(\Omega_U(s)[m]) \geq m/s - c.$$

Partial randomness

3

Definition

Let $\alpha \in (0, 1)$.

a) (Tadaki, Calude-Staiger-Terwijn) Let ε be a computable real, U a universal self-delimiting Turing machine.

- ▶ α is *Chaitin ε -random* if there is a constant c such that for each $n \geq 1$, $H_U(\alpha[n]) \geq \varepsilon \cdot n - c$,
- ▶ α is *Chaitin partially random* if it is Chaitin ε -random for some computable real $\varepsilon \in (0, 1)$.

b) Let T be a universal Turing machine and $s > 0$ be a computable real. The real α is $1/s$ - *K-random* if there exists a constant $c > 0$ such that for every $m \geq 1$: $K_T(\alpha[m]) \geq m/s - c$.

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Partial randomness

4

We give some concrete examples.

- ▶ Let T be a universal Turing machine and define $M(xx) = T(x)$, for every string x . The natural halting probability of M is $1/2$ – K -random.
- ▶ The natural halting probability of Barker's language iota is at least Chaitin $1/193$ -random.

Partial randomness

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- ▶ Let T be a universal Turing machine and define $M(xx) = T(x)$, for every string x . The natural halting probability of M is $1/2$ – K -random.
- ▶ The natural halting probability of Barker's language *lota* is at least Chaitin $1/193$ –random.

▶ *lota*

Partial randomness

5

An early result in algorithmic information theory states

Theorem (Chaitin, Martin-Löf)

For universal Turing machine T and every $c > 0$ the set

$$\{\alpha \in (0, 1) \mid K_T(\alpha[m]) \geq m - c, \text{ for all } m \geq 1\} = \emptyset.$$

Question: Given a computable $s > 1$, does there exist a $1/s$ - K -random real α ?

Answer: Random reals satisfy the condition, but are there other examples?

Partial randomness

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Partial randomness

6

Given a Turing machine M (which may or may not be self-delimiting), we define the halting function $\kappa_M(p) = 1$ if p halts on M , 0 otherwise. Then we define the “halting probability of M at s ” to be

$$\begin{aligned} \langle \kappa_M \rangle(s) &= \left(\sum_{p \in \text{dom}(M)} 2^{-s|p|} \right) / \left(\sum_{q \in \mathbf{N}} 2^{-s|\text{bin}(q)|} \right) \\ &= (1 - 2^{-s+1}) \cdot \Omega_M(s). \end{aligned}$$

Partial randomness

7

Theorem

For every computable real $s > 1$ and universal T , $\langle \kappa_T \rangle(s)$ is $1/s - K$ -random.

[▶ An example](#)[▶ Comment](#)

Partial randomness

8

Obviously, if α is $1/s - K$ -random, then it is also Chaitin $1/s$ -random.

The converse implication is false:

Theorem

There exists a Chaitin $1/2$ -random real which is not $1/2 - K$ -random.

Partial randomness

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The lower asymptotic complexity is

$$\underline{k}(\alpha) = \liminf_{n \rightarrow \infty} \frac{1}{n} K(\alpha_1 \dots \alpha_n).$$

Theorem (Staiger)

Let $s \geq 1$ be computable. Then, for a real α , the following statements are equivalent:

- 1) We have: $\underline{k}(\alpha) \geq 1/s$.
- 2) For every computable real $t > s > 1$, α is $1/t - K$ -random.
- 3) For every computable real $t > s > 1$, α is Chaitin $1/t$ -random.

Asymptotic randomness

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- 3) For every computable real $t > s > 1$, α is Chaitin $1/t$ -random.

Asymptotic randomness

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Definition

Let $s > 1$ be computable. We say that a real number $\alpha \in (0, 1)$ is **asymptotically $1/s$ -random** if one of the equivalent conditions in the above theorem is satisfied. If $s = 1$, then α is called **asymptotically random**.

Theorem

The notion of asymptotic $1/s$ -randomness induces a strict hierarchy on $s > 1$.

Example

The zeta number of Iota is at least $1/194$ – K -random and at least asymptotically $1/193$ -random.

Asymptotic randomness

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Asymptotic randomness

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Given an arbitrary Turing machine M , we define the “zeta function of M ” to be

$$\zeta_M(s) = \sum_{q \in \Upsilon[\text{dom}(M)]} q^{-s},$$

and the “natural halting probability of M at s ” to be

$$\langle \kappa_M^n \rangle(s) = \zeta_M(s) / \zeta(s),$$

where $\zeta(s)$ is Riemann’s zeta function.

Asymptotic randomness

4

Define the set

$$P = \{p_i \mid \text{bin}(i) \in \text{dom}(M)\},$$

where p_i is the i th prime in increasing order, and the set

$$S = \{n \mid \text{all prime factors of } n \text{ are in } P\}.$$

The set $\text{bin}(S)$ is the domain of a Turing machine $R(M)$ (*prime product machine*) that performs the following steps on $x \in \Sigma^*$:

1. Compute $n = \text{bin}^{-1}(x)$.
2. Compute the prime factors p_i of n .
3. For each p_i , simulate $M(\text{bin}(i))$.
4. Output the empty string.

Then,

$$\zeta_{R(M)}(s) = \sum_{n \in S} n^{-s} = \prod_{p \in P} 1/(1 - p^{-s}).$$

Asymptotic randomness

5

Let T be a universal Turing machine, and $R(T)$ as before.

Theorem

For every universal Turing machine T and computable $s > 1$, $\langle \kappa_{R(T)}^n \rangle(s)$ is asymptotically $1/s$ -random.

Theorem

If U is a universal self-delimiting Turing machine, then Ω_U is asymptotically random.

Theorem

- If U is a universal self-delimiting machine, then ζ_U is asymptotically random.
- There is a self-delimiting Turing machine V such that ζ_V is asymptotically random, but not random.

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Asymptotic randomness

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Theorem

Let U is a universal self-delimiting machine. There exists a real $\alpha \in (0, 1)$ such that for every pair of computable reals $r, t > 1$ and integer $c \geq 1$, there exists an integer $b \geq 1$ such that for every $m \geq 1$,

$$H_U(\alpha[m]) \geq \frac{1}{r} \cdot m + \frac{c}{t} \cdot \log_2 m - b.$$

So, $H_U(\alpha[m])$ is as close as we want, but never equal, to $\max_{|x|=m} H_U(x) - O(1)$.

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Open problems

Two open problems

- ▶ The zeta number of Ω is at least $1/194$ – K -random and at least asymptotically $1/193$ -random; we conjecture that natural halting probability of Ω is asymptotically random, but not random.
- ▶ Let U^K is a universal self-delimiting machine with an oracle to the Halting Problem, and $\Omega^K = \Omega_{U^K}$; $\Omega^K(2)$ is Chaitin $1/2 - 2$ -random. Is $\Omega^K(2)$ random or asymptotically random?

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Outline

Appendix

Tuatara



Tuatara (“peaks on the back” in Maori) is a reptile (not a lizard) found only in New Zealand. Tuatara is the only survivor of a large group of reptiles that roamed the earth at the time of dinosaurs. Tuatara has not changed its form much in over 225 million years! Its relatives died out about 60 million years ago. Tuatara has a ‘third eye’.

► Classification

Universality

Theorem

There effectively exists a universal tuatara machine.

Theorem

There effectively exists a universal tuatara machine for the class of convergent Turing machines.

► Classification

Uncertainty

Let $\Delta_s = 2^{-s}$. The randomness property of the real α can now be presented as a “formal uncertainty principle”: for every tuatara machine V there is a constant $\varepsilon > 0$ such that $\Delta_s \cdot \nabla_V(\alpha[n]) \geq \varepsilon$.

► Natural Complexity

Iota

The simplest way to define Iota is in terms of Church's λ -calculus: the universal basis $\{S = \lambda xyz.xz(yz), K = \lambda xy.x\}$ suffices to produce every lambda term, but for universality it is not necessary to have two combinators. There are one-combinator bases, known as *universal combinators*. Iota is a very simple universal combinator, $\lambda f.fSK$, denoted 0. To make Iota unambiguous, there is a prefix operator, 1, for application.

[▶ Examples](#)

Partial randomness

7

Theorem

For every computable real $s > 1$ and universal T , $\langle \kappa_T \rangle(s)$ is $1/s - K$ -random.

The number $\langle \kappa_T \rangle(s)$ is a halting probability.

One particularly nice value is $s = 2$ where $\sum_{n>0} 2^{-2\lfloor \log_2(n) \rfloor} = 2$.

If U is universal self-delimiting and $M(xx) = U(x)$, then $\Omega_M = \Omega_M(1)$ is Chaitin $1/2$ -random:

$$\Omega_M(1) = \sum_{x \in \text{dom}(M)} 2^{-s|x|} = \sum_{x \in \text{dom}(U)} 2^{-2s|x|} = \Omega_U(2) = 2\langle \kappa_U \rangle(2).$$

Partial randomness

7

Theorem

For every computable real $s > 1$ and universal T , $\langle \kappa_T \rangle(s)$ is $1/s - K$ -random.

The theorem shows a property true for partial random reals, but not for random reals. Here is an opposite phenomenon: The following characterisation of random reals is no longer true for partial random reals:

Theorem

A real $\alpha \in (0, 1)$ is random iff there exist a constant $c \geq 0$ and an infinite computable set $M \subseteq \mathbf{N}$ such that $H_U(\alpha[n]) \geq n - c$, for each $n \in M$.