Exercise 1

Ex 5.28: Rice’s Theorem: Let $P$ be any nontrivial property of the language of a Turing machine. Prove that the problem of determining whether a given TM’s language has property $P$ is undecidable.

In more formal terms, let $P$ be a language consisting of TM descriptions where $P$ fulfills two conditions. First, $P$ is non-trivial – it contains some, but not all, TM descriptions. Second, $P$ is a property of the TM’s language – whenever $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$. Here, $M_1$ and $M_2$ are any TMs. Prove that $P$ is an undecidable language.

Assume for the sake of contradiction that $P$ is a decidable language satisfying the properties and let $R_P$ be the TM that decides $P$. We show how to decide $A_{TM}$ using $R_P$ by constructing TM $S$. First let $T_\emptyset$ be a TM that always rejects, so $L(T_\emptyset) = \emptyset$. You may assume that $\langle T_\emptyset \rangle \notin P$ without loss of generality, because you could proceed with $P$ instead of $P$ if $\langle T_\emptyset \rangle \in P$. Because $P$ is not trivial, there exists a TM $T$ with $\langle T \rangle \notin P$. Design $S$ to decide $A_{TM}$ using $R_P$’s ability to distinguish between $T_\emptyset$ and $T$.

$S = \begin{array}{l}
\text{On input } \langle M, w \rangle: \\
1. \text{Use } M \text{ and } w \text{ to construct the following TM } M_w.
\quad M_w = \text{“On input } x:\n\qquad 1. \text{Simulate } M \text{ on } w. \text{ If it halts and rejects, reject. If it accepts, proceed to stage 2.}
\qquad 2. \text{Simulate } T \text{ on } x. \text{ If it accepts, accept.”}
\quad 2. \text{Use TM } R_P \text{ to determine whether } \langle M_w \rangle \in P. \text{ If YES, accept. If NO, reject.”}
\end{array}$

TM $M_w$ simulates $T$ if $M$ accepts $w$. Hence $L(M_w)$ equals $L(T)$ if $M$ accepts $w$ and $\emptyset$ otherwise. Therefore $\langle M, w \rangle \in P$ iff $M$ accepts $w$.

Exercise 2

Ex 5.29: Show that both conditions in Ex. 5.28 are necessary for proving that $P$ is undecidable.

First, let $P$ be the language $\{ \langle M \rangle \mid M \text{ is a TM with 5 states}\}$. $P$ is non-trivial, and so it satisfies the second condition of Rice’s Theorem but $P$ can be easily decided by checking the number of states of the input TM. Second, let $P$ be the empty set. Then it does not contain any TM and so it satisfies the first condition of Rice’s Theorem, but $P$ can be decided by a TM that always rejects. Therefore both properties are necessary for proving $P$ undecidable.
Exercise 3

Ex 5.30: Use Rice's Theorem to prove the undecidability of each of the following languages:

a. $\text{INFINITE}_{TM} = \{ \langle M \rangle | M \text{ is a TM and } L(M) \text{ is an infinite language} \}$.

b. $\{ \langle M \rangle | M \text{ is a TM and } 1011 \in L(M) \}$

c. $\text{ALL}_{TM} = \{ \langle M \rangle | M \text{ is a TM and } L(M) = \Sigma^* \}$

a. $\text{INFINITE}_{TM}$ is a language of TM descriptions. It satisfies the two conditions of Rice’s Theorem. First, it is non-trivial because some TMs have infinite languages and others do not. Second, it depends only on the language. If two TMs recognise the same language, either both have descriptions in $\text{INFINITE}_{TM}$ or neither do. Consequently, Rice’s theorem implies that $\text{INFINITE}_{TM}$ is undecidable.

b. Let $P = \{ \langle M \rangle | M \text{ is a TM and } 1011 \in L(M) \}$. $P$ is a language of TM descriptions. It satisfies the two conditions of Rice’s Theorem. First, it is non-trivial because some TMs contain the string 1011 in their language and others do not. Second, it depends only on the language. If two TMs recognise the same language, either both have descriptions in $P$ (because they both accept 1011), or neither do. Thus, Rice’s theorem implies that $P$ is undecidable.

c. $\text{ALL}_{TM}$ is a language of TM descriptions. It satisfies the two conditions of Rice’s Theorem. First, it is non-trivial because some TMs accept all possible strings of an alphabet $\Sigma$ and others do not. Second, it depends only on the language. If two TMs recognise the same language, either both have descriptions in $\text{ALL}_{TM}$ or neither do. Therefore, Rice’s theorem implies that $\text{ALL}_{TM}$ is undecidable.