Exercise 1

**Ex 3.1:** This exercise concerns TM $M_2$ whose description and state diagram appear in Example 3.7. In each of the parts, give the sequence of configurations that $M_2$ enters when started on the indicated input string.

  c. 000
  d. 000000

<table>
<thead>
<tr>
<th>c.  $q_1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\downarrow q_200$</td>
</tr>
<tr>
<td>$\downarrow xq_30$</td>
</tr>
<tr>
<td>$\downarrow x0q_4$</td>
</tr>
<tr>
<td>$\downarrow x0$ $q_{reject}$</td>
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</tbody>
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<table>
<thead>
<tr>
<th>d.  $q_1000000$</th>
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<tbody>
<tr>
<td>$\downarrow x0x0q_4$</td>
</tr>
<tr>
<td>$\downarrow xq_50x0$</td>
</tr>
<tr>
<td>$\downarrow xxq_50x$</td>
</tr>
<tr>
<td>$\downarrow x0q_400$</td>
</tr>
<tr>
<td>$\downarrow x0xq_50$</td>
</tr>
<tr>
<td>$\downarrow q_50x0$</td>
</tr>
<tr>
<td>$\downarrow xx0q_4$</td>
</tr>
<tr>
<td>$\downarrow x0q_40$</td>
</tr>
<tr>
<td>$\downarrow x0xq_50$</td>
</tr>
<tr>
<td>$\downarrow xq_20x$</td>
</tr>
<tr>
<td>$\downarrow xx0x$ $q_{reject}$</td>
</tr>
</tbody>
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Exercise 2

**Ex 3.5:** Examine the formal definition of a Turing machine to answer the following questions, and explain your reasoning.

  a. Can a TM ever write the blank symbol on its tape?
  b. Can the tape alphabet $\Gamma$ be the same as the input alphabet $\Sigma$?
  c. Can a TM’s head ever be in the same location in two successive steps?
  d. Can a TM contain just a single state?

<table>
<thead>
<tr>
<th>a. Yes. $\Gamma$ contains $\sqcup$. A TM can write any characters in $\Gamma$ on its tape.</th>
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<tbody>
<tr>
<td>b. No. $\Sigma$ never contains $\sqcup$, but $\Gamma$ always contains $\sqcup$.</td>
</tr>
<tr>
<td>c. Yes. If the TM attempts to move its head off the left-hand end of the tape, it remains on the same cell.</td>
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<tr>
<td>d. No. Any TM must contain two distinct states $q_{accept}$ and $q_{reject}$. So at least two states.</td>
</tr>
</tbody>
</table>
Exercise 3

Ex 3.15: Show that the collection of decidable languages is closed under the operation of:

a. union.
b. concatenation.
c. complementation. [denoted as c in the solution]

d. For any two decidable languages $L_1$ and $L_2$, let $M_1$ and $M_2$ be the TMs that decide them, respectively. We construct a TM $M'$ that decides $L_1 \cup L_2$:

$M' = \text{"On input } w:\$  
1. Run $M_1$ on $w$. If it accepts, accept.
2. Run $M_2$ on $w$. If it accepts, accept.
3. Otherwise reject.”

$M'$ accepts if either $M_1$ or $M_2$ accept the input $w$.

$M'$ is a decider because both $M_1$ and $M_2$ are deciders, so it will use a finite number of steps to accept or reject any $w$.

d. For any two decidable languages $L_1$ and $L_2$, let $M_1$ and $M_2$ be the TMs that decide them, respectively. We construct a multi-tape TM $M'$ that decides $L_1 \cdot L_2$:

$M' = \text{"On input } w:\$  
1. Copy $w$ onto the second tape and reset all heads to the front of the tapes.
2. Run $M_1$ on $w$ (first tape).
3. If $M_1$ accepts, run $M_2$ on the rest of $w$ (from where the second head is pointing at).
4. If $M_2$ accepts, accept.
5. Otherwise reject.”

$M'$ accepts if $M_1$ then $M_2$ accept the input $w$.

$M'$ is a decider because both $M_1$ and $M_2$ are deciders, so it will use a finite number of steps to accept or reject any $w$. There is no case where $M'$ will run forever.

c. For any decidable language $L$, let $M$ be the TM that decides it. We construct a TM $M'$ that decides the complement of $L$:

$M' = \text{"On input } w:\$  
1. Run $M$ on $w$. If it accepts, reject.
2. Otherwise accept.”

$M'$ accepts if $M$ rejects the input $w$.

$M'$ is a decider because $M$ is a decider, so it will use a finite number of steps to accept or reject any $w$. 

Exercise 4

Ex 3.16: Show that the collection of recognizable languages is closed under the operation of:
   a. union.
   b. concatenation.

a. For any two recognizable languages $L_1$ and $L_2$, let $M_1$ and $M_2$ be the TMs that respectively recognize them.
   We construct a TM $M'$ that recognizes $L_1 \cup L_2$:
   $M' = \text{“On input } w:\$
   1. Simultaneously run $M_1$ and $M_2$ on $w$.
   2. If either TMs accept, accept.
   $M'$ accepts if either $M_1$ or $M_2$ accept the input $w$.
   $M'$ is a recognizer (clearly not a decider) because both $M_1$ and $M_2$ are recognizers, so it will use a finite number of steps to accept any $w$ in its language (but will not reject strings which are not in its language).

b. For any two recognizable languages $L_1$ and $L_2$, let $M_1$ and $M_2$ be the TMs that respectively recognize them.
   We construct a TM $M'$ that recognizes $L_1 \cdot L_2$:
   $M' = \text{“On input } w:\$
   1. Copy $w$ onto the second tape and reset all heads to the front of the tapes.
   2. Run $M_1$ on $w$ (first tape).
   3. If $M_1$ accepts, run $M_2$ on the rest of $w$ (from where the second head is pointing at).
   4. If $M_2$ accepts, accept.
   $M'$ accepts if $M_1$ then $M_2$ accept the input $w$.
   $M'$ is a recognizer because both $M_1$ and $M_2$ are recognizers, so it will use a finite number of steps to accept any $w$ which is in its language. If either $M_1$ or $M_2$ run forever, so will $M'$.

Exercise 5

Ex 3.11: A **TM with doubly infinite tape** is similar to an ordinary TM, but its tape is infinite to the left as well as to the right. The tape is initially filled with blanks except for the portion that contains the input. Computation is defined as usual except that the head never encounters an end to the tape as it moves leftward. Show that this type of TM recognizes the class of Turing-recognizable languages.

Let $M$ be any ordinary TM, and $D$ be any doubly infinite tape TM.
We need to show that $M \iff D$.

1. $M \Rightarrow D$: Given a TM $M$, $D$ can simulate $M$ by marking on its doubly infinite tape the start of each of $M$’s tapes and make sure it cannot move left beyond each of these markers.

2. $D \Rightarrow M$: Given a doubly infinite tape TM $D$, $M$ can simulate $D$ by using several tapes (2 tapes actually). One tape for the right-hand side of the beginning of the tape. One tape for the left-hand side (you may have to change all ‘L’ instructions to ‘R’ and insure a correct move from one tape to another).