Exercise 1

State and prove the Pumping Lemma:

See lecture notes for Lecture 10.

Exercise 2

Textbook exercise 1.55: For each of the following languages, give the minimum pumping length and justify your answer:

1. $0001^*$
2. $0^*1^*$
3. $0^*1^+0^+1^* \cup 10^*1$

First of all: the pumping length is the length of the whole string you are pumping! [I thoroughly apologise for the confusion in the tutorial!!!]

1. $0001^*$ has a minimum pumping length of 4. The string $000$ is in the language but cannot be pumped, thus the minimum pumping length must be larger than 3 ($|000|$). If the string $s$ that we are pumping has length 4 or more, then it contains $1$s. By diving $s$ into $xyz$, where $x = 000$, $y = 1$ and $z$ is the remainder of the string, we satisfy the pumping lemma’s three conditions.

2. $0^*1^*$ has a minimum pumping length of 1. The pumping length cannot be 0 because the string $\epsilon$ (which is in the language) cannot be pumped. Every nonempty string in the language can be divided into $xyz$ where $x = \epsilon$, $y$ is the character in the string, and $z$ is the remainder. This division satisfies all three conditions.

3. $0^*1^+0^+1^* \cup 10^*1$ has a minimum pumping length of 3. The pumping length cannot be 2 because $11$ is in the language but cannot be pumped. Thus, let $s$ be any string in the language of length at least 3. Dividing $s$ into $xyz$ such that all three conditions of the pumping lemma hold we get:

   • If generated by $0^*1^+0^+1^*$, then $x = \epsilon$, $y$ is the first symbol of $s$ and $z$ is the remainder.
   • If generated by $10^*1$, then $x = 1$, $y = 0$ and $z$ is the remainder.

   In both these cases, $z$ has to be nonempty to satisfy the conditions.
Exercise 3

Textbook exercise 1.29 a. and c. : Us the Pumping Lemma and prove that the following languages are not regular, using the Pumping lemma (and you may use the closure of the class of regular languages under union, complement, and intersection).

There are two ways to solve this problem: using the closure under certain operations properties or the pumping lemma.

1. Let \( B = \{0^n1^n \mid m \neq n\} \). Observe that \( B \cap 0^*1^* = \{0^k1^k \mid k \geq 0\} \). If \( B \) were regular, then \( B \) would be regular and so would \( B \cap 0^*1^* \). But we know (from the proof done in class, which you should be able to repeat), that \( \{0^k1^k \mid k \geq 0\} \) is not regular, so \( B \) cannot be regular.

2. Assume that \( B = \{0^n1^n \mid m \neq n\} \) is regular. Let \( p \) be the pumping length given by the pumping lemma. Observe that \( p! \) is divisible by all integers from 1 to \( p \), where \( p! = p(p-1)(p-2) \cdots 1 \). The strings \( 0^p1^{p+p!} \in B \) and \( |s| \geq p \). Thus the pumping lemma implies that \( s \) can be divided into \( xyz \) with \( x = 0^p, y = 0^b \) and \( z = 0^c1^{b+p!} \), where \( b \geq 1 \) and \( a+b+c = p \). Let \( s' = xy^iz \), where \( i = p!/b \). Then, \( y^i = 0^{p!/b} \) so \( y^i+1 = 0^{p+p!} \). So, \( xyz = 0^p+b+c+p!1^{p+p!} = 0^{p+p!}1^{p+p!} \notin B \), a contradiction.

Exercise 4

Textbook exercise 1.46 b. : Prove that \( \{0^m1^n \mid m \neq n\} \) is not regular, using the Pumping lemma (and you may use the closure of the class of regular languages under union, complement, and intersection).

1. Assume that \( A_1 = \{0^n1^n2^n \mid n \geq 0\} \) is regular. Let \( p \) be the pumping length given by the pumping lemma. Let \( s = 0^p1^p2^p \). Because \( s \in A_1 \) and \( |s| > p \), the pumping lemma states that \( s \) can be split into \( xyz \), satisfying all three conditions of the pumping lemma.

   We need to consider two possibilities here.

   • \( y \) consists only of 0s, only 1s, or only 2s. Then pumping \( y \) gives you strings which do not belong to \( A_1 \) and thus violating the first condition. A contradiction.

   • \( y \) consists of more than one kind of symbol. In this case, the pattern of 0s then 1s then 2s will not be maintained as you pump \( y \). Hence you will never obtain a string belonging to \( A_1 \), a contradiction.

   Therefore, \( A_1 \) is not regular.

2. Assume \( A_3 = \{a^{2^n} \mid n \geq 0\} \) is regular. Let \( p \) be the pumping length given by the pumping lemma. Let \( s = a^p \). Because \( s \in A_3 \) and \( |s| > p \), the pumping lemma states that \( s \) can be split into \( xyz \), satisfying all three conditions of the pumping lemma.

   Take \( y = a^{p-1} \), hence \( 0 < |y| < p \). Now, \( |xyyz| = 2^p + (p-1) \) which is not a power of 2. Thus, \( xyyz \notin A_3 \), a contradiction.

   Hence, \( A_3 \) is not regular.