1. Show that \( \text{ALL}_{\text{DFA}} = \{\langle B \rangle \mid B \text{ is DFA and } L(B) = \Sigma^*\} \) is decidable. [10 marks]

Solution: The following Turing machine decides \( \text{ALL}_{\text{DFA}} \):
\[
M = \text{“on input } \langle B \rangle \text{ where } B \text{ is a DFA:}
1. Let } C \text{ be the DFA obtained by interchanging accepting and rejecting states of } B.
2. Run TM } T \text{ from Thm. 4.4 on input } \langle C \rangle \text{ to see whether } L(C) = \emptyset.
3. If so ACCEPT, otherwise REJECT.”
\]

2. Show that the subset problem for DFA is decidable. Namely,
\[
\{\langle B, C \rangle \mid B, C \text{ are DFA and } L(B) \subseteq L(C)\}
\]
is decidable. [10 marks]

Solution: (Sketch) This is similar to Theorem 4.5 that \( \text{EQ}_{\text{DFA}} \) is decidable.
From input \( \langle B, C \rangle \) a Turing machine constructs a DFA \( D \) recognizing \( L(B) \cap L(C) \). Then it accepts if \( L(D) \) is empty, otherwise it rejects.

3. (Sipser 4.28)
Let \( A \) be a Turing recognizable language consisting of descriptions \( \langle M \rangle \) of Turing machines \( M \) that are all deciders.
Prove that some decidable language \( D \) is not decided by any decider \( M \) such that \( \langle M \rangle \) is in \( A \).

Solution: We may assume that \( A \) is infinite since there are infinitely many decidable languages. By Thm 3.21 there is an enumerator \( E \) for the language \( A \).
Define a decider \( N \) as follows.
\( N = \text{“on input } \langle i \rangle \text{ where } i \text{ is a natural number}
1. Wait till for the } i \text{-time a description } \langle M \rangle \text{ of a machine is printed by } E.
2. Run } M \text{ on input } \langle i \rangle.
3. If } M \text{ accepts, REJECT. Otherwise ACCEPT.”}

By assumption on \( A \), the machine \( N \) is a decider. Let \( D \) be the language it decides. Assume for a contradiction that \( D = L(M) \) for some \( \langle M \rangle \) in \( A \). Then \( \langle M \rangle \) is the } i \text{-th machine description printed by } E \text{ for some } i \text{. But } N \text{ behaves the opposite of } M \text{ on input } \langle i \rangle, \text{ contrary to the assumption that } D = L(M).

[10 marks]

4. Prove that there is no onto function \( g : \mathcal{N} \rightarrow \mathcal{P}(\mathcal{N}) \). Here \( \mathcal{P}(\mathcal{N}) \) is the power set of \( \mathcal{N} \), the set of all subsets of \( \mathcal{N} \). [10 marks]

Solution: The proof from class actually shows that such a \( g \) is not onto. (The set \( S \) defined there is not in the range of \( g \).) See presentation material weeks 6-8, page 22.

5. Show that the collection of Turing recognizable languages is closed under union and intersection. [10 marks]

Solution: Union: answered in Sipser 3.16 (page 163 2d edition). Intersection:
Let TM \( M_1, M_2 \) recognize languages \( L_1, L_2 \). The following machine recognizes \( L_1 \cap L_2 \):
\( N = \text{“on input } w:\n1. Run } M_1 \text{ on } w. \text{ If it accepts goto 2.}
2. Run } M_2 \text{ on } w. \text{ If it accepts ACCEPT.”}"
6. Let $a, b$ be distinct symbols. (1) Write a (small) CFG generating the language $a^* b^*$. [3 marks]

(2) Write such a grammar that is also in Chomsky normal form. [5 marks]

(3) In both grammars, give a derivation of the string $aab$. [2 marks]

Solution:

(1) $S \rightarrow aS | Sb | \epsilon$.

(2) $S \rightarrow \epsilon | AT | TB | a | b; T \rightarrow AT | TB | a | b; A \rightarrow a; B \rightarrow b$.

(3) $S \Rightarrow aS \Rightarrow aaS \Rightarrow aaSb \Rightarrow aab$
   $S \Rightarrow AT \Rightarrow AAT \Rightarrow aAT \Rightarrow aaT \Rightarrow aab$

7. Do (1) and (2) above for the language $\{a^i b^i | i \geq 0\}$. That is, (1) first write a small grammar generating this language [4 marks], and then a (larger) one in Chomsky NF [6 marks].

Solution:

(1) $S \rightarrow 0S1 | \epsilon$

(2) $S \rightarrow AT' | \epsilon; AB; T \rightarrow AT'; T' \rightarrow TB; T \rightarrow AB; A \rightarrow a; B \rightarrow b$