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Most programs stop quickly or never halt

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Abstract

The aim of this paper is to provide a probabilistic, but non-quantum, analysis of the Halting Problem. Our approach is to have the probability space extend over both space and time and to consider the probability that a random N -bit program has halted by a random time. We postulate *an a priori computable probability distribution on all possible runtimes* and we prove that given an integer $k > 0$, we can effectively compute a time bound T such that the probability that an N -bit program will eventually halt given that it has not halted by T is smaller than 2^{-k} .

We also show that the set of halting programs (which is computably enumerable, but not computable) can be written as a disjoint union of a computable set and a set of effectively vanishing probability.

Finally, we show that “long” runtimes are effectively rare. More formally, the set of times at which an N -bit program can stop after the time $2^{N+\text{constant}}$ has effectively zero density.

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1. Introduction

The Halting Problem for Turing machines is to decide whether an arbitrary Turing machine M eventually halts on an arbitrary input x . As a Turing machine M can be coded by a finite string—say, $code(M)$ —one can ask whether there is a Turing machine M_{halt} which, given $code(M)$ and

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the input x , eventually stops and produces 1 if $M(x)$ stops, and 0 if $M(x)$ does not stop. Turing's famous result states that this problem cannot be solved by any Turing machine, i.e. there is no such M_{halt} . Halting computations can be recognised by simply running them; the main difficulty is to detect non-halting programs. In what follows time is discrete.

Since many real-world problems arising in the fields of compiler optimisation, automatised software engineering, formal proof systems, and so forth are equivalent to the Halting Problem, there is a strong interest—not merely academic!—in understanding the problem better and in providing alternative solutions.

There are two approaches we can take to calculating the probability that an N -bit program will halt. The first approach, initiated by Chaitin [10], is to have the probability space range only over programs; this is the approach taken when computing the Omega number [2,4]. In that case, the probability is $\text{Prob}_N = \#\{p \in \Sigma^N \mid p \text{ halts}\}/2^N$. For a self-delimiting machine, Prob_N goes to zero when N tends to infinity, since it becomes more and more likely that any given N -bit string is an extension of a shorter halting program. For a universal non-self-delimiting Turing machine, the probability is always nonzero for large enough N : after some point, the universal non-self-delimiting Turing machine will simulate a total Turing machine (one that halts on all inputs), so some fixed proportion of the space will always contribute. The probability in this case is uncomputable, machine-dependent; in general, 1 is the best computable upper bound one can find. In this approach it matters only whether a program halts or not; the time at which a halting program stops is irrelevant.

Our approach is to have the probability space extend over both space and time, and to consider the probability that a random N -bit program—which has not stopped by some given time—will halt by a random later time. In this approach, the stopping time of a halting program is paramount. The problem is that there is no uniform distribution on the integers, so we must choose some kind of distribution on times as well. Any distribution we choose must have that most long times are rare, so in the limit, which distribution we choose does not matter very much.

The new approach was proposed by Calude and Pavlov [7] (see also [1]) where a mathematical quantum “device” was constructed to probabilistically solve the Halting Problem. The procedure has two steps: (a) based on the length of the program and an a priori admissible error 2^{-k} , a finite time T is effectively computed, (b) a quantum “device,” designed to work on a randomly chosen test-vector is run for the time T ; if the device produces a click, then the program halts; otherwise, the program probably does not halt, with probability higher than $1 - 2^{-k}$. This result uses an unconventional model of quantum computing, an infinite dimensional Hilbert space. This quantum proposal has been discussed in Ziegler [17].

It is natural to ask whether the quantum mechanics machinery is essential for obtaining the result. In this paper we discuss a method to “de-quantize” the algorithm. We have been motivated by some recent approximate solutions to the Halting Problem obtained by Köhler, Schindelbauer and M. Ziegler [13] and experimental work [4,14].¹ Different approaches were proposed by Hamkins and Miasnikov [15], and D’Abramo [12].

Our working hypothesis, crucial for this approach, is to postulate *an a priori computable probability distribution on all possible runtimes*. Consequently, the probability space is the product of the space of programs of fixed length (or of all possible lengths), where programs are uniformly distributed, and the time space, which is discrete and has an a priori computable probability dis-

¹ For example, Langdon and Poli [14] suggest that, for a specific universal machine that they describe, about $N^{-1/2}$ programs of length N halt.

tribution. In this context we show that given an integer $k > 0$, we can effectively compute a time bound T such that the probability on the product space that an N -bit program will eventually halt given that it not stopped by T is smaller than 2^{-k} . This phenomenon is similar to the one described for proofs in formal axiomatic systems [5].

We also show that for every integer $k > 0$, the set of halting programs (which is computably enumerable, but not computable) can be written as a disjoint union of a computable set and a set of probability effectively smaller than 2^{-k} .

Of course, an important question is to what extent the postulated hypothesis is acceptable/realistic. The next part of the paper deals with this question offering an argument in favour of the hypothesis. First we note that for any computable probability distribution most long times are effectively rare, so in the limit they all have the same behaviour irrespective of the choice of the distribution. Our second argument is based on the common properties of the times when programs may stop. Our proof consists of three parts:

- (a) the exact time a program stops is algorithmically not too complicated; it is (algorithmically) nonrandom because most programs either stop ‘quickly’ or never halt,
- (b) an N -bit program which has not stopped by time $2^{N+\text{constant}}$ cannot halt at a later random time,
- (c) since nonrandom times are (effectively) rare, the density of times an N -bit program can stop vanishes.

We will start by examining a particular case in detail, the behaviour of all programs of length 3 for a certain Turing machine. This case study will describe various possibilities of halting/non-halting programs, the difference between a program stopping exactly *at* a time and a program stopping *by* some time, as well as the corresponding probabilities for each such event.

Finally, we show some differences between the halting probabilities for different types of universal machines.

Some comments will be made about the “practicality” of the results presented in this paper: can we use them to approximately solve any mathematical problems?

2. A case study

We consider all programs of length $N = 3$ for a simple Turing machine M whose domain is the finite set $\{000, 010, 011, 100, 110, 111\}$. The halting “history” of these programs, presented in Table 1, shows the times at which the programs in the domain of M halt. The program $M(p_1)$ halts at time $t = 1$, so it is indicated by an “h” on the row corresponding to p_1 and time $t = 1$; the program $M(p_4)$ has not halted at time $t = 5$, so on the row corresponding to p_4 and time $t = 4$ there is a blank. Finally, programs which have not stopped by time $t = 17$ never stop.

Here are a few miscellaneous facts we can derive from Table 1:

- the program $M(p_1)$ halts exactly at time $t = 1$,
- the set of 3-bit programs which halt exactly at time $t = 1$ consists of $\{p_1, p_7\}$, so the ‘probability’ that a randomly chosen 3-bit program halts at time $t = 1$ is $\#\{3\text{-bit programs halting at time } 1\} / \#\{3\text{-bit programs}\} = \#\{p_1, p_7\} / 8 = 2/8 = 1/4$,
- the set of 3-bit programs which halt by time $t = 8$ consists of $\{p_1, p_4, p_7\}$, so the ‘probability’ that a randomly picked 3-bit program halts by time $t = 8$ is $\#\{3\text{-bit programs halting by time } 8\} / \#\{3\text{-bit programs}\} = \#\{p_1, p_4, p_7\} / 8 = 3/8$,

Table 1
Halting “history” for 3-bit programs

Program/time	$t = 1$	$t = 2$	$t = 5$	$t = 8$...	$t = 14$	$t = 15$	$t = 16$	$t = 17$
$p_1 = 000$	h	h	h	h	h	h	h	h	h
$p_2 = 001$									
$p_3 = 010$							h	h	h
$p_4 = 011$				h	h	h	h	h	h
$p_5 = 100$						h	h	h	h
$p_6 = 101$									
$p_7 = 110$	h	h	h	h	h	h	h	h	h
$p_8 = 111$								h	h

- the ‘probability’ that a random 3-bit program eventually stops is $\frac{\#\{\text{3-bit programs that halt}\}}{\#\{\text{3-bit programs}\}} = \frac{6}{8}$,
- the program $M(p_4)$ has not stopped by time $t = 5$, but stops at time $t = 8$,
- the ‘probability’ that a 3-bit program does not stop by time $t = 5$ and that it eventually halts is $\frac{\#\{\text{3-bit programs that eventually halt that have not stopped by time } t = 5\}}{\#\{\text{3-bit programs}\}} = \frac{\#\{p_3, p_4, p_5, p_8\}}{8} = \frac{4}{8} = \frac{1}{2}$,
- the ‘probability’ that a 3-bit program eventually stops given that it has not halted by time $t = 5$ is $\frac{\#\{\text{3-bit programs that eventually halt that have not stopped by time } t = 5\}}{\#\{\text{3-bit programs that have not halted by time } t = 5\}} = \frac{4}{(8 - 2)} = \frac{2}{3}$,
- the ‘probability’ that a 3-bit program halts at time $t = 8$ given that it has not halted by time $t = 5$ is $\frac{\#\{\text{3-bit programs that halt by time } t = 8 \text{ but not by time } t = 5\}}{\#\{\text{3-bit programs that have not halted by time } t = 5\}} = \frac{1}{6}$.

We can express the above facts in a bit more formal way as follows. We fix a universal Turing machine U (see Section 3 for a definition) and a pair (N, T) , where N represents the length of the program and T is the “time-window”, i.e. the interval $\{1, 2, \dots, T\}$, where the computational time is observed. The probability space is thus

$$Space_{N,T} = \Sigma^N \times \{1, 2, \dots, T\}.$$

Both programs and times are assumed to be *uniformly distributed*. For $A \subseteq Space_{N,T}$ we define $Prob_{N,T}(A)$ to be $\frac{\#(A)}{2^N \cdot T}$.

Define

$$A_{N,T} = \{(p, t) \in Space_{N,T} \mid U(p) \text{ stops exactly at time } t\},$$

and

$$B_{N,T} = \{(p, t) \in Space_{N,T} \mid U(p) \text{ stops by time } t\}.$$

Fact 1. We have: $Prob_{N,T}(A_{N,T}) \leq \frac{1}{T}$ and $Prob_{N,T}(B_{N,T}) \leq 1$.

Proof. It is easy to see that $\#(A) \leq 2^N$, consequently,

$$Prob_{N,T}(A_{N,T}) = \frac{\#(A_{N,T})}{2^N \cdot T} \leq \frac{2^N}{2^N \cdot T} = \frac{1}{T}, \quad Prob_{N,T}(B_{N,T}) \leq \frac{\#(B_{N,T})}{2^N \cdot T} \leq \frac{2^N \cdot T}{2^N \cdot T} = 1.$$

□

Comment. The inequality $\text{Prob}_{N,T}(B_{N,T}) \leq 1$ does not seem to be very informative. However, for all N , one can construct a universal Turing machine U_N such that $\text{Prob}_{N,T}(B) = 1$; U_N cannot be self-delimiting (see, for a definition, Section 4). There is no universal Turing machine U such that $\text{Prob}_{N,T}(B) = 1$, for all N , so can we do better than stated in Fact 1?

More precisely, we are interested in the following problem:

We are given a universal Turing machine U and a randomly chosen program p of length N that we know not to stop by time t . Can we effectively evaluate the ‘probability’ that $U(p)$ eventually stops?

An obvious way to proceed is the following. Simply, run in parallel all programs of length N till the time $T_N = \max\{t_p \mid |p| = N, U(p) \text{ halts}\} = \min\{t \mid \text{for all } |p| = N, t_p \leq t\}$, where t_p is the exact time $U(p)$ halts (if indeed it stops). In other words, get the analogue of Table 1 for U and N , and then calculate directly all probabilities. This method, as simple as it may look, is not very useful, since the time T_N is *not computable* because of the undecidability of the Halting Problem.

Can we overcome this serious difficulty?

3. Notation

All strings are binary and the set of strings is denoted by Σ^* . The length of the string x is denoted by $|x|$. The logarithms are binary too. Let $\mathbf{N} = \{1, 2, \dots\}$ and let $\text{bin} : \mathbf{N} \rightarrow \Sigma^*$ be the computable bijection which associates to every $n \geq 1$ its binary expansion without the leading 1,

n	n_2	$\text{bin}(n)$	$ \text{bin}(n) $
1	1	λ	0
2	10	0	1
3	11	1	1
4	100	00	2
\vdots	\vdots	\vdots	\vdots

We will work with Turing machines M which process strings into strings. The domain of M , $\text{dom}(M)$, is the set of strings on which M halts (is defined). The *natural complexity* [9] of the string $x \in \Sigma^*$ (with respect to M) is $\nabla_M(x) = \min\{n \geq 1 \mid M(\text{bin}(n)) = x\}$. The Invariance Theorem [2] has the following form: we can effectively construct a machine U (called *universal*) such that for every machine M , there is a constant $\varepsilon > 0$ (depending on U and M) such that $\nabla_U(x) \leq \varepsilon \cdot \nabla_M(x)$, for all strings x . For example, if $U(0^i 1x) = M_i(x)$ (where (M_i) is an effective enumeration of all Turing machines), then $\nabla_U(x) \leq (2^{i+1} + 1) \cdot \nabla_{M_i}(x)$, because $0^i 1 \text{bin}(m) = \text{bin}(2^{i+1} + \lfloor \log(m) \rfloor + m)$, for all $m \geq 1$. In what follows we will fix a universal Turing machine U and we will write ∇ instead of ∇_U . There are some advantages in working with the complexity ∇ instead of the classical complexity K (see [2]); for example, for every $N > 0$, the inequality $\#\{x \in \Sigma^* : \nabla(x) < N\} \leq N$ is obvious; a better example appears in [8] where ∇ is a more natural measure to investigate the relation between incompleteness and uncertainty.

4. Halting according to a computable time distribution

We postulate an *a priori* computable probability distribution on all possible runtimes. Consequently, the probability space is the product of the space of programs—either taken to be all programs of a fixed length, where programs are uniformly distributed, or to be all programs of all possible lengths, where the distribution depends on the length—and the time space, which is discrete and has an *a priori* computable probability distribution.

In what follows we randomly choose a time i from according to a probability distribution $\rho(i)$ which effectively converges to 1, that is, there exists a computable function B such that for every $n \geq B(k)$,

$$\left| 1 - \sum_{i=1}^n \rho(i) \right| < 2^{-k}.$$

How long does it take for an N -bit program p to run without halting on U to conclude that the probability that $U(p)$ eventually stops is less than 2^{-k} ?

It is not difficult to see that the probability that an N -bit program which has not stopped on U by time t_k (which can be effectively computed) will eventually halt is not larger than $\sum_{i \geq t_k} \rho(i)$, which effectively converges to 0, that is, there is a computable function $b(k)$ such that for $n \geq b(k)$, $\sum_{i \geq n} \rho(i) < 2^{-k}$.

The probability distribution $\rho(i)$ may or may not be related to the computational runtime of a program for U . Here is an example of a probability distribution which effectively converges to 1 and relates the observer time to the computational runtime. This probability distribution is reminiscent of Chaitin's halting probability [2], but in contrast, is *computable*.

The idea is to define the distribution at moment i to be 2^{-i} divided by the exact time it takes $U(\text{bin}(i))$ to halt, or to be 0 if $U(\text{bin}(i))$ does not halt. Recall that t_p is the exact time $U(p)$ halts (or $t_p = \infty$ when $U(p)$ does not halt).

First we define the number

$$\Upsilon_U = \sum_{i \geq 1} 2^{-i} / t_{\text{bin}(i)}.$$

It is clear that $0 < \Upsilon_U < 1$. Moreover, Υ_U is computable. Indeed, we construct an algorithm computing, for every positive integer n , the n th digit of Υ_U . The proof is simple: only the terms $2^{-i} / t_{\text{bin}(i)}$ for which $U(\text{bin}(i))$ does not halt, i.e. $t_{\text{bin}(i)} = \infty$, produce 'false' information because at every finite step of the computation they appear to be non-zero when, in fact, they are zero! The solution is to run all non-stopping programs $U(\text{bin}(i))$ for enough time such that their cumulative contribution is too small to affect the n th digit of Υ_U : indeed, if $n > 2$, and $t_{\text{bin}(i)} = 1$, for $i \geq n$, then $\sum_{i=n}^{\infty} 2^{-i} / t_{\text{bin}(i)} < 2^{-n}$.

So, Υ_U induces a natural probability distribution on the runtime: to i we associate²

$$\rho(i) = \frac{2^{-i}}{t_{\text{bin}(i)} \cdot \Upsilon_U}. \tag{1}$$

² Of course, instead of $2^{-i} / t_{\text{bin}(i)}$ we can take $r_i / t_{\text{bin}(i)}$, where $\sum_{i \geq 1} r_i < \infty$, effectively.

The probability space is

$$Space_{N, \{\rho(i)\}} = \Sigma^N \times \{1, 2, \dots\},$$

where N -bit programs are assumed to be uniformly distributed, and we choose at random a runtime from distribution (1).

Theorem 2. Assume that $U(p)$ has not stopped by time $T > k - \lfloor \log \Upsilon_U \rfloor$. Then, the probability (according to the distribution (1)) that $U(p)$ eventually halts is smaller than 2^{-k} .

Proof. It is seen that

$$\frac{1}{\Upsilon_U} \sum_{i=T}^{\infty} \frac{2^{-i}}{t_{\text{bin}(i)}} \leq \frac{1}{\Upsilon_U \cdot 2^{T-1}} < 2^{-k},$$

for $T > k - \lfloor \log \Upsilon_U \rfloor$. The bound is computable because Υ_U is computable. \square

We now consider the probability space to be

$$Space_{\{\rho(i)\}} = \Sigma^* \times \{1, 2, \dots\},$$

where N -bit programs are assumed to be uniformly distributed, and the runtime is chosen at random from the computable probability distribution $\{\rho(i)\}$.

Theorem 3. Assume that U and $Space_{\{\rho(i)\}}$ have been fixed. For every integer $k > 0$, the set of halting programs for U can be written as a disjoint union of a computable set and a set of probability effectively smaller than 2^{-k} .

Proof. Let b be a computable function such that for $n \geq b(k)$ we have $\sum_{i \geq n} \rho(i) < 2^{-k}$. The set of halting programs for U can be written as a disjoint union of the computable set $\{(p, t_p) \mid t_p < 2^{b(k+|p|+2)}\}$ and the set $\{(p, t_p) \mid 2^{b(k+|p|+2)} \leq t_p < \infty\}$. The last set has probability effectively less than

$$\sum_{N=1}^{\infty} \sum_{n=b(k+N+2)}^{\infty} \rho_n \leq \sum_{N=1}^{\infty} 2^{-N-k-2} = 2^{-k-1}. \quad \square$$

Comment. A stronger (in the sense that the computable set is even polynomially decidable), but machine-dependent, decomposition theorem for the set of halting programs was proved in [15].

5. How long does it take for a halting program to stop?

The common wisdom says that it is possible to write short programs which stop after a very long time. However, it is less obvious that there are only a few such programs; these programs are “exceptions.”

Working with self-delimiting Turing machines, Chaitin [11] has given the following estimation of the complexity³ of the runtime of a program which eventually halts: there is a constant c such that if $U(\text{bin}(i))$ halts in time t , then

$$\nabla(\text{bin}(t)) \leq 2^{|\text{bin}(i)|} \cdot c \leq i \cdot c. \tag{2}$$

Here t is the first time $U(\text{bin}(i))$ halts.⁴ The above relation puts a limit on the *complexity* of the time t a program $\text{bin}(i)$, that eventually halts on U , has to run before it stops; this translates into a limit on the time t because only finitely many strings have complexity bounded by a constant. In view of (2), *the bound depends only upon the length of the program*; the program itself (i.e. $\text{bin}(i)$) does not matter.

Because $\lim_{t \rightarrow \infty} \nabla(\text{bin}(t)) = \infty$, there are only finitely many integers t satisfying the inequality (2). That is, there exists a critical value $T_{critical}$ (depending upon U and $|\text{bin}(i)|$) such that if for each $t < T_{critical}$, $U(\text{bin}(i))$ does not stop in time t , then $U(\text{bin}(i))$ never halts. In other words,

if $U(\text{bin}(i))$ does not stop in time $T_{critical}$, then $U(\text{bin}(i))$ never halts.

So, what prevents us from running the computation $U(\text{bin}(i))$ for the time $T_{critical}$ and deciding whether it halts or not? Obviously, the uncomputability of $T_{critical}$. Neither the natural complexity ∇ nor any other size complexity, like K or H , is computable (see [2]). Obviously, there are large integers t with small complexity $\nabla(\text{bin}(t))$, but they cannot be effectively “separated” because we cannot effectively compute a bound $b(k)$ such that $\nabla(\text{bin}(t)) > k$ whenever $t > b(k)$.

The above analysis suggests that *a program that has not stopped after running for a long time has smaller and smaller chances to eventually stop*. The bound (2) is not computable. Still, can we “extract information” from the inequality (2) to derive a computable probabilistic description of this phenomenon?

Without loss of generality, we assume that the universal Turing machine U has a built-in counting instruction. Based on this, there is an effective transformation which for each program p produces a new program $time(p)$ such that there is a constant $c > 0$ (depending upon U) for which the following three conditions are satisfied:

- (1) $U(p)$ halts iff $U(time(p))$ halts,
- (2) $|time(p)| \leq |p| + c$,
- (3) if $U(p)$ halts, then it halts at the step $t_p = \text{bin}^{-1}(U(time(p)))$.

Intuitively, $time(p)$ either calculates the number of steps t_p till $U(p)$ halts and prints $\text{bin}(t_p)$, or, if $U(p)$ is not defined, never halts. The constant c can be taken to be less than or equal to 2, as the counting instruction is used only once, and we need one more instruction to print its value; however, we do not need to print the value $U(p)$.

We continue with a proof of the inequality (2) for an arbitrary universal Turing machine.

³ Chaitin used program-size complexity.

⁴ Of course, if $U(\text{bin}(i))$ halts in time t , it stops also in time $t' > t$, but only finitely many t' satisfy the inequality (2). For the reader more familiar with the program-size complexity H —with respect to a universal self-delimiting Turing machine [2]—the inequality (2) corresponds to $H(\text{bin}(t)) \leq |\text{bin}(i)| + c$.

Theorem 4. Assume that $U(p)$ stops at time t_p , exactly. Then,

$$\nabla(\text{bin}(t_p)) \leq 2^{|p|+c+1}. \quad (3)$$

Proof. First we note that for every program p of length at most N , $\text{bin}^{-1}(p) < 2^{N+1}$. Indeed, $|p| = |\text{bin}(\text{bin}^{-1}(p))| \leq N$ implies

$$2^{|p|} \leq \text{bin}^{-1}(p) < 2^{|p|+1} \leq 2^{N+1}. \quad (4)$$

Since $U(p) = U(\text{bin}(\text{bin}^{-1}(p)))$ we have:

$$\nabla(U(p)) = \min\{i \geq 1: U(\text{bin}(i)) = U(p)\} \leq \text{bin}^{-1}(p),$$

hence

$$\nabla(\text{bin}(t_p)) = \nabla(U(\text{time}(p))) \leq \text{bin}^{-1}(\text{time}(p)) < 2^{|p|+c+1},$$

because $|\text{time}(p)| \leq |p| + c$ and (4). \square

6. Can a program stop at an algorithmically random time?

In this section we prove that no program of length $N \geq 2$ which has not stopped by time $2^{2N+2c+1}$ will stop at an algorithmically random time. Consequently, since algorithmically non-random times are (effectively) rare, there are only a few times an N -bit program can stop in a suitably large range. As a consequence, the set of times at which an N -bit program can stop after the time $2^{N+\text{constant}}$ has effectively zero density.

A binary string x is “algorithmically random” if $\nabla(x) \geq 2^{|x|}/|x|$. Most binary strings of a given length n are algorithmically random because they have high density: $\#\{x \in \Sigma^*: |x| = n, \nabla(x) \geq 2^n/n\} \cdot 2^{-n} \geq 1 - 1/n$ which tends to 1 when $n \rightarrow \infty$.⁵

A time t will be called “algorithmically random” if $\text{bin}(t)$ is algorithmically random.

Theorem 5. Assume that an N -bit program p has not stopped on U by time $2^{2N+2c+1}$, where $N \geq 2$ and c comes from Theorem 4. Then, $U(p)$ cannot exactly stop at any algorithmically random time $t \geq 2^{2N+2c+1}$.

Proof. First we prove that for every $n \geq 4$ and $t \geq 2^{2n-1}$, we have:

$$2^{|\text{bin}(t)|} > 2^n \cdot |\text{bin}(t)|. \quad (5)$$

Indeed, the real function $f(x) = 2^x/x$ is strictly increasing for $x \geq 2$ and tends to infinity when $x \rightarrow \infty$. Let $m = |\text{bin}(t)|$. As $2^{2n-1} \leq t < 2^{m+1}$, it follows that $m \geq 2n - 1$, hence $2^m/m \geq 2^{2n-1}/(2n - 1) \geq 2^n$. The inequality is true for every $|\text{bin}(t)| \geq 2n - 1$, that is, for every $t \geq 2^{2n-1}$.

⁵ In the language of program-size complexity, x is “algorithmically random” if $H(x) \geq |x| - \log(|x|)$.

Next we take $n = N + c + 1$ in (5) and we prove that every algorithmically random time $t \geq 2^{2N+2c+1}$, $N \geq 2$, does not satisfy the inequality (3). Consequently, no program of length N which has not stopped by time $2^{2N+2c+1}$ will stop at an algorithmically random time. \square

A time t is called “exponential stopping time” if there is a program p which stops on U exactly at $t = t_p > 2^{2|p|+2c+1}$. How large is the set of exponential stopping times? To answer this question we first need a technical result.

Lemma 6. *Let $m \geq 3$, $s \geq 1$. Then*

$$\frac{1}{2^s - 1} \cdot \sum_{i=0}^s \frac{2^i}{m + i} < \frac{5}{m + s - 1}.$$

Proof. Let us denote by x_s^m the left-hand side of the inequality below. It is easy to see that

$$x_{s+1}^m = \frac{2^s - 1}{2^{s+1} - 1} \cdot x_s^m + \frac{2^{s+1}}{m + s + 1} \leq \frac{x_s^m}{2} + \frac{2}{m + s + 1}.$$

Next we prove by induction (on s) the inequality in the statement of the lemma. For $s = 1$ we have $x_1^m = 1/m + 2/(m + 1) < 5/m$. Assume that $x_s^m < 5/(m + s - 1)$. Then:

$$x_{s+1}^m \leq \frac{x_s^m}{2} + \frac{2}{m + s + 1} < \frac{5}{2(m + s - 1)} + \frac{2}{m + s + 1} \leq \frac{5}{m + s}. \quad \square$$

The density of times in the set $\{1, 2, \dots, N\}$ satisfying the property P is the ratio $\#\{t \mid 1 \leq t \leq N, P(t)\}/N$. A property P of times has “effective zero density” if the density of times satisfying the property P effectively converges to zero, that is, there is a computable function $B(k)$ such that for every $N > B(k)$, the density of times satisfying the property P is smaller than 2^{-k} .

Theorem 7. *For every length N , we can effectively compute a threshold time θ_N (which depends on U and N) such that if a program of length N runs for time θ_N without halting, then the density of times greater than θ_N at which the program can stop has effective zero density. More precisely, if an N -bit program runs for time $T > \max\{\theta_N, 2^{2+5 \cdot 2^k}\}$, then the density of times at which the program can stop is less than 2^{-k} .*

Proof. We choose the bound $\theta_N = 2^{2N+2c+1} + 1$, where c comes from (3). Let $T > \theta_N$ and put $m = 2N + 2c + 1$, and $s = \lceil \log(T + 1) \rceil - m$. Then, using Theorem 5 and Lemma 6, we have:

$$\begin{aligned} & \frac{1}{T - 2^m + 1} \cdot \#\left\{2^m \leq t \leq T \mid \nabla(\text{bin}(t)) \geq \frac{2^{|\text{bin}(t)|}}{|\text{bin}(t)|}\right\} \\ & \geq \frac{1}{T - 2^m + 1} \cdot \sum_{i=0}^s \#\left\{2^{m+i} \leq t \leq 2^{m+i+1} - 1 \mid \nabla(\text{bin}(t)) \geq \frac{2^{|\text{bin}(t)|}}{|\text{bin}(t)|}\right\} \\ & = \frac{1}{T - 2^m + 1} \cdot \sum_{i=0}^s \#\left\{2^{m+i} \leq t \leq 2^{m+i+1} - 1 \mid \nabla(\text{bin}(t)) \geq \frac{2^{m+i}}{m+i}\right\} \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{T - 2^m + 1} \cdot \sum_{i=0}^s 2^{m+i} \left(1 - \frac{1}{m+i} \right) \\ &\geq 1 - \frac{1}{T - 2^m + 1} \cdot \sum_{i=0}^s \frac{2^{m+i}}{m+i} \\ &\geq 1 - \frac{1}{(2^s - 1)} \cdot \sum_{i=0}^s \frac{2^i}{m+i} \\ &> 1 - \frac{5}{m+s-1}, \end{aligned}$$

consequently, the density of algorithmically random times effectively converges to 1:

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{\#\{t \mid t > \theta_N, t \leq T, t \neq t_p, \text{ for all } p \text{ with } |p| = N\}}{T - 2^m + 1} \\ &\geq \lim_{T \rightarrow \infty} \frac{1}{T - 2^m + 1} \cdot \#\left\{ 2^m \leq t \leq T \mid \nabla(\text{bin}(t)) \geq \frac{2^{|\text{bin}(t)|}}{|\text{bin}(t)|} \right\} = 1, \end{aligned}$$

so the density of times greater than θ_N at which an N -bit program can stop effectively converges to zero. \square

The next result states that “almost any” time is not an exponential stopping time.

Corollary 8. *The set of exponential stopping times has effective zero density.*

Proof. It is seen that

$$\begin{aligned} &\{t \mid t = t_p, \text{ for some } p \text{ with } t > 2^{2|p|+2c+1}\} \\ &\subseteq \bigcup_{N \geq 1} \left\{ t \mid t > 2^{2|p|+2c-1}, |p| = N, \nabla(\text{bin}(t)) < \frac{2^{|\text{bin}(t)|}}{|\text{bin}(t)|} \right\} \\ &\subseteq \left\{ t \mid t > 2^{2c+1}, \nabla(\text{bin}(t)) < \frac{2^{|\text{bin}(t)|}}{|\text{bin}(t)|} \right\}, \end{aligned}$$

which has effectively zero density in view of Theorem 7. \square

7. Halting probabilities for different universal machines

In this section we show a significant difference between the halting probability of a program of a given length for a universal Turing machine and for a universal self-delimiting Turing machine: in the first case the probability is always positive, while in the second case the probability tends to zero when the length tends to infinity.

The probability that an arbitrary string of length N belongs to $A \subset \Sigma^*$ is $\text{Prob}_N(A) = \#(A \cap \Sigma^N) \cdot 2^{-N}$, where Σ^N is the set of N -bit strings.

If U is a universal Turing machine, then the probability that an N -bit program p halts on U is $\text{Prob}_N(\text{dom}(U))$.

Fact 9. *Let U be a universal Turing machine. Then, $\lim_{N \rightarrow \infty} \text{Prob}_N(\text{dom}(U)) > 0$.*

Proof. We consider the universal Turing machine $U(0^i x) = T_i(x)$, described in Section 3. If N is sufficiently large, then there exists an $i < N$ such that T_i is a total function, i.e. T_i is defined on each input, so $\#(\text{dom}(U) \cap \Sigma^N) \geq 2^{N-i-1}$. For all such N 's, $\text{Prob}_N(\text{dom}(U)) \geq 2^{-i-1} > 0$. The result extends to any universal Turing machine because of universality. \square

A convergent machine V is a Turing machine such that its ζ number is finite:

$$\zeta_V = \sum_{\text{bin}(n) \in \text{dom}(V)} 1/n < \infty,$$

see [9]. The Ω number of V is $\Omega_V = \sum_{N=1}^{\infty} \text{Prob}_N(\text{dom}(V))$. Because $\zeta_V < \infty$ if and only if $\Omega_V < \infty$, see [9], we get:

Fact 10. *Let V be a convergent machine. Then, $\lim_{N \rightarrow \infty} \text{Prob}_N(\text{dom}(V)) = 0$.*

Recall that a self-delimiting Turing machine V is a machine with a prefix-free domain. For such a machine, $\Omega_V < 1$, hence we have:

Corollary 11. *Let V be a universal self-delimiting Turing machine. Then*

$$\lim_{N \rightarrow \infty} \text{Prob}_N(\text{dom}(V)) = 0.$$

The probability that an N -bit program never stops on a convergent Turing machine tends to one when N tends to infinity; this is not the case for a universal Turing machine.

8. Final comments

We studied the halting probability using a new approach, namely we considered the probability space extend over both space and time, and the probability that a random N -bit program will halt by a random later time given that it has not stopped by some threshold time. We postulated *an a priori computable probability distribution on all possible runtimes*. Consequently, the probability space is the product of the space of programs—either taken to be all programs of a fixed length, where programs are uniformly distributed, or to be all programs of all possible lengths, where the distribution depends on the length—and the time space, which is discrete and has an a priori computable probability distribution. We proved that given an integer $k > 0$, we can effectively compute a time bound T such that the probability that an N -bit program will eventually halt, given that it has not stopped by time T , is smaller than 2^{-k} .

We also proved that the set of halting programs (which is computably enumerable, but not computable) can be written as a disjoint union of a computable set and a set of probability effectively smaller than any fixed bound.

Finally we showed that runtimes much longer than the lengths of their respective halting programs are (effectively) rare. More formally, the set of times at which an N -bit program can stop after the time $2^{N+\text{constant}}$ has effectively zero density.

Can we use this type of analysis for developing a probabilistic approach for proving theorems?

The class of problems which can be treated in this way are the “finitely refutable conjectures.” A conjecture is finitely refutable if verifying a finite number of instances suffices to disprove it [6]. The method seems simple: we choose a natural universal Turing machine U and to each such conjecture C we can effectively associate a program Π_C such that C is true iff $U(\Pi_C)$ never halts. Running $U(\Pi_C)$ for a time longer than the threshold will produce a good evidence of the likelihood validity of the conjecture. For example, it has been shown [3] that for a natural U , the length of the program validating the Riemann Hypothesis is 7780 bits, while for the Goldbach’s Conjecture the length of the program is 3484 bits.

Of course, the choice of the probability distribution on the runtime is paramount. Further, there are at least two types of problems with this approach.

First, the choice of the universal machine is essential. Pick a universal U and let p be a program such that $U(p)$ never stops if and only if a fixed finitely refutable conjecture (say, the Riemann Hypothesis) is true. Define W such that $W(1) = U(p)$ (tests the conjecture), and $W(0x) = U(x)$. The Turing machine W is clearly universal, but working with W “artificially” makes the threshold θ very small. Going in the opposite direction, we can write our simulator program in such a way that it takes a huge number of steps to simulate the machine—say Ackermann’s function of the runtime given by the original machine. Then the new runtime will be very long, while the program is very short. Or we could choose very powerful instructions so that even a ridiculously long program on the original machine would have a very short runtime on the new one.

The moral is that if we want to have some real idea about the probability that a conjecture has a counter-example, we should choose a simulator and program that are “honest”: they should not overcharge or undercharge for each time-step advancing the computation. This phenomenon is very similar to the fact that the complexity of a single string cannot be independent of the universal machine; here, the probability of halting cannot be independent of the machine whose steps we are counting.

Secondly, the threshold T will increase exponentially with the length of the program Π_C (of course, the length depends upon the chosen U). For most interesting conjectures the length is greater than 100, so it is hopeless to imagine that these computations can be effectively carried out (see [16] for an analysis of the maximum speed of dynamical evolution). It is an open question whether another type of computation (possibly, quantum) can be used to speed-up the initial run of the program.

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