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THE FIRST EXAMPLE OF A RECURSIVE FUNCTION WHICH IS NOT PRIMITIVE RECURSIVE

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SUMMARIES

The first example of a recursive function which is not primitive recursive is usually attributed to W. Ackermann. The authors of the present paper show that such an example can also be found in a paper by G. Sudan, published concomitantly with Ackermann's paper.

Le premier exemple d'une fonction récursive qui n'est pas recursive primitive est attribuê usuellement â W. Ackermann. Les auteurs du présent article montrent qu'un tel exemple se trouve aussi dans un article de G. Sudan, publiê en même temps que l'article de W. Ackermann.

Authors of the early classics of mathematical logic and the foundations of mathematics [for example Hilbert and Bernays 1934 1939; Péter 1967], and contemporary authors attribute the first example of a recursive function which is not primitive recursive to W. Ackermann [1928]. The purpose of the present paper is to show that the earliest examples of this type were discovered at the same time by W. Ackermann [1928] and G. Sudan [1927].

In his paper, Sudan dealt with a problem proposed by D. Hilbert [1926] in connection with the Cantor problem of the continuum: Does there exist, for any n, a variable of type n which is of no type smaller than n? (A variable which runs over the finite ordinal numbers is of type 0; a function of a variable of type 0 whose range is also a variable of type 0 is a variable of type 1; a function of a variable of type ≤ 1 whose range is of type ≤ 1 is a variable of type 2, etc. In this manner, we can define variables of types $1, 2, \ldots, n, \ldots, \omega, \ldots, \omega \cdot 2, \omega \cdot 2+1, \ldots, \omega^2, \ldots, \omega^n, \ldots, \omega^\omega, \ldots$). *Sudan obtained an example of a function which is a variable of type 2, but which is not a variable of type 1.* This function was constructed by *nested recursion*, a recursion which is not reducible to primi-

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tive recursion [1]. Sudan showed that any ordinal number smaller than ω^{ω} can be reached by means of a variable of type 1 (i.e., by means of ordinary induction), whereas ω^{ω} can be reached only by means of a variable of type 2 (i.e., by means of transfinite induction). This implies that any ordinal number smaller than ω^{ω} can be obtained by means of primitive recursion, whereas ω^{ω} can be obtained only by means of a recursive function which is not primitive recursive.

Before presenting Sudan's result, we introduce the appropriate symbolism, which was first used by Hilbert [1926]. Given a set $M = \{f(1), f(2), \ldots, f(n), \ldots\}$ of ordinal numbers of the second class, there exists an ordinal number $\lambda_n f(n)$ with the following two properties:

1. $\lambda_n f(n)$ is greater than any $f(n)$ in M;
2. any ordinal number satisfying condition 1 is greater than or equal to $\lambda_n f(n)$.

Thus $\lambda_n n = \omega$, $\lambda_n (\omega + n) = \omega \cdot 2$, and $\lambda_n \omega^n = \omega^{\omega}$. Sudan's function is given by

$$\psi(a,b,0) = a + b,$$

$$\Psi(a,b,n+1) = t_c(a, \lambda_m \Psi(c,m,n), b),$$

where

$$t_c(a, f(c), 0) = a$$

and

$$t_c(a, f(c), n+1) = f(t_c(a, f(c), n))$$

[Sudan 1927, 1929].

In these equations, b is a finite ordinal, a is a transfinite ordinal, $f(c)$ is a variable of type 1, and λ_m is the Hilbert operator. This function is defined by a recursion which is not reducible to primitive recursion.

Sudan knew the contents of Ackermann's paper (he cited the ideas of Ackermann which had been described in Hilbert [1926]. However, at the time that he wrote his paper Ackermann was already acquainted with Sudan's ideas and results. Indeed, after describing his result Ackermann wrote:

> *Eine Arbeit, die mit der vorliegenden manche Berührungspunkte hat, wird von Herrn G. Sudan publiziert werden. Es handelt sich bei ihr um die Definition von Zahlen der zweiten Zahlklasse, die man in ähnlicher Weise klassifizieren kann wie die Definitionen der reellen Zahlen* [Ackermann 1928, 119].

Of course, Ackermann gave no reference here. His paper, dated January 20, 1927, was written before Sudan's paper was published. However, Ackermann and Sudan were together with Hilbert in

Germany, from 1922 until 1925, after which Sudan returned to Romania.

The reason that Hilbert cited only Ackermann's example of a recursive function which is not primitive recursive and failed to mention Sudan's result, seems to be the following: In his argument, Hilbert used a number-theoretic function (similar to the one discussed by Ackermann), not an ordinal function (like Sudan's), because he believed, at that time, that any definition by transfinite recursion could be replaced by a definition using ordinary recursion. Later this conjecture was proved to be false [van Heijenoort 1967, 368]. We also observe that Sudan's equations expressing the nested recursion are mentioned in Hilbert [1926; see van Heijenoort 1967, 388] without explicitly relating it to the preceding construction of Ackermann's function. This reinforces the hypothesis that these equations were borrowed from Sudan. The same equations, $\rho_c(f(c),a,o) = a$, $\rho_c(f(c),a,n+1) = f(\rho_c(f(c),a,n))$, are considered by van Heijenoort [1967, 493].

Like Sudan's function, Ackermann's function is obtained as the superposition of two functions: The first function, which Sudan denotes by t_c, appears in Ackermann's paper in a different notation and with a permutation of the first two arguments. More precisely, Ackermann's function is given by

$$\phi(a,b,o) = a+b$$

$$\phi(a,b,n+1) = g_c(\phi(a,c,n), \alpha(a,n), b),$$

where

$$g_c(f(c), a, o) = a,$$

$$g_c(f(c), a, n+1) = f(g_c(f(c),a,n)),$$

and

$$\alpha(a,n) = 0,\ n=0$$

$$= 1,\ n=1$$

$$= a,\ n=2.$$

[Ackermann 1928, 119].

Some other points in Ackermann's paper are also related to Sudan's paper [Marcus 1975, 19], [2].

Ackermann's function is an example of a function of type 2 which is not of type 1. Later, Péter [1956] proved that it is not a primitive recursive function. The recursiveness of a slight modification of Ackermann's function was proved by Eilenberg and Elgot [1970]. Sudan's result is stronger than Ackermann's because he also gave a characterization of variables of type 2 which are not of type 1.

Péter asserts that Ackermann gave the first example of a function which is not primitive recursive [Péter 1956, 106]. A similar assertion can be found in Hilbert and Bernays [1934-1939] and in many other books and papers. These authors may not have seen Sudan's paper which, although written in French, was published in a Romanian journal. However, they could easily have learned the content of Sudan's work from Fraenkel's review [1927]. Unfortunately Ackermann's paper was reviewed by someone else [Skolem 1928], so that no comparison with Sudan's article was made. Finally, we note that Sudan's article appears in the bibliography of Péter's book [1967], but no reference is made to it in the text.

Thus, despite the fact that only Ackermann is cited by many authors [Davis 1958; Rogers 1967; Esenin-Volpin 1969; Eilenberg and Elgot 1970], Ackermann and Sudan have to share authorship of the first example of a recursive function which is not primitive recursive.

NOTES

1. Let us denote by basic functions the projection function, the identity function, the zero function, and the successor function. A function f of n variables is defined by *primitive recursion* from the functions g and h of n-1 and n+1 variables, respectively, if

$$f(x_1,\dots,x_{n-1},0) = g(x_1,\dots,x_{n-1}),$$
$$f(x_1,\dots,x_{n-1},y+1) = h(x_1,\dots,x_{n-1},y\,f(x_1,\dots,x_{n-1},y)).$$

A function is said to be *primitive recursive* if it can be obtained from the basic functions by a finite number of serial and parallel compositions and primitive recursions; it is *partial recursive* if it can be obtained from the basic functions by a finite number of serial and parallel compositions, primitive recursions, and minimizations. A partial recursive function which is total is called a *recursive function*.

Nested (*eingeschachtelte*) *recursion* occurs when the value $f(x_1,\dots,x_n,0)$ of a function f is defined as a given function of the variables $x_1,\dots,x_n$, and $f(x_1,\dots,x_n,y+1)$ is defined explicitly in terms of given functions and values $f(a_{j1},\dots,a_{jn},b_j)$, $j=1,\dots,p$. Here $a_{j1},\dots,a_{jn}$ and b_j are positive integers obtained by means of the variables $x_1,\dots,x_n,y$ and some given constant functions, whereas for all $x_1,\dots,x_n,y$, we have $b_j < y+1$ $(j=1,\dots,p)$ [Tait 1961, 236].

2. For example, the function

$$\psi(a,0) = A(a),$$
$$\psi(a,b+1) = \mathcal{L}_C(a,b,\psi(c,b)),$$

used by Ackermann [1928, 129] in his proof is precisely the Sudan function

$$\phi\,(a,0) = \alpha(a),$$

$$\phi\,(a,n+1) = \ell_{c}(a,\,\phi\,(c,n),n)$$

[Sudan 1927, 21].

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