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Representation of left-computable ε-random reals ☆

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ABSTRACT

In this paper we introduce the notion of ε-universal prefix-free Turing machine (ε is a computable real in $(0,1]$) and study its halting probability. The main result is the extension of the representability theorem for left-computable random reals to the case of ε-random reals: *a real is left-computable ε-random iff it is the halting probability of an ε-universal prefix-free Turing machine*. We also show that left-computable ε-random reals are provable ε-random in the Peano Arithmetic. The theory developed here parallels to a large extent the classical theory, but not completely. For example, random reals are Borel normal (in any base), but for $\varepsilon \in (0,1)$, some ε-random reals do not contain even arbitrarily long runs of 0s.

1. Introduction

A real α is left-computable (or recursively/computably enumerable) if there is a computable increasing sequence of rationals which converges to α. Left-computable random reals can be characterised using various tools including prefix-complexity, Martin–Löf tests, martingales, Chaitin Omega numbers and universal probability [1,3,5,6,8,11,15].

Some left-computable reals are not random, but "partially random." For example, inserting a 0 in between adjacent bits of a (left-computable) random sequence produces a non-random sequence, having some weak randomness properties: this sequence is, as intuition suggests, left-computable (because it is left-approximated by approximations of the original sequence in which a 0 was inserted in between each adjacent bits) and 1/2-random.

The papers [4,12,16–19] have studied the degree of randomness of reals (or sequences) by measuring their "degree of compression." In what follows ε is a fixed computable real number with $0 < \varepsilon \leqslant 1$. We study ε-randomness of reals, both intrinsically and in relation to the classical notion of randomness (which corresponds to $\varepsilon = 1$, here referred to as 1-randomness or simply randomness).

Our main tool is the ε-universal prefix-free Turing machine, a machine that can simulate any other prefix-free machine: the length of the simulating program on the ε-universal machine is bounded up to a fixed constant by the length of the simulated program divided by ε. In case $\varepsilon = 1$ we get the classical notion of universal machine.

We show that the halting probability of an ε-universal prefix-free Turing machine is left-computable and ε-random. Generalising the corresponding representability theorem of left-computable random reals [1,3,8,11] we show that the converse is also true: every left-computable ε-random real is the halting probability of an ε-universal prefix-free Turing machine. A specific ε-universal Turing machine V_ε is obtained via Eq. (1) below; the main principle is to "dilute" a universal Turing

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machine V. This machine plays an important role as its halting probability is the least with respect to H-reducibility of all ε-random reals.

The theory developed here parallels to a large extent the classical theory, but not completely. The following two results show interesting differences: (a) the prefix-free complexities induced by universal machines differ by at most an additive constant, but the difference between prefix-free complexities induced by ε-universal machines may be unbounded, (b) random reals are Borel normal (in any base), but some ε-random reals do not contain even arbitrarily long runs of 0s.

The paper is organised as follows. In Section 2 we present the necessary notation and previous results. In Section 3 we introduce and study ε-universal machines and their halting probabilities. In Section 4 we study left-computable ε-random reals and in Section 5 we present the representability theorem for left-computable ε-random reals. In Section 6 we discuss the provability in the Peano Arithmetic of ε-randomness for left-computable reals. In Section 7 we disprove Stay's conjecture regarding the 1-randomness (with respect to U) of the halting probability of an ε-universal machine U. We end with a few conclusions.

2. Notation and background

Let $\Sigma = \{0, 1\}$ and denote by Σ^n and Σ^* the set of all bit-strings of length n and the set of all bit-strings, respectively. The length of $\sigma \in \Sigma^*$ is denoted by $|\sigma|$. By $\log n$ we abbreviate the function $\lfloor \log_2(n+1) \rfloor$. Let $\mathbb{N} = \{1, 2, \ldots\}$ and let $\mathrm{bin}: \mathbb{N} \to \Sigma^*$ be the bijection which associates to every $n \geqslant 1$ its binary expansion without the leading 1.

To every infinite binary sequence $\alpha_1\alpha_2\cdots\alpha_n\cdots$ we associate the real number $\alpha = 0.\alpha_1\alpha_2\cdots\alpha_n\cdots$ in $(0, 1]$. We denote by $\alpha \upharpoonright n = \alpha_1\alpha_2\cdots\alpha_n$ the prefix of length n of α's expansion. In this way, reals are identified with infinite binary sequences. Similarly, if $\mathbf{x} = x_1x_2\cdots x_n\cdots$ is an infinite sequence, $\mathbf{x} \upharpoonright n = x_1x_2\cdots x_n$.

We assume that the reader is familiar with algorithmic information theory, cf. [1,8] and present only a few notions to fix the notation.

If the Turing machine T is defined on σ we write $T(\sigma) < \infty$; the domain of T is the set $\mathrm{dom}(T) = \{\sigma \in \Sigma^*\colon\ T(\sigma) < \infty\}$. A prefix-free (Turing) machine is a Turing machine whose domain is a prefix-free set of strings. The prefix complexity of a string induced by a prefix-free machine W is $H_W(\sigma) = \inf\{|p|\colon\ W(p) = \sigma\}$. From now on *all Turing machines will be prefix-free and will be referred to as machines*.

We use several times the Kraft–Chaitin Theorem: given a computable enumeration of positive integers n_i such that $\sum_i 2^{-n_i} \leqslant 1$, we can effectively construct a prefix-free set of binary strings $\{x_i\}$ such that $|x_i| = n_i$, for all $i \geqslant 1$.

Throughout the whole paper ε is assumed to be a computable real in the interval $(0, 1]$. Fix a machine W. A sequence $\mathbf{x}$ is Chaitin (ε, W)-random if there is a constant $c > 0$ such that for every $n \geqslant 1$, $H_W(\mathbf{x} \upharpoonright n) \geqslant \varepsilon \cdot n - c$; $\mathbf{x}$ is strictly Chaitin (ε, W)-random if $\mathbf{x}$ is Chaitin (ε, W)-random, but not Chaitin (δ, W)-random for any δ with $\varepsilon < \delta \leqslant 1$.

If W is universal (from now on called 1-universal), then we get Tadaki's definition of weak Chaitin ε-randomness (see [4,18]). If W is 1-universal and $\varepsilon = 1$, then we get Chaitin's classical definition of randomness [5,6]. A real is Chaitin (ε, W)-random (shortly, (ε, W)-random) if its binary expansion is Chaitin (ε, W)-random.

For any prefix-free set $A \subset \Sigma^*$ we define $\Omega_A = \sum_{x \in A} 2^{-|x|}$. The halting probability of a machine W is $\Omega_W = \sum_{x \in \mathrm{dom}(W)} 2^{-|x|}$.

Following Tadaki [18], for any (not necessarily prefix-free) set $W \subseteq \Sigma^*$ and computable $\delta > 0$ we write $\mu^\delta(W) = \sum_{w \in W} 2^{-\delta \cdot |w|}$. If $\delta > 1$ and W is prefix-free, then $\mu^\delta(W) \leqslant \Omega_W \leqslant 1$. However, if $0 < \delta < 1$ then we can have $\mu^\delta(W) = \infty$ even for prefix-free W (for example, for $W = \{1^{\log n}0\mathrm{bin}(n)\colon\ n > 0\}$ and $0 < \delta < 1/2$).

3. ε-universal machines

In this section we introduce and study the notion of ε-universal machine.

In analogy with the classical case we say, following Stay [14], that a machine U is *ε-universal* if for every machine T there exists a constant $c_{U,T}$ such that for each program $\sigma \in \mathrm{dom}(T)$ there exists a program $p \in \mathrm{dom}(U)$ such that

$$U(p) = T(\sigma) \text{ and } \varepsilon \cdot |p| \leqslant |\sigma| + c_{U,T}.$$

If $\varepsilon = 1$ we get the classical notion of universal machine. Every universal machine is ε-universal, but the converse is not true (see Theorem 2).

A machine U is *strictly* ε-universal if U is ε-universal but not δ-universal for any δ with $\varepsilon < \delta \leqslant 1$.

Lemma 1. *The machine U is ε-universal iff there exists a 1-universal machine V and a constant $c_{U,V}$ such that for all $\sigma \in \Sigma^*$ we have $\varepsilon \cdot H_U(\sigma) \leqslant H_V(\sigma) + c_{U,V}$.*

Theorem 2. *Let V be a 1-universal machine and define*

$$V_\varepsilon\left(p0^{\lfloor (1/\varepsilon - 1)|p| \rfloor}\right) = V(p). \tag{1}$$

Then:

(a) V_ε *is a machine and for all* $\sigma \in \Sigma^*$ *we have* $H_{V_\varepsilon}(\sigma) = \lfloor H_V(\sigma)/\varepsilon \rfloor$,
(b) V_ε *is strictly* ε*-universal.*

Proof. Clearly V_ε is a machine and the equality in (a) can be directly checked. From (a) and Lemma 1 we deduce the ε-universality of V_ε. If there were a constant c such that for all $\sigma \in \Sigma^*$, $\delta \cdot H_{V_\varepsilon}(\sigma) \leqslant H_V(\sigma) + c$, for some $\varepsilon < \delta \leqslant 1$, then in view of (a) we would have $(\delta/\varepsilon - 1) \cdot H_V(\sigma) \leqslant c + \delta$, for all $\sigma \in \Sigma^*$, a contradiction (H_V is unbounded). So, V_ε is strictly ε-universal. □

Theorem 3. *Let* V *be a* 1*-universal machine. Then for every* ε*-universal machine* U, Ω_U *is* (ε, V)*-random.*

Proof. Let f be a computable one-to-one function which enumerates $\mathrm{dom}(U)$. Let $\omega_k = \sum_{j=1}^{k} 2^{-|f(j)|}$. Clearly, (ω_k) is a computable, increasing sequence of rationals converging to Ω_U. Consider the binary expansion of $\Omega_U = 0.\Omega_1\Omega_2\cdots$.

We define a machine T as follows: on input $\sigma \in \Sigma^*$, T first "tries to compute" the smallest number t with $\omega_t \geqslant 0.\sigma$. If successful, $T(\sigma)$ is the first (in quasi-lexicographical order) string not belonging to the set $\{U(f(1)), U(f(2)), \ldots, U(f(t))\}$; if no such t exists then $T(\sigma) = \infty$.

Fix a number $m \geqslant 1$ and note that T is defined on $\Omega_U \upharpoonright m$. Let t be the smallest number (computed in the first step of the computation of T) with $\omega_t \geqslant 0.\Omega_U \upharpoonright m$. We have

$$0.\Omega_U \upharpoonright m \leqslant \omega_t < \omega_t + \sum_{s=t+1}^{\infty} 2^{-|f(s)|} = \Omega_U \leqslant 0.\Omega_U \upharpoonright m + 2^{-m}.$$

Hence, $\sum_{s=t+1}^{\infty} 2^{-|f(s)|} \leqslant 2^{-m}$, which implies $|f(s)| \geqslant m$, for every $s \geqslant t+1$. From the construction of T we conclude that

$$H_U\big(T(\Omega_U \upharpoonright m)\big) \geqslant m. \tag{2}$$

Since T is a partially computable function, we get a constant c' such that for all $\sigma \in \Sigma^*$ for which $T(\sigma) < \infty$ we have:

$$H_V\big(T(\sigma)\big) \leqslant H_V(\sigma) + c'. \tag{3}$$

Using (2), the ε-universality of U, and (3) we obtain

$$\begin{aligned} \varepsilon \cdot m &\leqslant \varepsilon \cdot H_U\big(T(\Omega_U \upharpoonright m)\big) \\ &\leqslant H_V\big(T(\Omega_U \upharpoonright m)\big) + c \\ &\leqslant H_V(\Omega_U \upharpoonright m) + c + c', \end{aligned}$$

which proves that Ω_U is (ε, V)-random. □

Corollary 4. *If* V *be a* 1*-universal machine, then* Ω_{V_ε} *is* (ε, V)*-random and* $(1, V_\varepsilon)$*-random.*

Proof. The halting probability Ω_{V_ε} is (ε, V)-random because of Theorem 2(b) and Theorem 3. Using this fact and Theorem 2(a) we deduce that Ω_{V_ε} is $(1, V_\varepsilon)$-random. □

Next we present a mechanism for producing examples of ε-universal machines.

Let A, B be infinite, prefix-free (recursively/computably) enumerable sets. Generalising the strong simulation in [3], we say that the set A *ε-strongly simulates* the set B (write $B \leqslant_\varepsilon A$) if there is a constant $c > 0$ and a partial computable function $f : \Sigma^* \xrightarrow{o} \Sigma^*$ satisfying the following three conditions:

(a) $A = \mathrm{dom}(f)$,
(b) $B = f(A)$ and
(c) $\varepsilon \cdot |\sigma| \leqslant |f(\sigma)| + c$, for all $\sigma \in A$.

The function f is called an ε-strong simulation of A onto B.

Proposition 5. *If* V *is a* 1*-universal machine and* f *is an* ε*-strong simulation of* $\mathrm{dom}(V)$ *onto a prefix-free computably enumerable set* A*, then* $V \circ f$ *is an* ε*-universal machine with domain* A*.*

Proof. Recall that $(V \circ f)(p) = V(f(p))$ for all $p \in \Sigma^*$. Fix a machine T. Since V is 1-universal there exists a constant c_T such that for each $p \in \mathrm{dom}(T)$ there exists a $\sigma \in \mathrm{dom}(V)$ satisfying $|\sigma| \leqslant |p| + c_T$ and $V(\sigma) = T(p)$. Since f is onto there exists $\tau \in A$ such that $f(\tau) = \sigma$. Since f is an ε-strong simulation we have $\varepsilon \cdot |\tau| \leqslant |f(\tau)| + c = |\sigma| + c$. Combining the previous two equations we deduce that for every $p \in \mathrm{dom}(T)$ there exists a $\tau \in A$ such that $\varepsilon \cdot |\tau| \leqslant |p| + c_T + c$ and $V(f(\tau)) = T(p)$, so $V \circ f$ is ε-universal. $\square$

It may seem that the difference between the cases $\varepsilon = 1$ and $0 < \varepsilon < 1$ is just technical. Here is a deeper difference. If V and V' are 1-universal machines, then their complexities H_V and $H_{V'}$ differ by at most an additive constant [1]. *This result is not true for ε-universal machines.* To prove the claim we construct the following sequence of machines $V_{\varepsilon,k}$ by means of a fixed 1-universal machine V. We let

$$f_{\varepsilon,k}(p) = \begin{cases} p0^{\lfloor (1/\varepsilon - 1)|p| - k\cdot\log(|p|)\rfloor}, & \text{if } (1/\varepsilon - 1)|p| - k \cdot \log|p| \geqslant 1, \\ p1, & \text{otherwise,} \end{cases} \tag{4}$$

$$V_{\varepsilon,k} \circ f_{\varepsilon,k} = V. \tag{5}$$

Note that only for finitely many strings p the value $f_{\varepsilon,k}(p)$ is defined by the otherwise-case. Furthermore, Eq. (5) means that $V_{\varepsilon,k}(f_{\varepsilon,k}(p)) = V(p)$ for all $p \in \mathrm{dom}(V)$ and $V_{\varepsilon,k}(q)$ is undefined for all $q \notin \{f_{\varepsilon,k}(p)\colon\ p \in \mathrm{dom}(V)\}$.

Theorem 6. *The following properties are true*:

(a) $V_{\varepsilon,k}$ *is a machine and* $H_{V_{\varepsilon,k}}(\sigma) = \lfloor H_V(\sigma)/\varepsilon - k \cdot \log H_V(\sigma)\rfloor$, *for almost all strings* σ,
(b) $V_{\varepsilon,k}$ *is strictly* ε*-universal,*
(c) *we have* $H_{V_{\varepsilon,k}}(\sigma) - H_{V_{\varepsilon,k+1}}(\sigma) \geqslant \log H_V(\sigma) - 1 \to \infty$ *whenever* $|\sigma| \to \infty$,
(d) $\Omega_{V_{\varepsilon,k}}$ *is* (ε, V)*-random.*

Proof. Properties (a)–(c) follow from (4) and (5) using the technique presented in the proof of Theorem 2. In detail, the equality in (a) can be directly checked; ε-universality follows from (a) and Lemma 1. To show that $V_{\varepsilon,k}$ is strictly ε-universal we suppose, by absurdity, that there exist two constants c, δ such that $c > 0$, $1 > \delta > \varepsilon$ and $\delta \cdot H_{V_{\varepsilon,k}}(\sigma) \leqslant H_V(\sigma) + c$ for all $\sigma \in \Sigma^*$. Then given the equality (a) we would have $(\delta/\varepsilon - 1) \leqslant H_V(\sigma) \leqslant \delta \cdot k \cdot \log H_V(\sigma) + c + \delta$, for almost all strings σ, a contradiction since H_V is unbounded. Property (c) follows from (a) and property (d) follows from (b) and Theorem 3. $\square$

4. Left-computable (ε, V)-random reals

We now study (ε, V)-random reals with the following reducibility relation: a real α is *H-reducible* to a real β, written $\alpha \leqslant_H \beta$, if there exists a 1-universal machine V and a constant $c > 0$ such that for all $n \geqslant 1$, we have $H_V(\alpha \restriction n) \leqslant H_V(\beta \restriction n) + c$. Of course, the choice of the 1-universal machine V is irrelevant. Two reals α, β are *H-equivalent* if $\alpha \leqslant_H \beta$ and $\beta \leqslant_H \alpha$.

Recall that a real γ is *ε-convergent* [18] if there exists an increasing computable sequence of rationals $\{a_n\}$ converging to γ such that $\sum_{n=1}^{\infty}(a_{n+1} - a_n)^{\varepsilon} < \infty$.

Theorem 7. *Let V be a* 1*-universal machine. For every left-computable* (ε, V)*-random real* α, $\Omega_{V_\varepsilon} \leqslant_H \alpha$.

Proof. Tadaki [19, Theorem 4.6(i) and (iv)] shows the following equivalence: a left-computable real α is (ε, V)-random iff for every left-computable ε-convergent real β there exists a constant c such that for all n, $H_V(\beta \restriction n) \leqslant H_V(\alpha \restriction n) + c$.

Now start with left-computable (ε, V)-random real α. Because Ω_{V_ε} is left-computable and ε-convergent we can apply the above mentioned equivalence to deduce the existence of a constant c such that $H_V(\Omega_{V_\varepsilon} \restriction n) \leqslant H_V(\alpha \restriction n) + c$, i.e. $\Omega_{V_\varepsilon} \leqslant_H \alpha$. $\square$

Comment 8. Theorem 7 shows that Ω_{V_ε} is up to H-equivalence the least of all (ε, V)-random reals. In fact, there is one left-computable real below all other left-computable (ε, V)-random reals.

Proposition 9. *Let V be a* 1*-universal machine. Assume that* $\varepsilon \in (0, 1)$ *is computable. Then, for almost all constants c, and for every string x there exist two strings y, z of length c such that*

1. $H_V(xy) \geqslant H_V(x) + \varepsilon \cdot c + 1$,
2. $H_V(x) - \varepsilon \cdot c + 1 \leqslant H_V(xz) \leqslant H_V(x) + \varepsilon \cdot c - 1$.

Furthermore, z can be chosen as 0^c.

Proof. The proof follows mainly along the lines of Lemma 1 in [12] (with $\rho(x) = 2^{-\varepsilon|x|}$).

For item 1, given x and c we find y of length c such that $H_V(y|(x, H_V(x))) \geqslant c$; such an y exists by the pigeon hole principle. Then $H_V((x, y)) \geqslant H_V(x) + c - d$ and $H_V(xy) \geqslant H_V(x) + c - H_V(c) - d$, for some constant d independent of c. The first inequality follows from Theorem 2.3.6 in [8] and the second inequality follows from the first one by noting that (x, y)

can be computed from xy and c; the constant d is taken such that it satisfies both inequalities. Now all sufficiently large c satisfy $c - H_V(c) - d \geqslant \varepsilon \cdot c + 1$.

For item 2, note that $H_V(x)$ and $H_V(x0^c)$ differ at most by $H_V(c) + d'$ from each other, where d' is again a constant independent of c. The function $c \mapsto H_V(c) + d'$ is dominated by the function $c \mapsto \varepsilon \cdot c - 1$, hence the given inequalities hold. □

Theorem 10. *Let V be a 1-universal machine. Assume that ε is a computable real in $(0, 1)$. There exists a left-computable α and a constant C such that for all $n \geqslant 1$, $|H_V(\alpha \upharpoonright n) - n \cdot \varepsilon| \leqslant C$.*

Proof. In view of Proposition 9 there is a constant c such that for all $\sigma \in \Sigma^*$:

1. σ has an extension τ of length $|\sigma| + c$ such that $H_V(\tau) > H_V(\sigma) + \varepsilon \cdot c + 1$,
2. $H_V(\sigma) - c < H_V(\sigma 0^c) < H_V(\sigma) + \varepsilon \cdot c - 1$.

Let T be the tree of all strings $\sigma \in \Sigma^*$ whose prefixes η with $|\eta|$ are a multiple of c have the property $H_V(\eta) \geqslant \varepsilon \cdot |\eta|$. Note that whenever σ is a node of length $n \cdot c$, by the first condition, there is an extension of σ in T of length $n \cdot c + c$.

Let α be the left-most infinite branch of T, hence left-computable. If $H_V(\alpha \upharpoonright (n \cdot c)) > n \cdot c \cdot \varepsilon + 2c + 1$, then $\alpha \upharpoonright (n \cdot c)0^c$ is in T as

$$H_V\big(\alpha \upharpoonright (n \cdot c)0^c\big) > n \cdot c \cdot \varepsilon + c + 1 > (n \cdot c + c) \cdot \varepsilon.$$

As α is the leftmost infinite branch, $\alpha \upharpoonright (n \cdot c + c) = \alpha \upharpoonright (n \cdot c)0^c$. Consequently, by the second condition,

$$H_V\big(\alpha \upharpoonright (n \cdot c + c)\big) < H_V\big(\alpha \upharpoonright (n \cdot c)\big) + \varepsilon \cdot c - 1,$$

hence $H_V(\alpha \upharpoonright (n \cdot c + c))$ is at least by 1 less than the target $H_V(\alpha \upharpoonright (n \cdot c))$. From this it follows that $|H_V(\alpha \upharpoonright (n \cdot c)) - n \cdot c \cdot \varepsilon|$ is bounded by a constant.

The tree T is the intersection of trees $T_0, T_1, T_2, \ldots$ where each T_s contains an infinite branch β iff the initial segments σ of β of length $0 \cdot c, 1 \cdot c, \ldots, s \cdot c$ satisfy the inequality $H_{V,s}(\sigma) \geqslant \varepsilon \cdot |\sigma|$. The left-most branches α_s of T_s are uniformly computable and approximate α from the left, hence α is left-computable. □

Comment 11. Note that in [12] an essentially similar construction for the real α in Theorem 10 was given; the new fact in Theorem 10 is the property of α to be left-computable. If one does not need the left-computable part of the result, one can construct α by a simple induction: append the corresponding strings previously obtained and keep the complexity of $\alpha(0)\alpha(1)\ldots\alpha(n)$ to be $\varepsilon \cdot n$ up to an additive constant. This method does not work with $\varepsilon = 1$ as it is known that whenever $H_V(\alpha(0)\alpha(1)\ldots\alpha(n)) \geqslant n - c$ for all n then

$$\forall d\, \forall^\infty n\, \big[H_V\big(\alpha(0)\alpha(1)\ldots\alpha(n)\big) \geqslant n + d\big].$$

Hence the existence of α in Theorem 10 holds only $0 < \varepsilon < 1$, another difference between 1-randomness and ε-randomness.

Corollary 12. *Assume that ε is computable in $(0, 1)$ and V is a 1-universal machine.*

a) *There is a constant C such that for all n, $|H_V(\Omega_{V_\varepsilon} \upharpoonright n) - \varepsilon \cdot n| \leqslant C$.*
b) *The real Ω_{V_ε} is strictly (ε, V)-random.*

Proof. From Corollary 4, Ω_{V_ε} is (ε, V)-random. In view of Theorem 10 there exists a left-computable real α and a constant C such that for all n, $|H_V(\alpha \upharpoonright n) - \varepsilon \cdot n| \leqslant C$. In particular, α is left-computable and (ε, V)-random, so by Theorem 7 there exists a constant c such that for all n:

$$H_V(\Omega_{V_\varepsilon} \upharpoonright n) \leqslant H_V(\alpha \upharpoonright n) + c \leqslant \varepsilon \cdot n + c + C. \tag{6}$$

The converse inequality comes from Corollary 4.

Finally, b) is a consequence of (6). □

It is well known that Ω_V is Borel absolutely normal[3] [1]. If $\alpha = 0.\alpha_1\alpha_2\cdots$ is $(1, V)$-random then the real $\beta = 0.\alpha_1 0 \alpha_2 0 \cdots$ is $(1/2, V)$-random and not Borel normal (because in its binary expansion, in the limit, the frequency of 0s is three times larger than the frequency of 1s).

We show now that Ω_{V_ε} is more than not Borel normal:

[3] A real is absolutely Borel normal if its digits, in every base, follow the uniform distribution: all digits are equally likely, all pairs of digits are equally likely, all triplets of digits are equally likely, etc.

Proposition 13. *Let V be a 1-universal machine. Assume that ε is computable in $(0,1)$ and α is a left-computable real such that there is a constant C such that for all n, $|H_V(\alpha \upharpoonright n) - \varepsilon \cdot n| \leqslant C$. Then, for every binary string τ there is a constant c such that τ^c is not an infix of α.*

Proof. The proof follows along the lines of the proof of Proposition 9. To see this, note that for every strings σ, τ and positive integer c we have

$$H_V\left(\sigma\tau^c\right) \leqslant H_V(\sigma) + H_V(\tau) + H_V(c) + d,$$

for some constant d independent of σ, τ, c. Hence, if all prefixes σ of α satisfy the inequality

$$|\sigma| \cdot \varepsilon - c' \leqslant H_V(\sigma) \leqslant |\sigma| \cdot \varepsilon + c',$$

then for every τ there is a value for c such that

$$H_V(\tau) + H_V(c) + d < \varepsilon \cdot c - 2c',$$

and thus whenever $H_V(\sigma) \leqslant |\sigma| \cdot \varepsilon + c'$ we have $H_V(\sigma\tau^c) < |\sigma\tau^c| \cdot \varepsilon - c'$ and $\sigma\tau^c$ is not a prefix of α. □

Corollary 14. *For every binary string τ there is a constant c such that τ^c does not occur in Ω_{V_ε} as a substring.*

Proof. Use Corollary 12(a) and Proposition 13. □

Comment 15. If $\alpha = 0.\alpha_1\alpha_2\cdots$ is $(1, V)$-random then the real $\beta = 0.\alpha_1\alpha_1\alpha_2\alpha_2\cdots$ is $(1/2, V)$-random but does not satisfy the hypothesis of Proposition 13.

5. Representability of left-computable (ε, V)-random reals

In this section we generalise the representability of left-computable random reals [3,11] for the case of left-computable (ε, V)-random reals.

Theorem 16. *Let V be a 1-universal machine. Every left-computable (ε, V)-random number in $(0,1]$ is the halting probability of an ε-universal machine.*

Proof. Given V and ε we consider the machine V_ε defined by (1). Recall that $\mathrm{dom}(V_\varepsilon)$ is the set of all strings $p0^{\lfloor(1/\varepsilon-1)|p|\rfloor}$ with $p \in \mathrm{dom}(V)$. Now Ω_{V_ε} can be represented by the sum $\sum_{q\in\mathrm{dom}(V_\varepsilon)} 2^{-|q|}$. This sum is ε-convergent, as

$$\sum_{q\in\mathrm{dom}(V_\varepsilon)} \left(2^{-|q|}\right)^\varepsilon \leqslant \sum_{p\in\mathrm{dom}(V)} \left(2^{1-|p|/\varepsilon}\right)^\varepsilon \leqslant \sum_{p\in\mathrm{dom}(V)} 2^{\varepsilon-|p|} \leqslant 2^\varepsilon \cdot \Omega_V < \infty.$$

Hence Ω_{V_ε} is ε-convergent.

By Theorem 4.6 (i,v) in [19], given a left-computable and (ε, V)-random real α we can construct a left-computable real $\beta \geqslant 0$ and a rational $q > 0$ (in fact, we can take q to be 2^{-m}, for some $m > 0$) such that $\alpha = \beta + 2^{-m} \cdot \Omega_{V_\varepsilon}$, hence

$$\begin{aligned}\alpha &= \beta + 2^{-m} \cdot \sum_{p\in\mathrm{dom}(V_\varepsilon)} 2^{-|p|} \\ &= 2 \cdot \sum_{r\in\mathrm{dom}(T)} 2^{-|r|-1} + \sum_{p\in\mathrm{dom}(V_\varepsilon)} 2^{-|p|-m}\end{aligned}$$

where the machine T is constructed from the left-computable real β using the Kraft–Chaitin Theorem.

Define now the ε-universal machine W by the formula:

$$W(s) = \begin{cases} 0, & \text{if } s = 1r \text{ and } r \in \mathrm{dom}(T), \\ V_\varepsilon(s), & \text{if } s = 0^m p \text{ and } p \in \mathrm{dom}(V_\varepsilon), \\ \infty, & \text{otherwise,} \end{cases}$$

and notice that its domain is the disjoint union of the sets $\{1r\colon\ r \in \mathrm{dom}\, T\} \cup \{0^m p\colon\ p \in \mathrm{dom}(V_\varepsilon)\}$, hence

$$\alpha = \sum_{s\in\mathrm{dom}(W)} 2^{-|s|} = \Omega_W. \qquad \square$$

6. Provability of left-computable (ε, V)-random reals

Peano Arithmetic (see [10], shortly, PA) is the first-order theory given by a set of 15 axioms defining discretely ordered rings, together with induction axioms for each formula $\varphi(x, y_1, \dots, y_n)$: $\forall \bar{y}(\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \varphi(x+1, \bar{y})) \to \forall x(\varphi(x, \bar{y})))$.

The proof in [2] can be adapted to show that *every left-computable* (ε, V)*-random real is provable* (ε, V)*-random in* PA. This means the following: if PA is given an algorithm for computing the computable real ε, an algorithm for a machine U, a proof that U is prefix-free and ε-universal, then it can prove that Ω_U is left-computable and (ε, V)-random. This proof requires ε to be defined in terms of primitive recursive functions, which is always possible by a result in [13].[4]

Another representation which can be used to prove (ε, V)-randomness is the following: if PA is given an algorithm for computing the computable real ε, an algorithm for a machine V, a proof that V is prefix-free and ε-universal, a positive integer c, and a computable increasing sequence of rationals converging to a real $\gamma > 0$, then PA can prove that $\alpha = 2^{-c} \cdot \Omega_V + \gamma$ is (ε, V)-random.

Is any "representation" of an (ε, V)-random real enough to guarantee PA provability of (ε, V)-randomness? To answer this question we fix an effective enumeration of all left-computable reals in $(0, 1)$, $\{\gamma_i\}$. Such an enumeration can be based on an enumeration of all increasing primitive recursive sequences of rationals in $(0, 1)$. Our question becomes: based solely on the index i can we always prove in PA that "γ_i is (ε, V)-random real" in case γ_i is (ε, V)-random real? We answer this question in the negative. To this aim we define the following sets:

$$\mathfrak{R}_{\mathrm{lc}} = \{\gamma \in (0,1)\colon \gamma \text{ is left-computable}\} = \{\gamma_i\},$$
$$\mathfrak{R}_{\mathrm{lc}}(\varepsilon, V) = \{\gamma \in \mathfrak{R}_{\mathrm{lc}}\colon \gamma \text{ is } (\varepsilon, V)\text{-random}\},$$
$$\mathfrak{R}^{\mathrm{PA}}_{\mathrm{lc}}(\varepsilon, V) = \{\gamma \in \mathfrak{R}_{\mathrm{lc}}\colon \gamma \text{ is provable } (\varepsilon, V)\text{-random in PA}\}.$$

By enumerating proofs in PA we deduce that the set $\mathfrak{R}^{\mathrm{PA}}_{\mathrm{lc}}(\varepsilon, V)$ is computably enumerable.[5] Is $\mathfrak{R}_{\mathrm{lc}}(\varepsilon, V)$ computably enumerable?

We use Lemma 26 from [2]:

Lemma 17. *If $A \subseteq \mathfrak{R}_{\mathrm{lc}}$ is computably enumerable, then for every left-computable reals $\alpha, \beta \in A$ such that $\beta > \alpha$, we have $\beta \in A$.*

Theorem 18. *The set $\mathfrak{R}_{\mathrm{lc}}(\varepsilon, V)$ is not computably enumerable, so there exists $\alpha \in \mathfrak{R}_{\mathrm{lc}}(\varepsilon, V) \setminus \mathfrak{R}^{\mathrm{PA}}_{\mathrm{lc}}(\varepsilon, V)$.*

Proof. Consider $\alpha \in \mathfrak{R}_{\mathrm{lc}}(\varepsilon, V)$ and define the left-computable real β in the following way. If $\alpha \geqslant 1/2$, then the real $\beta = (\alpha \restriction n)11 \cdots 1 \cdots$ (where $\alpha \restriction (n+1) = 1^n 0$); if $\alpha < 1/2$ consider the left-computable real $\beta = (\alpha \restriction n)11 \cdots 1 \cdots$ (where $\alpha \restriction (n+1) = 0^m 1^{n-m} 0$). In both cases $\beta > \alpha$ and $\beta \notin \mathfrak{R}_{\mathrm{lc}}(\varepsilon, V)$, which shows, by Lemma 17, that $\mathfrak{R}_{\mathrm{lc}}(\varepsilon, V)$ is not computably enumerable, thus concluding the proof. $\square$

In fact, a more precise result is true:

Theorem 19. *For every $\alpha \in \mathfrak{R}_{\mathrm{lc}}(\varepsilon, V)$ there exists an index i such that $\alpha = \gamma_i$ and* PA *cannot prove the statement "γ_i is (ε, V)-random."*

Proof. The set $A_\alpha = \{\gamma_i\colon \alpha = \gamma_i\} \subset \mathfrak{R}_{\mathrm{lc}}(\varepsilon, V) \subset \mathfrak{R}_{\mathrm{lc}}$ is not computably enumerable. $\square$

7. Stay's conjecture

Stay [14] studied generalisations of the statement that Ω_U is random for every 1-universal machine U. In particular he conjectured that Ω_U is $(1, U)$-random for every ε-universal machine U. Although our results show that Ω_U is (ε, V)-random (for a 1-universal machine V; Theorem 3) and the conjecture is true for V_ε (Corollary 4), it turns out that the conjecture itself is too general and does not hold. We provide now a strong counterexample.

Theorem 20. *There exists a $\frac{1}{16}$-universal machine U such that Ω_U is not $(\frac{1}{2}, U)$-random, hence not $(1, U)$-random.*

Proof. Let V be a 1-universal machine. From V and input σ we define $U(\sigma)$ using a parameter τ which satisfies the right-hand side conditions in the following definition:

[4] The proof in [2] has been precisely formalised and mechanically proved in the interactive theorem prover Isabelle [7]. It should be straight forward to adapt this proof to the more general ε-random case.

[5] Recall that a set $A \subseteq \mathfrak{R}_{\mathrm{lc}}$ is computably enumerable if the set $\{i \in \mathbb{N}\colon \gamma_i \in A\}$ is computable enumerable (as a set of non-negative integers). In such a set we enumerate all indices for all elements in A [9].

$$U(\sigma) = \begin{cases} \tau 0^{16n}, & \text{if } \exists n > 0\ [\sigma = 1^n 0\tau \text{ and } |\tau| = 8n], \\ V(\tau), & \text{if } \exists n, m > 0,\ \tau \in \operatorname{dom}(V) \\ & [\sigma = 0\tau 0^n 1,\ |\sigma| = 4^{m+1} \text{ and } |\tau| \leqslant 4^m], \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly, U is a machine. Given $\tau \in \operatorname{dom}(V)$, let $m_\tau = \min\{k > 0\colon\ |\tau| \leqslant 4^k\}$ and $n_\tau = 4^{m_\tau+1} - |\tau| - 2$. Then $U(0\tau 0^{n_\tau} 1) = V(\tau)$ and $|0\tau 0^{n_\tau} 1| \leqslant 16 \cdot |\tau|$, hence U is $\frac{1}{16}$-universal.

Now consider the binary expansion of the halting probability Ω_U. The first bit after the dot is 1 as the strings starting with 1 contribute $\frac{1}{2}$ to the halting probability of U. Furthermore, the strings of length 4^{m+1} starting with a 0 in the domain of U contribute $4^{-m-1} \cdot a_m$ to the halting probability of U; here a_m is the number of strings up to the length 4^m in the domain of V. Because $a_m < 2^{4^m}$ it follows that a_m can be written with 4^m bits. So, in the binary expansion of Ω_U, the bits from the positions $4^m + 1$ until $3 \cdot 4^m$ are all 0; the bits from the positions $3 \cdot 4^{m+1} + 1$ to 4^{m+1} describe the binary value of a_m.

Let $m \geqslant 4$, $8n = 4^m$ and let τ be the string of the first $8n$ bits of Ω_U after the dot. Then $U(1^n 0\tau) = \tau 0^{16n}$ is a prefix of Ω_U of length $24n$ which is generated by the program $1^n 0\tau$ of length $9n + 1$ as Ω_U has 0s on the positions $4^m + 1, \ldots, 3 \cdot 4^m$ and $24n \leqslant 3 \cdot 4^m$. Consequently, Ω_U is not $(\frac{1}{2}, U)$-random. □

8. Conclusion

In this paper we have introduced the notion of ε-universal machine and studied its halting probability. An ε-universal machine is capable of simulating every other machine, but less efficiently than a universal machine V. More precisely, the length of the simulating program on the universal machine is bounded up to a fixed constant by the length of the simulated program divided by ε. The halting probability of an ε-universal machine is left-computable and (ε, V)-random. The main result of this paper is the extension of the representability theorem for left-computable random reals to the case of ε-random reals: *a real is left-computable and (ε, V)-random iff it is the halting probability of an ε-universal machine*. Furthermore, we showed that left-computable ε-random reals are provable (ε, V)-random in Peano Arithmetic, for some, but not all of their representations. Finally we refuted Stay's conjecture stating that Ω_U is $(1, U)$-random provided U is ε-universal.

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