Theoretical Computer Science 87 (1991) 347-352
Elsevier

Note

Relativized topological size of sets of partial recursive functions

Cristian Calude

Department of Mathematics, University of Bucharest, 14 Academiei Str., 70109 Bucharest, Romania

Communicated by G. Ausiello
Received June 1989
Revised June 1990

Abstract

Calude, C., Relativized topological size of sets of partial recursive functions (Note), Theoretical Computer Science 87 (1991) 347-352.

In [1], a recursive topology on the set of unary partial recursive functions was introduced and recursive variants of Baire topological notions of nowhere dense and meagre sets were defined. These tools were used to measure the size of some classes of partial recursive (p.r.) functions. Thus, for example, it was proved that measured sets or complexity classes are recursively meagre in contrast with the sets of all p.r. functions or recursive functions, which are sets of recursively second Baire category. In this paper we measure the size of sets of p.r. functions using the above Baire notions relativized to the topological spaces induced by these sets. In this way we strengthen, in a uniform way, most results of [4, 5, 6, 3, 2], and we also obtain new results. For many sets of p.r. functions, strong differences between "local" and "global" topological size are established.

1. Introduction

Let $N=\{0,1,2,\ldots\}$ be the set of naturals and let $(\varphi_i)_{i\geqslant 0}$ be an acceptable gödelization of P, the set of unary partial (p.r.) recursive functions. For $\varphi\in P$, we put $\operatorname{dom}(\varphi)=\{x\in N \mid \varphi(x) \text{ is defined}\}$, $\operatorname{range}(\varphi)=\{\varphi(x)\mid x\in\operatorname{dom}(\varphi)\}$, and the support of φ, $\operatorname{supp}(\varphi)=\{x\in\operatorname{dom}(\varphi)\mid\varphi(x)\neq 0\}$. A recursively enumerable (r.e.) set is the domain of a p.r. function. By μ we denote the minimization operator and by K, L the (primitive) recursive components of the inverse of Cantor's pairing function. See [2] for more details.

By R and F we denote, respectively, the sets of unary recursive functions and unary recursive functions of finite support. The set F is r.e., i.e. $F=\{\varphi_{h(n)}\mid h\geq 0\}$ for some $h\in R$. The function h will be fixed throughout the paper as well as the recursive function $l(n)=\mathrm{card}(\mathrm{supp}(\varphi_{h(n)}))$.

For φ, θ in P we put $\varphi\subseteq\theta$ in case $\mathrm{supp}(\varphi)\subset\mathrm{supp}(\theta)$ and $\varphi(x)=\theta(x)$, for every x in $\mathrm{supp}(\varphi)$. If, in addition, $\varphi\neq\theta$ then we write $\varphi\subset\theta$.

For $t\in F$ and $X\subset P$ we define the following sets:

$$U_t=\{\varphi\in P\mid t\subseteq\varphi\},\qquad U_t^X=X\cap U_t,$$

$$F(X)=\{t\in F\mid U_t^X\neq\emptyset\}.$$

The sequence $(U_t^X)_{t\in F(X)}$ yields a system of basic neighborhoods in X, thus defining a topology $\tau(X)$ on X. For example, in [4] one studies the space $(R,\tau(R))$, while in [2] one works in $(P,\tau(P))$.

Definition 1. (a) We say that a set $X\subset P$ *has property* (F) if two recursive functions $h(X)$ and $l(X)$ exist, satisfying the following two conditions:

(1) $F(X)=\{\varphi_{h(X)(n)}\mid n\geq 0\}$,

(2) for every natural n, $l(X)(n)=\mathrm{card}(\mathrm{supp}(\varphi_{h(X)(n)}))$.

(b) A set $X\subset P$ with property (F) is called an F_1*-space* iff

(3) there exists a recursive function $s(X)=s$ such that for every natural n, $\varphi_{h(X)(n)}\subset\varphi_{h(X)(s(n))}$.

The set $X=\{f_i\mid i\geq 0\}$, where $f_i(i)=1$ and $f_i(x)=0$, for $x\neq i$, has property (F), but does not satisfy (3). In contrast, P is an F_1-space with respect to $h(P)=h$, $l(P)=l$ (defined above) and $s(P)=s$ coming from the equation

$$\varphi_{h(s(n))}(x)=\begin{cases}\varphi_{h(n)}(x) & \text{if } x\in\mathrm{supp}(\varphi_{h(n)}),\\ 1 & \text{if } x=1+\max\{i\in\mathrm{supp}(\varphi_{h(n)})\},\\ 0 & \text{otherwise.}\end{cases}$$

The fact that the following sets are F_1-spaces is a routine verification: R, $X_{\mathrm{fin}}=\{\varphi\in P\mid\mathrm{range}(\varphi)\text{ is finite}\}$, $M_k=\{\varphi\in P\mid\mathrm{range}(\varphi)\subset\{0,1,\ldots,k\}\}$, $X_k=\{\varphi\in P\mid k\notin\mathrm{range}(\varphi)\}$ (for $k\in N$, $k>0\}$, $P^s=\{\varphi\in P\mid\mathrm{dom}(\varphi)\neq N\}$.

Starting from the definition of recursively meagre sets in $(P,\tau(P))$ (see [1, 2]) and $(R,\tau(R))$ (see [4]) we obtain the following relativized Baire notions.

Definition 2. Let $X\subset P$ be a set with property (F).

(a) A set $Y\subset X$ is *recursively nowhere dense* with respect to $\tau(X)$ (r.n.d. in $\tau(X)$) if $f,g\in R$ exist, satisfying the following four conditions:

(4) for every natural n, $\varphi_{f(n)}\in F(X)$,

(5) for all natural m, n, if $m>g(n)$, then $\varphi_{f(n)}(m)=0$,

(6) for every natural n, $\varphi_{h(X)(n)}\subseteq\varphi_{f(n)}$,

(7) there exists a natural i such that $U^Y_{\varphi_{f(n)}}=Y\cap U^X_{\varphi_{f(n)}}=\emptyset$, whenever $l(X)(n)>i$.

(b) A set $Y \subset X$ is *recursively meagre* with respect to $\tau(X)$ (r.m. in $\tau(X)$) if there exists a sequence $(Y_n)_{n\geq 0}$ of subsets of X and two r.e. sequences of recursive functions $(f_n)_{n\geq 0}$, $(g_n)_{n\geq 0}$ such that

(8) $Y = \bigcup_{n\geq 0} Y_n$,

(9) for every natural n, Y_n is r.m. in $\tau(X)$ under f_n and g_n.

(c) A non-r.m. set in $\tau(X)$ is called a *set of recursively second Baire category* in $\tau(X)$ (s.r.s. Baire c., in $\tau(X)$).

2. Results

First we show that the concepts in Definition 2 do not depend upon the recursive functions $h(X)$ and $l(X)$ satisfying (1) and (2).

Lemma 3. *Let $X \subset P$ and suppose that both pairs of recursive functions $(h(X), l(X))$ and (h', l') satisfy (1) and (2). If $Y \subset X$ is r.n.d. in $\tau(X)$ with respect to the first pair of functions, then Y keeps this property under the second pair.*

Proof. The p.r. function $r(x) = \mu j[\varphi_{h(X)(j)} = \varphi_{h'(x)}]$ is recursive and $\varphi_{h(X)(r(n))} = \varphi_{h'(n)}$, $l'(n) = l(X)(r(n))$. If Y satisfies (4)–(7) under $h(X)$, $l(X)$, f, g, then Y will satisfy these properties under h', l', $f \circ r$ and $g \circ r$. □

Remark. Let $X \subset P$ satisfy (F). Then the family of r.m. subsets in $\tau(X)$ is closed under subset and union, and the family of s.r.s. Baire c. is closed under superset.

Proposition 4. *Let $X \subset Z \subset P$ satisfy (F). Assume that $F(X)$ is a recursive subset of $F(Z)$, i.e. the predicate "there exists a natural m such that $\varphi_{h(Z)(n)} = \varphi_{h(X)(m)}$" is recursive. If $Y \subset X$ is r.n.d. (r.m.) set in $\tau(X)$, then Y is r.n.d. (r.m.) set in $\tau(Z)$.*

Proof. It is sufficient to deal with r.n.d. sets $Y \subset X$ in $\tau(X)$. Clearly, $F(X) \subset F(Z)$ and the recursive function $r(m) = \mu n[\varphi_{h(X)(m)} = \varphi_{h(Z)(n)}]$ satisfies the relation $\varphi_{h(X)(m)} = \varphi_{h(Z)(r(m))}$.

If Y satisfies the properties (4)–(7) under the recursive functions f and g, in $\tau(X)$, then Y is r.n.d. set in $\tau(Z)$ under the recursive functions f^* and g^* defined as follows:

$$\varphi_{f^*(n)}(x) = \begin{cases} \varphi_{f(m)}(x) & \text{if } \varphi_{h(Z)(n)} \in F(X) \text{ and } m = \mu i[\varphi_{h(Z)(n)} = \varphi_{h(X)(i)}], \\ \varphi_{h(Z)(n)} & \text{otherwise.} \end{cases}$$

$$g^*(n) = \begin{cases} g(m) & \text{if } \varphi_{h(Z)(n)} \in F(X) \text{ and } \varphi_{f^*(n)} = \varphi_{h(X)(m)}, \\ \max\{i \in \operatorname{supp}(\varphi_{h(Z)(n)})\} & \text{otherwise.} \end{cases}$$ □

Remark. One can easily check the validity of the following equalities: $F(R) = F(P) = F(X_{\text{fin}}) = F(P^s) = F$, $F(X_k) = F \cap X_k$, $F(M_k) = F \cap M_k$. For various combinations of the sets P, R, X_{fin}, P, X_k and M_k, Proposition 4 applies: it asserts that "small" sets remain "small" when passing to supersets.

Lemma 5. *Let $X \subset P$ be an F_1-space and let $Y \subset X$ be r.n.d. set in $\tau(X)$. Then we can effectively find two recursive functions f', g' satisfying* (4), (5), (7) *and*

(6′) *for every natural n*, $\varphi_{h(X)(n)} \subset \varphi_{f'(n)}$.

Proof. Assume $f, g \in R$ satisfy (4)–(7). There exists a recursive function p such that $\varphi_{f(n)} = \varphi_{h(X)(p(n))} \subset \varphi_{h(X)(s(p(n)))}$, by (3). Put $f'(n) = h(X)(s(p(n))$ and $g'(n) = \max\{i \in \operatorname{supp}(\varphi_{f'(n)})\}$. □

Remark. Obviously, Lemma 5 fails to be true for sets with property (F), not satisfying (3).

Theorem 6. *Let $X \subset P$ be an F_1-space with $F(X)$ infinite and let $Y \subset X$ be r.m. in $\tau(X)$. Then, for every $t \in F(X)$ we can effectively construct a function $f \in U_t^R \setminus Y$.*

Proof. Suppose, by Definition 2(b), that $Y = \bigcup_{n \geq 0} Y_n$, where Y_n is r.m. in $\tau(X)$ under f_n and g_n. Assume, by Lemma 5, that $\varphi_{h(X)(n)} \subset \varphi_{f(n)}$, for every natural n. Let $t \in F(X)$, i.e. $t = \varphi_{h(X)(q)}$, for some natural q. The predicate $Q(i, j, n) = 1$ iff $\varphi_{f_i(n)} = \varphi_{h(X)(j)}$, is recursive. The p.r. function r defined by

$$r(0) = q, \qquad r(x+1) = \mu j[Q(K(x), j, r(x)) = 1],$$

is recursive because for all naturals i and n, there exists $j \in N$ such that $Q(i, j, n) = 1$.

Next we construct the sequence $(t_m)_{m \geq 0}$, $t_m \in F(X)$ as follows:

$$t_0(x) = t(\dot{x}), \qquad t_{m+1}(x) = \varphi_{f_{K(m)}(r(m))}(x), \quad m \geq 0.$$

Clearly, $t_m \subset t_{m+1}$, for all $m \in N$. Accordingly, the function $f: N \to N$ given by $f(x) = t_m(x)$, if $x \leq g_{K(m)}(r(m))$ is well defined and $f \in R \cap U_t$. To show that $f \notin Y$, assume, to the contrary, that $f \in Y_i$, for some natural i. For every n with $l(X)(n) > i$,

$$Y_i \cap U^X_{\varphi_{f_i}(n)} = \emptyset.$$

Taking m such that $l(X)(r(m)) > n_i$ and $K(m) = i$, we have a contradiction. □

Remark. The above result generalizes Theorem 1 in [1] (see also Theorem 9.12 in [2]). It can be used to strengthen the basic results in [2, 5, 6, 4] (in particular, the recursive variant of Baire Category Theorem).

Corollary 7. *Let $X \in \{P, R, X_k, M_k\}$, $k \geq 1$, and let $t \in F(X)$. Then, the sets* (i) X, (ii) $X \cap R$, (iii) U_t^X, (iv) *every non-empty open set in $\tau(X)$, are s.r.s. Baire c. in $\tau(X)$.*

Corollary 8. *For every natural $k \geq 1$, X_k and M_k are s.r.s. Baire c. in $\tau(X_{k+1})$, but they are r.n.d. in $\tau(P^s)$ and $\tau(P)$.*

Proposition 9. (a) *For every natural $k > 0$, M_k is r.n.d. in $\tau(X_{\text{fin}})$, $\tau(P^s)$ and $\tau(P)$.*

(b) *The set X_{fin} is r.m. in $\tau(X_{\text{fin}})$, $\tau(P^s)$ and $\tau(P)$.*

(c) *The set P^s is r.m. in $\tau(P^s)$ and $\tau(P)$.*

Proof. One has $X_{\text{fin}}=\bigcup_{k\geq 0} M_k$ and every M_k is r.n.d. in $\tau(X_{\text{fin}})$ under the functions $f(k, n)$ and $g(k, n)$ given by

$$\varphi_{f(k,n)}(x)=\begin{cases}\varphi_{h(n)}(x) & \text{if } x\leq l(n),\\ k+1 & \text{if } x=l(n)+1,\\ 0 & \text{otherwise,}\end{cases}$$

and $g(k, n)=l(n)+1$. The last assertions in (a), (b) and (c) follow from Proposition 4 and the proof that P^s is r.m. in $\tau(P^s)$ can be obtained mutatis mutandis from the proof of Theorem 2 in [5], because $F(X_{\text{fin}})=F$. □

Remarks. (a) Proposition 4 states an intuitive fact. By contrast, some later results point out facts which are somewhat against the intuition. For example, M_k is s.r.s. Baire c. in $\tau(X_{k+1})$ and $\tau(M_k)$, but it is r.n.d. in $\tau(P^s)$ and $\tau(P)$.

(b) The set R is a recursive residual in $\tau(P)$, because $P^s=P\backslash R$ is r.m. in $\tau(P)$ (see [5, 6]). From Proposition 9(c), P^s is r.m. in $\tau(P^s)$; this fact reinforces our intuition that R is a topologically "big" set.

We recall that a measured set is a $X=\{m_i \mid i\geq 0\}\subset P$ for which the predicate $M(i, x, y)=1$ iff $m_i(x)=y$, is recursive. In what follows we are interested in measured sets X such that $F(X)=F$. For example, the set of primitive recursive functions satisfies the above conditions. Clearly, measured sets for which $F(X)\subsetneq F$ exist. From [1, 2] one knows that every measured set is r.m. in $\tau(P)$. This result can be strengthened for our class of measured sets as follows.

Theorem 10. *Let $X\subset P$ be a measured set with $F(X)=F$. Then X is r.m. in $\tau(X)$. If $X\subset R$, then X is r.m. in $\tau(R)$.*

Proof. Let $X=\{m_i \mid i\geq 0\}$ and define the p.r. function

$$p(i, n, x)=\begin{cases}\varphi_{h(n)}(x) & \text{if } x\in \operatorname{supp}(\varphi_{h(n)}),\\ 1 & \text{if } x=l(n)+1 \text{ and } \sum_{y=0}^{x+2} M(i, x, y)=0,\\ x+3 & \text{if } x=l(n)+1 \text{ and } \sum_{y=0}^{x+2} M(i, x, y)\geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

In view of the s-m-n theorem we obtain a recursive function s such that $p(i, n, x)=\varphi_{s(i,n)}(x)$. Put $f_i(n)=s(i, n)$ and $g_i(n)=l(n)+1$, $X_i=\{m_i\}$. Each X is r.n.d. in $\tau(X)$ under f_i and g_i; properties (4)–(6) are obviously fulfilled and for (7) we note that for every natural n with $l(n)>0$, $0\neq \varphi_{f_i(n)}(l(n)+1)\neq m_i(l(n)+1)$, so $\varphi_{f_i(n)}\subsetneq m_i$. □

Consider the following sets: PR = the set of primitive recursive functions; PR(f) = the set of primitive recursive functions in $f\in R$.

Corollary 11. *The following assertions are true*:

(a) *The set F is r.m. in $\tau(F)$, $\tau(\text{PR})$, $\tau(\text{PR}(f))$, $\tau(R)$* and *$\tau(P)$.*

(b) *the set* PR *is r.m. in $\tau(\text{PR})$, $\tau(\text{PR}(f))$, $\tau(R)$ and $\tau(P)$.*

(c) *each set* PR(f) *is r.m. in $\tau(\text{PR}(f))$, $\tau(R)$, $\tau(P)$.*

Corollary 12. *Every complexity class $C \subset R$ for which $F \subset C$ is r.m. in $\tau(C)$, $\tau(R)$ and $\tau(P)$.*

Proof. A complexity class C for which $F \subset C$ is a measured set and $F(C) = F$. □

Corollary 13. *Both sets of rational and algebraic numbers are r.m. in $\tau(R)$ and $\tau(P)$.*

Proof. These sets are contained in the complexity class containing F (see [2]). □

Some open problems naturally arise. We present some of them:

(i) Does Proposition 4 remain true in case we delete the hypothesis "$F(X)$ is a recursive subset of $F(Z)$"?

(ii) Let $X \subset P$ be an arbitrary measured set; is it true that X is r.m. in $\tau(X)$?

(iii) Let $X \subset Z \subset P$, X, Z satisfying property (F). Assume that X is a s.r.s. Baire c. in $\tau(X)$ and $Y \subset Z$ is a set r.s. Baire in $\tau(Z)$. Is it true that if $X \cap Y \neq \emptyset$, then $X \cap Y$ is a s.r.s. Baire c. in $\tau(X)$?

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