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Note

Determining and stationary sets for some classes of partial recursive functions

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Abstract

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In analogy with the case of real functions we introduce and study the determining and stationary sets for some classes of unary p.r. functions, including the recursive and primitive recursive functions. As a by-product, a new characterization of Post simple sets is obtained, which offers a natural motivation for their name.

1. Introduction

Let $X=\{a_1,\ldots,a_p\}$, $p\geqslant 1$, be a fixed set and let X^* be the free monoid generated by X (e is the empty string). We shall deal only with unary partial functions $f:X^*\xrightarrow{o}X^*$. The domain and the range of a partial function f are respectively denoted by dom(f) and range(f). Two partial functions f, g are equal in case $\mathrm{dom}(f)=\mathrm{dom}(g)$ and $f(x)=g(x)$ for all $x\in\mathrm{dom}(f)$; they are equal almost everywhere (written $f=g$ a.e.) if $\mathrm{dom}(f)\,\Delta\,\mathrm{dom}(g)$ is finite (Δ refers to the set-theoretic symmetric difference) and $f(x)=g(x)$, for all but finitely many x in $\mathrm{dom}(f)\cap\mathrm{dom}(g)$; finally, if $E\subset\mathrm{dom}(f)\cap\mathrm{dom}(g)$ and $f(x)=g(x)$ for all but finitely many x in E, then we say that f and g are equal almost everywhere on E (written $f=g$ a.e. on E).

For recursive function theory we refer to [4, 8, 9, 10]; we shall use the usual abbreviations (i.e., p.r. and r.e. denote, respectively, partial recursive and recursively enumerable).

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The determining and stationary sets were introduced in [1] for classes of real functions (see for a monographical presentation [2]). In analogy with these studies we introduce the determining and stationary sets for a class F of partial functions $f: X^* \xrightarrow{0} X^*$. A *determining set* for F is a subset E of X^* such that two arbitrary elements f, g in F are equal a.e. whenever they are equal a.e. on E. A subset E of X^* with the property that a partial function f in F which is constant a.e. on E is constant a.e. is called a *stationary set* for F. The discrete nature of our partial functions motivates the introduction of the "a.e." restriction, not present in the case of real functions. Clearly, every cofinite set is both a determining and a stationary set; it will be called a *trivial determining* (*stationary*) set. Accordingly, we will be mainly interested in the study of nontrivial determining (stationary) sets.

2. Results

Let $W \subset X^*$ be an infinite set and consider F to be a subset of all partial functions $f: X^* \xrightarrow{0} X^*$ such that $\mathrm{dom}(f) \,\Delta\, W$ is finite. We shall assume that F contains the *identity function* $I: W \to X^*$, $I(x) = x$, the *constant functions* $C_y: W \to X^*$, $C_y(x) = y$, $y \in X \cup \{e\}$, and is closed under *concatenation* (if $f, g \in F$, then $h: \mathrm{dom}(f) \cap \mathrm{dom}(g) \to X^*$, $h(x) = f(x)g(x)$ is in F) and *definition by cases* (if $f, g \in F$, then $h: \mathrm{dom}(f) \cap \mathrm{dom}(g) \to X^*$, $h(x) = a_1$ if $f(x) = g(x)$ and $h(x) = e$ otherwise, is in F).

The following classes of partial functions satisfy the above requirements: (i) the class of recursive functions, (ii) the class of primitive recursive functions, (iii) each class in Ackermann-Peter's hierarchy (i.e., each class of unary functions in Grzegorczyk's hierarchy; see [4]), (iv) the class of all p.r. functions $f: X^* \xrightarrow{0} X^*$ for which $\mathrm{dom}(f) \,\Delta\, W$ is finite (W is a fixed infinite r.e. subset of X^*). Notice that in the case $p = 1$ concatenation coincides with addition.

Fix now a family F satisfying the above requirements. First we prove two technical facts.

Lemma 1. *The following partial functions are in* F:

(i) $C_y: W \to X^*$, $C_y(x) = y$ *for every* $y \in X^*$,

(ii) $\overline{\mathrm{sg}}$, $\mathrm{sg}: W \cup \{e\} \to X^*$, $\overline{\mathrm{sg}}(e) = \mathrm{sg}(x) = a_1$, $\overline{\mathrm{sg}}(x) = \mathrm{sg}(e) = e$ *for every* $x \in W - \{e\}$.

Proof. (i) Use the closure under concatenation.

(ii) One has $\overline{\mathrm{sg}}(x) = a_1$ if $I(x) = C_e(x)$, $\overline{\mathrm{sg}}(x) = e$ otherwise, $\mathrm{sg}(x) = a_1$ if $\overline{\mathrm{sg}}(x) = C_e(x)$, $\mathrm{sg}(x) = e$ otherwise; the closure under definition by cases assures that $\overline{\mathrm{sg}}$, $\mathrm{sg} \in F$. □

Lemma 2. (i) *If* $f \in F$, *then the composition* $\mathrm{sg} \circ f$, $\overline{\mathrm{sg}} \circ f: \mathrm{dom}(f) \cup \{e\} \to X^*$ *are in* F.

(ii) *The characteristic function* (*with respect to* W) *of each finite subset* $A \subset X^*$, $\chi_A: W \to X^*$, $\chi_A(x) = a_1$ *if* $x \in A$ *and* $\chi_A(x) = e$ *otherwise, lies in* F.

Proof. (i) Obviously, $\overline{\mathrm{sg}}(f)(x)=a_1$ if $f(x)=C_e(x)$, $\overline{\mathrm{sg}}(f)(x)=e$ otherwise, and similarly for $\mathrm{sg}(f)$.

(ii) In case $A=\{x_1,\dots,x_k\}\subset X^*$ we define $\chi_A(x)=\mathrm{sg}(h_1(x)\dots h_k(x))$, where $h_i(x)=a_1$, if $C_{x_i}(x)=I(x)$ and $h_i(x)=e$ otherwise. □

Theorem 3. *Let $E\subset W$. The following assertions are equivalent*:

(1) *The set E is a stationary set for F.*

(2) *The set E is a determining set for F.*

(3) *The set E is infinite and no infinite subset $G\subset W-E$ has the characteristic function $\chi_G: W\to X^*$ in F.*

(4) *The set E is infinite and no set $H\supset E$ with $W-H$ infinite has the characteristic function $\chi_H: W\to X^*$ in F.*

Proof. (1)⇒(2). Assume E is a stationary set for F. Let f, g be two partial functions in F such that $f=g$ a.e. on E. Define $h:\mathrm{dom}(f)\cap\mathrm{dom}(g)\to X^*$ by $h(x)=a_1$ if $f(x)=g(x)$ and $h(x)=e$ otherwise, and notice that h lies in F. Clearly, $f=h$ a.e. iff $h=a_1$ a.e., so E is a determining set for F.

(2)⇒(3). First we prove that E is infinite. If, ab absurdo, E would be finite, then, by Lemma 2, $\chi_E: W\to X^*$ would be in F. So $\chi_E=C_{a_1}$ on E, $C_{a_1}\in F$, but $\chi_E(x)\neq C_{a_1}(x)$ for infinitely many x in $W-E$, contradicting the hypothesis.

Suppose now that $G\subset W-E$ is an infinite set and $\chi_F: W\to X^*$ is in F. Let $f\in F$ and define $g:\mathrm{dom}(f)\to X^*$ by concatenation: $g(x)=f(x)\chi_F(x)$; clearly, $g\in F$. For every x in E, $\chi_G(x)=e$, so $g(x)=f(x)$ $(G\subset W-E)$. On the other hand, $g(y)=f(y)a_1\neq f(y)$ for infinitely many $y\in G$, a contradiction.

(3)⇒(4). Let $E\subset H$ with $W-H$ infinite. Since $G=W-H\subset W-E$, by (3), $\chi_G\notin F$. It follows that $\chi_H\notin F$ because $\chi_H=\overline{\mathrm{sg}}(\chi_G)$ and Lemma 2.

(4)⇒(1). If $W-E$ is finite, then E is clearly a stationary set. Assume that $W-E$ is infinite and suppose ab absurdo that there exist f in F and u in X^* such that the set $A=\{x\in W\cap\mathrm{dom}(f)\mid f(x)\neq u\}$ is infinite but the set $B=\{x\in E\cap\mathrm{dom}(f)\mid f(x)\neq u\}$ is finite. Notice that the set $D=A\cap(W-E)$ is infinite, $E\subset W-D$ and $\chi_{W-D}: W\to X^*$ belongs to F (reason: $\chi_{W-D}(x)=\mathrm{sg}(h(x)\chi_{A\cap E}(x))$, where $h: W\to X^*$ is given by $h(x)=a_1$ if $f(x)=C_u(x)$, $h(x)=e$ otherwise, and $A\cap E=B$ is finite, so by Lemma 2 its characteristic function is in F).

The set $H=W-D$, $W-H=D$ is infinite and $\chi_H\in F$, a contradiction. □

Corollary 4. *Let $E\subset W$. If E is a stationary (or, equivalently, determining) set for F, then χ_E is in F iff $W-E$ is finite.*

Proof. By Lemma 2, if $W-E$ is finite, then $\chi_E=\overline{\mathrm{sg}}(\chi_{W-E})$ is in F. Conversely, if E is a determining set for F and $\chi_E\in F$, then $\chi_E=C_{a_1}$ on E, so $\chi_E=C_{a_1}$ a.e., i.e., $W-E$ is finite. □

Recall that a co-immune set is a set with immune complement (i.e., no infinite subset of its complement can be r.e. [7, 8, 9]).

Theorem 5. *The nontrivial determining (stationary) sets for the class of recursive functions are exactly the co-immune subsets of X^*.*

Proof. Let $E \subset X^*$ be a nontrivial determining set for the class of recursive functions. Suppose $A \subset X^* - E$ is an infinite r.e. set; in view of a well known result [4, 7, 8] there exists an infinite recursive subset $B \subset A$, thus contradicting Theorem 3(3). Conversely, if E is co-immune, then no infinite subset of $X^* - E$ can be r.e., hence recursive, so by Theorem 3(3) E is a determining set. □

Example 6. Every Post simple set (i.e. every r.e. co-immune set [8, 9]) is a determining set for the class of recursive functions. For example, the set of Kolmogorov nonrandom strings [4, 7] is such a set.

Example 7. Every bi-immune set (i.e. a set both immune and co-immune [7, 8]) is a determining set for the class of recursive functions. Accordingly, the family of determining sets for the class of recursive functions has the power of the continuum.

Theorem 8. *The family of nontrivial determining sets for the class of recursive predicates coincides with the family of co-immune sets. Again, "determining" is equivalent to "stationary" for this class.*

Proof. In contrast with the proof of Theorem 5, we cannot rely on Theorem 3 in this case. Given a nontrivial determining set $E \subset X^*$ and an infinite recursive subset $G \subset X^* - E$ we pick an arbitrary recursive predicate $f: X^* \to \{a_1, e\}$ and we construct the recursive predicate

$$g(x) = \begin{cases} f(x) & \text{if } x \notin G, \\ a_1 & \text{if } x \in G \text{ and } f(x) = e, \\ e & \text{if } x \in G \text{ and } f(x) = a_1. \end{cases}$$

Clearly, $f(x) = g(x)$ for all $x \in E \subset X^* - G$, but $f(x) \neq g(x)$ for infinitely many x in G. The converse implication follows from Theorem 5. □

Remark. Theorems 5 and 8 show that classes having different recursive topological size (the recursive functions form a recursively second Baire category set, while the set of recursive predicates is recursively meagre [3, 4]) have the same family of determining sets. This is not a general phenomenon as Theorem 10 will prove.

Corollary 9. *A set $E \subset X^*$ is Post simple iff E is a nontrivial r.e. determining (stationary) set for the class of recursive functions (predicates).*

Remark. We may ask for the "simplest" determining sets for a class F. Corollary 9 shows that Post simple sets are really the "simplest" nontrivial determining sets for the family of recursive functions (predicates).

Theorem 10. *The nontrivial determining (stationary) sets for the class of primitive recursive functions are exactly the sets containing in their complement no infinite primitive recursive set.*

Proof. Use Theorem 3(3). □

Remark. It is interesting to notice that there exist co-infinite recursive sets $A \subset X^*$ for which there is no infinite primitive recursive set $B \subset A$ or $B \subset X^* - A$ [6]. Accordingly, the class of primitive recursive functions (a recursively meagre set [4]) has more determining sets than the family of recursive functions.

Final comment. It will be interesting to characterize the determining and stationary sets for other classes of partial functions, particularly for various types of continuous functions [5] or morphisms.

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