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A Combinatorial Characterization of Sequential P. Martin-Löf Tests

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Dedicated to Professor S. Marcus, for his 60th anniversary

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We give a combinatorial characterization of sequential P. Martin-Löf tests within the class of all P. Martin-Löf tests.

KEY WORDS: (Sequential) P. Martin-Löf tests, prefix-free set.
C.R. CATEGORIES: F.4.1, G.3

1. PREREQUISITES

The set of natural numbers is \( \mathbb{N} = \{0, 1, 2, \ldots \} \).

Let \( X = \{a_1, a_2, \ldots, a_p\} \), \( p \geq 2 \) be a finite alphabet. Denote by \( X^* \) the free monoid generated by \( X \) (the elements of \( X^* \) are called strings; \( \lambda \) is the empty string). If \( x = x_1 x_2 \ldots x_n \) is in \( X^* \), then the length of \( x \) is \( l(x) = n; \quad l(\lambda) = 0 \). For all \( x \) and \( y \) in \( X^* \) we write \( x \leq y \) in case there exists a string \( z \) in \( X^* \) such that \( y = xz \). If \( a \in X \) and \( n \) is natural, we write \( a^n = \lambda \) (if \( n = 0 \)) and \( a^n = aa \ldots a \) (\( n \) times, if \( n > 0 \)).

A non-empty r.e. set \( V \subset X^* \times (\mathbb{N} - \{0\}) \) is called P. Martin-Löf test (ML test) if it possesses the following two properties (see [3] and [1]):

1) For every natural \( m \geq 1 \), \( V_{m+1} \subset V_m \). Here \( V_m = \{ x \in X^* | (x, m) \in V \} \).
2) For all naturals $n, m, m \geq 1$, one has
\[
\text{card } \{x \in X^* | l(x) = n, (x, m) \in V\} < p^{n-m}/(p-1).
\]

We shall agree upon the fact that the empty set $\emptyset$ is a M-L test.

The critical level induced by a M-L test $V$ is the function $m_V : X^* \rightarrow \mathbb{N}$ given by $m_V(x) = \max (m \geq 1 | (x, m) \in V)$, in case such an $m$ exists, and $m_V(x) = 0$, in the opposite case.

2. RESULTS

The P. Martin-Löf tests were introduced in [3] in order to give a statistical interpretation of the Kolmogorov complexity-theoretic notion of random string [2]. The sequential P. Martin-Löf tests ([3]) are designated to play the same role in the study of random sequences (see [4, 5, 6, 7]).

Our aim is to give a complete characterization of the class of sequential P. Martin-Löf tests within the larger class of P. Martin-Löf tests.

Firstly, we give the following definition (see [3] and [4]):

DEFINITION 1 A r.e. set $V \subseteq X^* \times (\mathbb{N} - \{0\})$ satisfying the properties:

1) If $(x, m) \in V, y \gg x$ and $1 \leq n \leq m$, then $(y, n) \in V$,
2) For all natural $m, n \geq 1$, we have
\[
\text{card } \{x \in X^* | l(x) = n, (x, m) \in V\} < p^{n-m}/(p-1),
\]

is called a sequential P. Martin-Löf test (s. M-L test in the sequel).

We shall agree upon the fact that the empty set $\emptyset$ is a s. M-L test.

It is easy to see that every s. M-L test is an infinite M-L test (of course, the converse is not true).

In the sequel we shall constantly use the following notations:

a) For every $x \in X^*$ and natural $m \geq 1$, $H(x, m) = \{(x, 1), (x, 2), \ldots, (x, m)\}$. It is easy to see that $H(x, m)$ is a M-L test iff $l(x) > m$.

b) For every $A \subseteq X^* \times \mathbb{N}$, we put $\bar{A} = \emptyset$, in case $A$ is empty, and
\( \mathcal{A} = \{(y, m) \mid \text{there exists } (x, m) \in \mathcal{A} \text{ such that } y \supseteq x \} \). It is easy to see that the map \( A \mapsto \mathcal{A} \) is a closure operator which preserves recursive enumerability. Moreover, one has \( \bigcup_{i \in I} \mathcal{A}_i = \bigcup_{i \in I} A_i \), for each family \( \{A_i\}_{i \in I} \) of subsets of \( X^* \times \mathbb{N} \).

**Lemma 2** Let \( x \in X^* \) and \( m \in \mathbb{N}, m \geq 1 \). The following assertions are equivalent:

1) The set \( H(x, m) \) is a s. M-L test.
2) The set \( H(x, m) \) is a M-L test.

**Proof** Only the implication "(2) \( \Rightarrow \) (1)" needs a careful check. Our hypothesis is \( l(x) > m \) and we must only prove property (2) in Definition 1.

Let \( n \geq 1 \) and \( q \geq 1 \). We must prove the inequality:

\[
C = \text{card} \{ y \in X^* \mid (y(q) = n, (y, q) \in H(x, m)) \} < p^{n - q} / (p - 1),
\]

which is obvious for \( q > m \) or \( n \geq m \). Consequently, (1) must be checked only for \( n > m \geq q \), and it is sufficient to prove that

\[ C < p^{n - m} / (p - 1). \]

The last inequality can be written

\[
p^{n - l(x)} < p^{n - m} / (p - 1),
\]

(because \( (y, q) \in H(x, m) \) iff \( y \supseteq x \)).

Finally, inequality (2) is equivalent to \( l(x) > m \). ❑

The following example shows that what happens in Lemma 2 cannot happen in more general conditions:

a) The closure \( \mathcal{P} \) of a M-L test is not always a (sequential) M-L test.

b) The closure of a finite union of M-L tests of the form \( H(x, m) \) is not always a (sequential) M-L test.

**Example 3** Take \( X = \{0, 1\} \) and \( H = H(00, 1) \cup H(11, 1) \cup H(10, 1) = \{(00, 1), (11, 1), (10, 1)\} \). Clearly, \( H \) is a M-L test, but \( \mathcal{P} \) is not a M-L test, containing the elements \( (00, 1), (00, 1), (11, 1), (10, 1) \). ❑
In order to be shorter in the sequel, we introduce

**Definition 4.** A non-empty subset \( A \subset X^* \) is called **prefix-free** if for all \( x, y \) in \( A \) such that \( x \subseteq y \) we have \( x = y \).

It is seen that every non-empty set \( A \subset X^* \) contains prefix-free subsets, e.g., its singletons.

The following theorem furnishes necessary and sufficient conditions for the closure of a finite union of M-L tests \( H(x, m) \) to be a s. M-L test.

**Theorem 5.** Let \( x_1, \ldots, x_k \) in \( X^* \) and \( m_1, m_2, \ldots, m_k \geq 1 \) be natural numbers. Put \( H = \bigcup_{i=1}^{k} H(x_i, m_i) \).

The following conditions are equivalent:

1) The set \( H \) is a s. M-L test.

2) We have simultaneously:
   a) The set \( H \) is a M-L test.
   b) For every prefix-free subset \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \subset \{x_1, x_2, \ldots, x_k\} \) one has

   \[
   \sum_{i=1}^{r} p^{-l(x_{i_i})} < p^{-\min(m_1, m_2, \ldots, m_r)}/(p-1).
   \]

**Proof.**

"(1)\Rightarrow(2)." Let \( 1 \leq i_1 < i_2 < \ldots < i_r \leq k \) be such that the set \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \) is prefix-free. Let a natural \( n \geq \max(l(x_{i_1}), l(x_{i_2}), \ldots, l(x_{i_r})), m = \min(m_1, m_2, \ldots, m_r) \). Then, because the sets \( H(x_{i_u}, m_{i_u}), u = 1, 2, \ldots, r, \) are disjoint, we have

\[
\text{card} \left\{ x \in X^* \mid l(x) = r, (x, m) \in B = \bigcup_{j=1}^{n} H(x_{i_j}, m_{i_j}) \right\} = \sum_{j=1}^{n} p^{-l(x_{i_j})} < p^{-m}/(p-1),
\]

since \( B \) is a M-L-test a.s.o.

"(2)\Rightarrow(1)." We shall proceed by induction upon \( k \). For \( k = 1 \) the result has been obtained in Lemma 2.

Consider the result to be valid for some natural number \( k > 1 \) and let us prove it for \( k+1 \). So, let \( \bar{H} = \bigcup_{j=1}^{k+1} H(x_j, m_j) \). We may assume that \( l(x_{k+1}) = \max(l(x_1), l(x_2), \ldots, l(x_{k+1})) \).
Recall that \( B(m, n) = \{ x \in X^* | l(x) = n \text{ and } (x, m) \in \mathcal{H} \} \), for all natural \( m, n \geq 1 \). All it remains to be proved is that for all \( n > m \geq 1 \) one has

\[
\text{card } B(m, n) < p^{n-m}(p-1).
\]

(3)

The first case: \( n < l(x_{k+1}) \). In this situation

\[
B(m, n) = \left\{ x \in X^* | l(x) = n, (x, m) \in \bigcup_{i=1}^{k} H(x_i, m) \right\}
\]

and (3) holds using the induction hypothesis.

The second case: \( n \geq l(x_{k+1}) \). The situation \( m > m_{k+1} \) is similar to the first case, because \( B(m, n) \) will be the same.

From now on, we shall consider that

\[
l(x_1) \leq l(x_2) \leq \ldots \leq l(x_k) \leq l(x_{k+1}).
\]

Therefore, let us suppose that \( m \leq m_{k+1} \). Under these conditions we divide the proof according to the fact that \( x_{k+1} \rightarrow x_i \) for some \( 1 \leq i \leq k \), or not.

\( a ) \) Assume first that \( x_{k+1} \not\rightarrow x_i \) for all \( 1 \leq i \leq k \). If \( m \leq \min (m_1, m_2, \ldots, m_k) \), then

\[
B(m, n) = \bigcup_{i=1}^{k+1} \{ x \in X^* | l(x) = n, (x, m) \in H(x_i, m) \}.
\]

To simplify facts, notice that in case \( x_u \not\rightarrow x_i \), we have \( H(x_u, m) \subseteq H(x_i, m) \). Using this remark we eliminate all \( H(x_i, m) \) such that \( x_i \subset x_u \) (if such \( x_i \) does exist) a.s.0. and finally we get a prefix-free subset

\[
\{ y_1, y_2, \ldots, y_j \} \text{ of } \{ x_1, x_2, \ldots, x_k, x_{k+1} \}
\]

with the property that

\[
\bigcup_{q=1}^{j} H(y_q, m) = \bigcup_{i=1}^{k+1} H(x_i, m).
\]
Then

\[ B(m, n) = \bigcup_{q=1}^{q} \{ x \in X^* | l(x) = n, (x, m) \in H(y_q, m) \}. \]

Because the sets \( H(y_q, m) \) are disjoint, one gets

\[
\text{card } B(m, n) \sum_{q=1}^{q} p^{n-1(y_q)} = p^n \left( \sum_{q=1}^{q} p^{-1(y_q)} \right) < p^n \cdot p^{-\min(m_1, m_2, \ldots, m_k)}/(p-1) \leq p^n \cdot p^{-m}/(p-1) = p^n/(p-1).
\]

Here we have considered that

\[ y_1 = x_{u_1}, y_2 = x_{u_2}, \ldots, y_j = x_{u_j}. \]

If \( m > \min(m_1, m_2, \ldots, m_k) \), then we get

\[ B(m, n) = \bigcup_{a=1}^{q} \{ x \in X^* | l(x) = n, (x, m) \in H(x_{i_a}, m_{i_a}) \} \]

\[ \cup \{ x \in X^* | l(x) = n, (x, m) \in H(x_{k+1}, m_{k+1}) \}, \]

where \( \{ m_1, m_2, \ldots, m_k \} = \{ m_i | m_i \geq m, i = 1, 2, \ldots, k \} \).

Proceeding in the same manner, we eliminate some \( H(x_{i_a}, m_{i_a}) \) and we get again a prefix-free subset

\[ \{ y_1, y_2, \ldots, y_j, x_{k+1} \} \]

such that

\[ \bigcup_{u=1}^{q} H(x_{i_u}, m) \cup H(x_{k+1}, m_{k+1}) = \bigcup_{r=1}^{r} H(y_r, m) \cup H(x_{k+1}, m_{k+1}). \]

The computation of card \( B(m, n) \) is similar to that one in the case \( m \leq \min(m_1, m_2, \ldots, m_k) \).
Assume now that there exists $1 \leq j \leq k$ such that $x_{k+1} \supset x_j$.

Put $I = \{ i \in \mathbb{N} | 1 \leq i \leq k + 1, m \leq m_i \}$ and notice that $k + 1 \in I$, according to the hypothesis. It is seen that

$$B(m, n) = \begin{cases} x \in X^* | l(x) = n, (x, m) \in \bigcup_{i \in I} H(x_i, m_i) \\ x \in X^* | l(x) = n, (x, m) \in \bigcup_{i \in I} \overline{H(x_i, m)} }.$$  

In any case, the set $\bigcup_{i \in I} H(x_i, m)$ is a M-L test, being contained in the M-L test $\bigcup_{j=1}^{n+1} H(x_j, m_j)$.

There are two possibilities:

1. There exists $j$ in $I$ such that $x_j \subset x_{k+1}$. We have $H(x_{k+1}, m) \subset H(x_j, m)$ and the union $\bigcup_{i \in I} H(x_i, m)$ has in fact at most $k$ terms.

2. For all $i \in I$, one has $x_i \supset x_{k+1}$. In this case we pick some $1 \leq j \leq k$ such that $x_j \subset x_{k+1}$ and we notice that $j \notin I$. Consequently, the union $\bigcup_{i \in I} H(x_i, m)$ has also at most $k$ terms.

In both cases, the induction hypothesis, applied to the M-L test $\bigcup_{i \in I} H(x_i, m)$ furnishes the inequality $\text{card} B(m, n) < p^{n-m}(p-1)$.

This completes the proof.

**Corollary 6** Condition 1) in Theorem 5 can be effectively checked.

We have seen that every non-empty set $A \subset X^*$ contains prefix-free subsets. The following lemma gives more precise results in the case when the only prefix-free subsets are the singletons, i.e. $r$ in the statement of Theorem 5 must be equal to one. The reader can relate the results obtained in Scolium 7, Theorem 5 and Lemma 2.

**Scolium 7** Let $y_1, y_2, \ldots, y_k$ in $X^*$ and $n_1, n_2, \ldots, n_k \geq 1$ in $\mathbb{N}$. We assume that for all $1 \leq i < j \leq k$ we have either $y_i \subset y_j$ or $y_j \subset y_i$. Put $H = \bigcup_{i=1}^{k} H(y_i, n_i)$. Then, the following conditions are equivalent:

1) The set $H$ is a s.M-L test.
2) The set $H$ is a M-L test.
3) The sets $H(y_i, n_i)$ are all M-L tests.
Proof Because the case $k=1$ has already been discussed (see Lemma 2), we shall consider that $k>1$.

The implication "(1)⇒(3)" being obvious, we confine ourselves to prove "(3)⇒(2)" and "(2)⇒(1)".

The set $\{y_1, y_2, \ldots, y_k\}$ being totally ordered by $\preceq$, we rewrite it in the form $\{x_1, x_2, \ldots, x_k\}$, where $x_1 \preceq x_2 \preceq \ldots \preceq x_k$, and we write also $(x_i, m_i)$ instead of $(y_j, n_j)$ in case $x_i = y_j$ in the new form. We can also suppose that $x_1, x_2, \ldots, x_k$ are distinct (separate trivial considerations for (1), (2) and 3))

"(3)⇒(2)". We have to check that $\text{card } A(m, n) < p^n - m / (p-1)$, for $n > m$, where $A(m, n) = \{x \in X^* | l(x) = n, (x, m) \in H\}$. This is obvious in case $n \neq l(x_i), i = 1, 2, \ldots, k$. In case $n = l(x_i)$, this $i$ is unique (because of total ordering). If $m > m_i$ then $A(m, n) = \{x_i\}$ and $1 < p^n - m / (p-1)$, because $n = l(x_i) > m$.

"(2)⇒(1)". Assume $H$ is a M-L test. Only condition (2) in Definition 1 is to be checked.

For all natural $m, n \geq 1$, we put $B(m, n) = \{x \in X^* | l(x) = n, (x, m) \in H\}$. We must prove that card $B(m, n) < p^n - m / (p-1)$, only in the non trivial case $m \leq \max (m_1, m_2, \ldots, m_k)$ (i.e. $B(m, n) \neq \emptyset$). Two non trivial cases are to be considered:

I) There exists $1 \leq i \leq k-1$, such that $l(x_i) \leq n < l(x_{i+1})$.

II) One has $l(x_k) \leq n$.

I) In case $m > m_u$, for all natural $u \leq i$, one has $B(m, n) = \emptyset$. Assume the contrary case, and put $j = \min (u \in \mathbb{N} | u \leq i$ and $m \leq m_u)$. One has:

$$B(m, n) = \{x_j | j \in X^* \} \cap H(x_j, m_j).$$

Consequently, card $B(m, n) = p^n - (l(x_j))$ and it is obvious that $p^n - (l(x_j)) < p^n - m / (p-1)$ (because $(x_j, m_j) \in H(x_j, m_j)$, which is a M-L test) and $p^n - m / (p-1) \leq p^n - m / (p-1)$.

II) In case $m > m_u$, for all $u = 1, 2, \ldots, k$, one has $B(m, n) = \emptyset$. Assume the contrary and put $j = \min (u \in \mathbb{N} | u \leq k$ and $m \leq m_u)$. The remainder of the proof is exactly as in case I).

We shall extend our result obtained in Theorem 5 to more general cases, namely we shall replace finite sets with r.e. sets.

**Theorem 8** Let $A \subset X^*$ be a r.e. set. For every $x \in A$ we consider the natural number $m_x \geq 1$. Put $H = \bigcup_{x \in A} H(x, m_x)$, and $\mathcal{A} = \bigcup_{x \in A} H(x, m_x)$.
The following assertions are equivalent:

1) The set \( \mathcal{H} \) is a s. M-L test.

2) The set \( \mathcal{H} \) is a M-L test and for every r.e. prefix-free set \( Y \subset A \) one has

\[
\sum_{y \in Y} p^{-1(y)} \leq p^{-\min(m_y; y \in Y)}/(p-1),
\]

where the equality can occur only for infinite \( Y \).

Proof "(1)\( \Rightarrow \) (2)". Assuming \( \mathcal{H} \) is a s.M-L test, it is clear that its subset \( \mathcal{H} \) is a M-L test.

Now let \( Y \subset A \) be a r.e. prefix-free set. In case \( Y \) is finite, then \( \bigcup_{y \in Y} \mathcal{H}(y, m_y) \) is a M-L test and, according to Theorem 5, one has

\[
\sum_{y \in Y} p^{-1(y)} < p^{-\min(m_y; y \in Y)}/(p-1).
\]

In case \( Y \) is infinite, write \( Y = \{y_1, y_2, \ldots, y_n, \ldots\} \), and, for every \( n \geq 1 \) put \( H_n = \bigcup_{i=1}^n \mathcal{H}(y_i, m_y) \), \( H_n = \bigcup_{i=1}^n \mathcal{H}(y_i, m_y) \). Because \( H_n \) is a s. M-L test, one has, again by Theorem 5, the inequality:

\[
u_n = \sum_{i=1}^n p^{-1(y_i)} < p^{-\min(m_y; y \in Y)}/(p-1) = v_n.
\]

The sequences \( (u_n) \) and \( (v_n) \) are increasing and \( (v_n) \) can take only the following values:

\[
p^{-m_1}/(p-1), p^{-m_1+1}/(p-1), \ldots, p^{-1}/(p-1).
\]

Consequently, \( (v_n) \) is stationary and \( \lim_n u_n \leq \lim_n v_n \). But

\[
\lim_n u_n = \sum_{y \in Y} p^{-1(y)},
\]

\[
\lim_n v_n = p^{-\min(m_y; y \in Y)}/(p-1).
\]

"(2)\( \Rightarrow \) (1)". It is clear that \( \mathcal{H} \) is a r.e. set, satisfying condition (1) in Definition 1. All it remains to be proved is that
for all natural \( m, n \geq 1 \).

It is seen that \( \|x\| = n \) and \( (x, m) \in \mathcal{R} \) imply \( m < n \), because \( H \) is a M-L test. Consequently, the set

\[
M = \{(x, m) \in \mathcal{R} | \|x\| = n\}
\]

is finite. For every \( x \) in \( X^* \) with \( \|x\| = n \) and \( (x, m) \in \mathcal{R} \) we can find \( y_x \) in \( A \) such that \( x \supset y_x \), and \( (y_x, m) \in H \). The set

\[
B = \{(y_x, 1), (y_x, 2), \ldots, (y_x, m) | \|x\| = n, (x, m) \in \mathcal{R} \} \subset \mathcal{H}
\]

is a finite M-L test. We can write

\[
B = \bigcup_{i=1}^{n} H(z_i, m), \quad \text{with } z_i \in A.
\]

We shall divide the remainder of the proof in two steps:

a) We prove that for all natural \( m, n \geq 1 \), we have

\[
\{x \in X^* | \|x\| = n, (x, m) \in \mathcal{R} \} = \{x \in X^* | \|x\| = n, (x, m) \in \mathcal{B} \}.
\]

One must check only the inclusion "\( \subset \)". But, if \( x \) is such that \( \|x\| = n \) and \( (x, m) \in \mathcal{R} \), we get \( y_x \in A \), \( y_x \subset x \), and \( (y_x, m) \in \mathcal{H} \), so \( (y_x, m) \in \mathcal{B} \), and \( (x, m) \in \mathcal{B} \).

b) We prove that \( B \) is a s.M.L test. In any case \( B \) is a finite M-L test. On the other hand, every prefix-free subset \( Y \subset \{z_1, z_2, \ldots, z_t\} \subset A \) satisfies the inequality in the hypothesis

\[
\sum_{y \in Y} p^{-\|y\|} < p^{-\min \{\|y\| \} - 1} \leq p^{-m/(p - 1)},
\]

because \( m \leq m_y \), for all \( y \) in \( Y \) (we put the condition that \( (y, m) \in \mathcal{H} \), for all \( y = y_i \) in \( Y \)). Applying Theorem 5 for \( B \) and \( \mathcal{H} \) we see that \( B \) is a s.M.L test.

In view of (a), inequality (5) follows from (b).

Remarks (1) We assert that every non-empty s.M.L test \( W \) is of the form \( W = \bigcup_{x \in A} H(x, m_x) \), where \( A \subset X^* \) is a r.e. set for every
\( x \in \mathcal{A} \), \( m_x \) is a natural number with \( 1 \leq m_x < l(x) \). Indeed, consider a recursive function \( f: \mathbb{N} \to \mathcal{X}^* \times \mathbb{N} \) such that \( \mathcal{W} = \{ f(i) | i \geq 0 \} \). Put \( f(i) = (x_i, m_i) \) for every \( i \) in \( \mathbb{N} \), and let \( \mathcal{A} = \{ x_i | i \geq 0 \} \). It is seen that

\[
\mathcal{W} = \bigcup_{i \geq 0} \mathcal{H}(x_i, m_i) = \bigcup_{x \in \mathcal{A}} \mathcal{H}(x, m_{\mathcal{W}}(x));
\]

here \( m_\mathcal{W} \) is the critical level induced by the s.M-L test \( \mathcal{W} \).

2) There exist infinite non r.e. sets which are prefix-free. Take, for example, the set \( \{0^n | n \in I \} \), where \( I \subseteq \mathbb{N} \) is a non r.e. set (here \( \mathcal{X} = \{0, 1\} \)).

3) There are situations when in (4) occurs equality, for infinite \( \mathcal{Y} \). Again our example is in the case \( \mathcal{X} = \{0, 1\} \). Take

\[
\mathcal{H} = \{01, 1\}, (001, 1), \ldots, (0^{k+1}1, 1)\ldots = \bigcup_{k=2}^{\infty} \mathcal{H}(0^{k+1}1, 1),
\]

and \( \mathcal{Y} = \{0^{k-1}1 | k \geq 2 \} \). One can see that \( \sum_{x \in \mathcal{Y}} 2^{-i(x)} = 1/2 \).

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References
