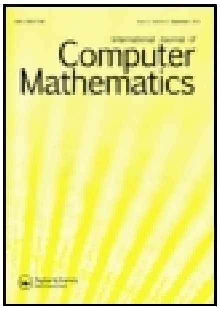

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A Combinatorial Characterization of Sequential P. Martin-Löf Tests

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Dedicated to Professor S. Marcus, for his 60th anniversary

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We give a combinatorial characterization of sequential P. Martin-Löf tests within the class of all P. Martin-Löf tests.

KEY WORDS: (Sequential) P. Martin-Löf tests, prefix-free set.
C.R. CATEGORIES: F.4.1, G.3

1. PREREQUISITES

The set of natural numbers is $\mathbb{N}=\{0,1,2,\ldots\}$.

Let $X=\{a_1,a_2,\ldots,a_p\}$, $p\geqq 2$ be a finite alphabet. Denote by X^* the free monoid generated by X (the elements of X^* are called strings; λ is the empty string). If $x=x_1x_2\ldots x_n$ is in X^*, then the length of x is $l(x)=n$; $l(\lambda)=0$. For all x and y in X^* we write $x\subset y$ in case there exists a string z in X^* such that $y=xz$. If $a\in X$ and n is natural, we write $a^n=\lambda$ (if $n=0$) and $a^n=aa\ldots a$ (n times, if $n>0$).

A non-empty r.e. set $V\subset X^*\times(\mathbb{N}-\{0\})$ is called *P. Martin-Löf test* (*M-L test*) if it possesses the following two properties (see [3] and [1]):

1) For every natural $m\geqq 1$, $V_{m+1}\subset V_m$. Here $V_m=\{x\in X^* \mid (x,m)\in V\}$.

2) For all naturals n, m, $m \geqq 1$, one has

$$\operatorname{card}\{x \in X^* \mid l(x)=n, (x,m) \in V\} < p^{n-m}/(p-1).$$

We shall agree upon the fact that the empty set $\varnothing$ is a M-L test.

The *critical level* induced by a M-L test V is the function $m_V: X^* \to \mathbb{N}$ given by $m_V(x) = \max(m \geqq 1 \mid (x,m) \in V)$, in case such an m exists, and $m_V(x)=0$, in the opposite case.

2. RESULTS

The P. Martin-Löf tests were introduced in [3] in order to give a statistical interpretation of the Kolmogorov complexity-theoretic notion of random string [2]. The sequential P. Martin-Löf tests ([3]) are designated to play the same role in the study of random sequences (see [4, 5, 6, 7]).

Our aim is to give a complete characterization of the class of sequential P. Martin-Löf tests within the larger class of P. Martin-Löf tests.

Firstly, we give the following definition (see [3] and [4]):

DEFINITION 1 A r.e. set $V \subset X^* \times (\mathbb{N} - \{0\})$ satisfying the properties:

1) If $(x,m) \in V$, $y \supset x$ and $1 \leqq n \leqq m$, then $(y,n) \in V$,

2) For all natural $m, n \geqq 1$, we have

$$\operatorname{card}\{x \in X^* \mid l(x)=n,\ (x,m) \in V\} < p^{n-m}/(p-1),$$

is called a *sequential P. Martin-Löf test* (*s. M-L test* in the sequel).

We shall agree upon the fact that the empty set $\varnothing$ is a s. M-L test.

It is easy to see that every s. M-L test is an infinite M-L test (of course, the converse is not true).

In the sequel we shall constantly use the following notations:

a) For every $x \in X^*$ and natural $m \geqq 1$, $H(x,m) = \{(x,1), (x,2), \ldots, (x,m)\}$. It is easy to see that $H(x,m)$ is a M-L test iff $l(x) > m$.

b) For every $A \subset X^* \times \mathbb{N}$, we put $\bar{A} = \varnothing$, in case A is empty, and

$\bar{A}=\{(y,m)|$ there exists $(x,m)\in A$ such that $y\supset x\}$. It is easy to see that the map $A\mapsto\bar{A}$ is a closure operator which preserves recursive enumerability. Moreover, one has $\bigcup_{i\in I}\bar{A}_i=\overline{\bigcup_{i\in I}A_i}$, for each family $(A_i)_{i\in I}$ of subsets of $X^*\times\mathbb{N}$.

LEMMA 2 *Let $x\in X^*$ and $m\in\mathbb{N}$, $m\geqq 1$. The following assertions are equivalent:*

1) *The set $\overline{H(x,m)}$ is a s. M-L test.*

2) *The set $H(x,m)$ is a M-L test.*

Proof Only the implication "(2)⇒(1)" needs a careful check. Our hypothesis is $l(x)>m$ and we must only prove property (2) in Definition 1.

Let $n\geqq 1$ and $q\geqq 1$. We must prove the inequality:

$$c=\mathrm{card}\,\{y\in X^*|l(y)=n,(y,q)\in\overline{H(x,m)}\}<p^{n-q}/(p-1), \tag{1}$$

which is obvious for $q>m$ or $n\leqq m$. Consequently, (1) must be checked only for $n>m\geqq q$, and it is sufficient to prove that

$$c<p^{n-m}/(p-1).$$

The last inequality can be written

$$p^{n-l(x)}<p^{n-m}/(p-1), \tag{2}$$

(because $(y,q)\in\overline{H(x,m)}$ iff $y\supset x$).

Finally, inequality (2) is equivalent to $l(x)>m$. ☒

The following example shows that what happens in Lemma 2 cannot happen in more general conditions:

a) The closure $\bar{V}$ of a M-L test is not always a (sequential) M-L test.

b) The closure of a finite union of M-L tests of the form $H(x,m)$ is not always a (sequential) M-L test.

Example 3 Take $X=\{0,1\}$ and $H=H(00,1)\cup H(111,1)\cup H(101,1)=\{(00,1),\ (111,1),\ (101,1)\}$. Clearly, H *is* a M-L test, but $\bar{H}$ *is not* a M-L test, containing the elements (000, 1), (001, 1), (111, 1), (101, 1). ☒

In order to be shorter in the sequel, we introduce

DEFINITION 4 A non-empty subset $A \subset X^*$ is called *prefix-free* if for all x, y in A such that $x \subset y$ we have $x = y$.

It is seen that every non-empty set $A \subset X^*$ contains prefix-free subsets, e.g., its singletons.

The following theorem furnishes necessary and sufficient conditions for the closure of a finite union of M-L tests $H(x, m)$ to be a s. M-L test.

THEOREM 5 *Let $x_1, \ldots, x_k$ in X^* and $m_1, m_2, \ldots, m_k \geqq 1$ be natural numbers. Put $H = \bigcup_{i=1}^{k} H(x_i, m_i)$.*

The following conditions are equivalent:

1) *The set $\bar{H}$ is a s. M-L test.*

2) *We have simultaneously:*

a) *The set H is a M-L test.*

b) *For every prefix-free subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \subset \{x_1, x_2, \ldots, x_k\}$ one has*

$$\sum_{u=1}^{r} p^{-l(x_{i_u})} < p^{-\min(m_{i_1}, m_{i_2}, \ldots, m_{i_r})}/(p-1).$$

Proof "(1)⇒(2)", Let $1 \leqq i_1 < i_2 < \ldots < i_r \leqq k$ be such that the set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ is prefix-free. Let a natural $n \geqq \max(l(x_{i_1}), l(x_{i_2}), \ldots, l(x_{i_r}))$ and $m = \min(m_{i_1}, m_{i_2}, \ldots, m_{i_r})$. Then, because the sets $\overline{H(x_{i_u}, m_{i_u})}$, $u = 1, 2, \ldots, r$, are disjoint, we have

$$\operatorname{card}\left\{x \in X^* \mid l(x) = r,\ (x, m) \in B = \bigcup_{j=1}^{n} \overline{H(x_{i_j}, m_{i_j})}\right\}$$

$$= \sum_{j=1}^{n} p^{r - l(x_{i_j})} < p^{r-m}/(p-1),$$

since B is a M-L-test a.s.o.

"(2)⇒(1)". We shall proceed by induction upon k. For $k = 1$ the result has been obtained in Lemma 2.

Consider the result to be valid for some natural number $k > 1$ and let us prove it for $k + 1$. So, let $\bar{H} = \bigcup_{i=1}^{k+1} \overline{H(x_i, m_i)}$. We may assume that $l(x_{k+1}) = \max(l(x_1), l(x_2), \ldots, l(x_{k+1}))$.

Recall that $B(m,n)=\{x\in X^*|l(x)=n$ and $(x,m)\in \bar{H}\}$, for all natural m, $n\geqq 1$. All it remains to be proved is that for all $n>m\geqq 1$ one has

$$\operatorname{card} B(m,n)<p^{n-m}/(p-1). \tag{3}$$

The first case: $n<l(x_{k+1})$. In this situation

$$B(m,n)=\left\{x\in X^*|l(x)=n,\ (x,m)\in\bigcup_{i=1}^{k}\overline{H(x_i,m_i)}\right\}$$

and (3) holds using the induction hypothesis.

The second case: $n\geqq l(x_{k+1})$. The situation $m>m_{k+1}$ is similar to the first case, because $B(m,n)$ will be the same.

From now on, we shall consider that

$$l(x_1)\leqq l(x_2)\leqq\ldots\leqq l(x_k)\leqq l(x_{k+1}).$$

Therefore, let us suppose that $m\leqq m_{k+1}$. Under these conditions we divide the proof according to the fact that $x_{k+1}\supset x_i$ for some $1\leqq i\leqq k$, or not.

α) Assume first that $x_{k+1}\not\supset x_i$, for all $1\leqq i\leqq k$. If $m\leqq\min(m_1,m_2,\ldots,m_k)$, then

$$B(m,n)=\bigcup_{i=1}^{k+1}\{x\in X^*|l(x)=n,\quad\text{and}\quad (x,m)\in\overline{H(x_i,m)}\}.$$

To simplify facts, notice that in case $x_u\supset x_v$, we have $\overline{H(x_u,m)}\subset\overline{H(x_v,m)}$. Using this remark we eliminate all $\overline{H(x_v,m)}$ such that $x_1\subset x_v$ (if such x_v does exist) a.s.o. and finally we get a prefix-free subset

$$\{y_1,y_2,\ldots,y_j\}\quad\text{of}\quad\{x_1,x_2,\ldots,x_k,x_{k+1}\}$$

with the property that

$$\bigcup_{q=1}^{j}\overline{H(y_q,m)}=\bigcup_{i=1}^{k+1}\overline{H(x_i,m)}.$$

Then

$$B(m,n)=\bigcup_{q=1}^{j}\{x\in X^*\,|\,l(x)=n,\ (x,m)\in\overline{H(y_q,m)}\}.$$

Because the sets $\overline{H(y_q,m)}$ are disjoint, one gets

$$\begin{aligned}\operatorname{card} B(m,n) &\sum_{q=1}^{j} p^{n-l(y_q)}\\ &=p^n\left(\sum_{q=1}^{j} p^{-l(y_q)}\right)<p^n\cdot p^{-\min(m_{u_1},m_{u_2},\dots,m_{u_j})}/(p-1)\\ &\leq p^n\cdot p^{-m}/(p-1)=p^{n-m}/(p-1).\end{aligned}$$

Here we have considered that

$$y_1=x_{u_1},\,y_2=x_{u_2},\dots,y_j=x_{u_j}.$$

If $m>\min(m_1,m_2,\dots,m_k)$, then we get

$$B(m,n)=\bigcup_{u=1}^{q}\{x\in X^*\,|\,l(x)=n,\ (x,m)\in\overline{H(x_{i_u},m_{i_u})}\}$$

$$\cup\{x\in X^*\,|\,l(x)=n,\ (x,m)\in\overline{H(x_{k+1},m_{k+1})}\},$$

where $\{m_{i_1},m_{i_2},\dots,m_{i_q}\}=\{m_i\,|\,m_i\geq m,\ i=1,2,\dots,k\}$.

Proceeding in the same manner, we eliminate some $H(x_{i_u},m_{i_u})$ and we get again a prefix-free subset

$$\{y_1,y_2,\dots,y_r,x_{k+1}\}\quad\text{of}\quad\{x_{i_1},x_{i_2},\dots,x_{i_q},x_{k+1}\}$$

such that

$$\bigcup_{u=1}^{q}\overline{H(x_{i_u},m)}\cup\overline{H(x_{k+1},m_{k+1})}=\bigcup_{t=1}^{r}\overline{H(y_t,m)}\cup\overline{H(x_{k+1},m_{k+1})}.$$

The computation of card $B(m,n)$ is similar to that one in the case $m\leq\min(m_1,m_2,\dots,m_k)$.

β) Assume now that there exists $1 \leqq j \leqq k$ such that $x_{k+1} \supset x_j$.

Put $I = \{i \in \mathbb{N} \mid 1 \leqq i \leqq k+1, m \leqq m_i\}$ and notice that $k+1 \in I$, according to the hypothesis. It is seen that

$$B(m,n) = \left\{ x \in X^* \middle| l(x) = n, (x,m) \in \bigcup_{i \in I} \overline{H(x_i, m_i)} \right\}$$

$$= \left\{ x \in X^* \middle| l(x) = n,\ (x,m) \in \bigcup_{i \in I} \overline{H(x_i, m)} \right\}.$$

In any case, the set $\bigcup_{i \in I} H(x_i, m)$ is a M-L test, being contained in the M-L test $\bigcup_{i=1}^{k+1} H(x_i, m_i)$.

There are two possibilities:

β.1) There exists j in I such that $x_j \subset x_{k+1}$. We have $\overline{H(x_{k+1}, m)} \subset \overline{H(x_j, m)}$ and the union $\bigcup_{i \in I} H(x_i, m)$ has in fact at most k terms.

β.2) For all $i \in I$, one has $x_i \not\subset x_{k+1}$. In this case we pick some $1 \leqq j \leqq k$ such that $x_j \subset x_{k+1}$ and we notice that $j \notin I$. Consequently, the union $\bigcup_{i \in I} H(x_i, m)$ has also at most k terms.

In both cases, the induction hypothesis, applied to the M-L test $\bigcup_{i \in I} H(x_i, m)$ furnishes the inequality card $B(m,n) < p^{n-m}/(p-1)$.

This completes the proof. ☒

Corollary 6 *Condition* 1) *in Theorem* 5 *can be effectively checked.* ☒

We have seen that every non-empty set $A \subset X^*$ contains prefix-free subsets. The following lemma gives more precise results in the case when the only prefix-free subsets are the singletons, i.e. r in the statement of Theorem 5 must be equal to one. The reader can relate the results obtained in Scolium 7, Theorem 5 and Lemma 2.

Scolium 7 *Let* $y_1, y_2, \ldots, y_k$ *in* X^* *and* $n_1, n_2, \ldots, n_k \geqq 1$ *in* $\mathbb{N}$. *We assume that for all* $1 \leqq i < j \leqq k$ *we have either* $y_i \subset y_j$ *or* $y_j \subset y_i$. *Put* $H = \bigcup_{i=1}^{k} H(y_i, n_i)$. *Then, the following conditions are equivalent:*

1) *The set* $\bar{H}$ *is a s.M-L test.*
2) *The set* H *is a M-L test.*
3) *The sets* $H(y_i, n_i)$ *are all M-L tests.*

Proof Because the case $k=1$ has already been discussed (see Lemma 2), we shall consider that $k>1$.

The implication "(1)⇒(3)" being obvious, we confine ourselves to prove "(3)⇒(2)" and "(2)⇒(1)".

The set $\{y_1, y_2, \ldots, y_k\}$ being totally ordered by $\subset$, we rewrite it in the form $\{x_1, x_2, \ldots, x_k\}$, where $x_1 \subset x_2 \subset \ldots \subset x_k$, and we write also (x_i, m_i) instead of (y_j, n_j) in case $x_i = y_j$ in the new form. We can also suppose that $x_1, x_2, \ldots, x_k$ are distinct (separate trivial considerations for (1), (2) and 3)).

"(3)⇒(2)". We have to check that card $A(m,n) < p^{n-m}/(p-1)$, for $n>m$, where $A(m,n) = \{x \in X^* \mid l(x) = n,\ (x,m) \in H\}$. This is obvious in case $n \neq l\ (x_i)$, $i = 1, 2, .., k$. In case $n = l(x_i)$, this i is unique (because of total ordering). If $m > m_i$ then $A(m,n) = \emptyset$. If $m \leqq m_i$, then $A(m,n) = \{x_i\}$ and $1 < p^{n-m}/(p-1)$, because $n = l(x_i) > m$.

"(2)⇒(1)". Assume H is a M-L test. Only condition (2) in Definition 1 is to be checked.

For all natural $m, n \geqq 1$, we put $B(m,n) = \{x \in X^* \mid l(x) = n,\ (x,m) \in \bar{H}\}$. We must prove that card $B(m,n) < p^{n-m}/(p-1)$, only in the non trivial case $m \leqq \max(m_1, m_2, \ldots, m_k)$ (i.e. $B(m,n) \neq \emptyset$). Two non trivial cases are to be considered:

I) There exists $1 \leqq i \leqq k-1$, such that $l(x_i) \leqq n < l(x_{i+1})$.

II) One has $l(x_k) \leqq n$.

I) In case $m > m_u$, for all natural $u \leqq i$, one has $B(m,n) = \emptyset$. Assume the contrary case, and put $j = \min\ (u \in \mathbb{N} \mid u \leqq i$ and $m \leqq m_u)$. One has:

$$B(m,n) = \{x_j y \mid y \in X^*,\ l(y) = n - l(x_j)\}.$$

Consequently, card $B(m,n) = p^{n-l(x_j)}$ and it is obvious that $p^{n-l(x_j)} < p^{n-m_j}/(p-1)$ (because $(x_j, m_j) \in H(x_j, m_j)$, which is a M-L test) and $p^{n-m_j}/(p-1) \leqq p^{n-m}/(p-1)$.

II) In case $m > m_u$, for all $u = 1, 2, \ldots, k$, one has $B(m,n) = \emptyset$. Assume the contrary and put $j = \min(u \in \mathbb{N} \mid u \leqq k$ and $m \leqq m_u)$. The remainder of the proof is exactly as in case I). ☒

We shall extend our result obtained in Theorem 5 to more general cases, namely we shall replace finite sets with r.e. sets.

THEOREM 8 *Let $A \subset X^*$ be a r.e. set. For every $x \in A$ we consider the natural number $m_x \geqq 1$. Put $H = \bigcup_{x \in A} H(x, m_x)$, and $\bar{H} = \bigcup_{x \in A} \overline{H(x, m_x)}$.*

The following assertions are equivalent:

1) *The set* $\bar{H}$ *is a* s. *M-L test.*

2) *The set* H is a *M-L test and for every r.e. prefix-free set* $Y \subset A$ *one has*

$$\sum_{y \in Y} p^{-l(y)} \leqq p^{-\min(m_y \mid y \in Y)}/(p-1), \tag{4}$$

where the equality can occur only for infinite Y.

Proof "(1)⇒(2)". Assuming $\bar{H}$ is a s.M-L test, it is clear that its subset H is a M-L test.

Now let $Y \subset A$ be a r.e. prefix-free set. In case Y is finite, then $\bigcup_{y \in Y} H(y, m_y)$ is a M-L test and, according to Theorem 5, one has

$$\sum_{y \in Y} p^{-l(y)} < p^{-\min(m_y \mid y \in Y)}/(p-1).$$

In case Y is infinite, write $Y = \{y_1, y_2, \ldots, y_n, \ldots\}$, and, for every $n \geqq 1$ put $H_n = \bigcup_{i=1}^{n} H(y_i, m_{y_i})$, $\overline{H_n} = \bigcup_{i=1}^{n} \overline{H(y_i, m_{y_i})}$. Because $\overline{H_n}$ is a s. M-L test, one has, again by Theorem 5, the inequality:

$$u_n = \sum_{i=1}^{n} p^{-l(y_i)} < p^{-\min(m_{y_1}, m_{y_2}, \ldots, m_{y_n})}/(p-1) = v_n.$$

The sequences $(u_n)_n$ and $(v_n)_n$ are increasing and $(v_n)_n$ can take only the following values:

$$p^{-m_{y_1}}/(p-1),\ p^{-m_{y_1}+1}/(p-1), \ldots, p^{-1}/(p-1).$$

Consequently, $(v_n)_n$ is stationary and $\lim_n u_n \leqq \lim_n v_n$. But

$$\lim_n u_n = \sum_{y \in Y} p^{-l(y)},$$

$$\lim_n v_n = p^{-\min(m_y \mid y \in Y)}/(p-1).$$

"(2)⇒(1)". It is clear that $\bar{H}$ is a r.e. set, satisfying condition (1) in Definition 1. All it remains to be proved is that

$$\text{card}\,\{x\in X^*\,|\,l(x)=n,\ (x,m)\in\bar{H}\}<p^{n-m}/(p-1), \tag{5}$$

for all natural $m, n \geqq 1$.

It is seen that $l(x)=n$ and $(x,m)\in\bar{H}$ imply $m<n$, because H is a M-L test. Consequently, the set

$$M=\{(x,m)\in\bar{H}\,|\,l(x)=n\}$$

is finite. For every x in X^* with $l(x)=n$ and $(x,m)\in\bar{H}$ we can find y_x in A such that $x\supset y_x$, and $(y_x,m)\in H$. The set

$$B=\{(y_x,1),(y_x,2),\dots,(y_x,m)\,|\,l(x)=n,\ (x,m)\in\bar{H}\}\subset H$$

is a finite M-L test. We can write

$$B=\bigcup_{i=1}^{t} H(z_i,m),\quad \text{with } z_i\in A.$$

We shall divide the remainder of the proof in two steps:

a) We prove that for all natural $m, n\geqq 1$, we have

$$\{x\in X^*\,|\,l(x)=n,\ (x,m)\in\bar{H}\}=\{x\in X^*\,|\,l(x)=n,\ (x,m)\in\bar{B}\}.$$

One must check only the inclusion "$\subset$". But, if x is such that $l(x)=n$ and $(x,m)\in\bar{H}$, we get $y_x\in A$, $y_x\subset x$, and $(y_x,m)\in H$, so $(y_x,m)\in B$, and $(x,m)\in\bar{B}$.

b) We prove that B is a s.M-L test. In any case B is a finite M-L test. On the other hand, every prefix-free subset $Y\subset\{z_1,z_2,\dots,z_t\}\subset A$ satisfies the inequality in the hypothesis

$$\sum_{y\in Y} p^{-l(y)}<p^{-\min(m_y|y\in Y)}/(p-1)\leqq p^{-m}/(p-1),$$

because $m\leqq m_y$, for all y in Y (we put the condition that $(y,m)\in H$, for all $y=y_x$ in Y). Applying Theorem 5 for B and $\bar{B}$ we see that $\bar{B}$ is a s.M-L test.

In view of (a), inequality (5) follows from (b). ☒

Remarks (1) We assert that every non-empty s.M-L test W is of the form $W=\overline{\bigcup_{x\in A} H(x,m_x)}$, where $A\subset X^*$ is a r.e. set for every

x in A, m_x is a natural number with $1 \leqq m_x < l(x)$. Indeed, consider a recursive function $f: \mathbb{N} \to X^* \times \mathbb{N}$ such that $W = \{f(i) | i \geqq 0\}$. Put $f(i) = (x_i, m_i)$, for every i in $\mathbb{N}$, and let $A = \{x_i | i \geqq 0\}$. It is seen that

$$W = \bigcup_{i \geqq 0} \overline{H(x_i, m_i)} = \bigcup_{x \in A} \overline{H(x, m_W(x))};$$

here m_W is the critical level induced by the s.M-L test W.

2) There exist infinite non r.e. sets which are prefix-free. Take, for example, the set $\{0^n 1 | n \in I\}$, where $I \subset \mathbb{N}$ is a non r.e. set (here $X = \{0, 1\}$).

3) There are situations when in (4) occurs equality, for infinite Y. Again our example is in the case $X = \{0, 1\}$. Take

$$H = \{(01, 1), (001, 1), \ldots, (0^{k-1}1, 1), \ldots\} = \bigcup_{k=2}^{\infty} H(0^{k-1}1, 1),$$

and $Y = \{0^{k-1}1 | k \geqq 2\}$. One can see that $\sum_{y \in Y} 2^{-l(y)} = 1/2$.

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