COLLOQUIA MATHEMATICA
SOCIETATIS JÁNOS BOLYAI, 23.

TOPOLOGY

Vol. I.

Edited by:

Á. CSÁSZÁR

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM – OXFORD – NEW YORK

Budapest, Hungary, 1980

ISBN North-Holland: 0444 85406 1
ISSN Bolyai: 0139 3383
ISBN Bolyai: 963 8021 30 6 Vol. I-II
ISBN Bolyai: 963 8021 31 4 Vol. I

Joint edition published by

JÁNOS BOLYAI MATHEMATICAL SOCIETY
and
NORTH-HOLLAND PUBLISHING COMPANY
Amsterdam – Oxford – New York

In the U.S.A. and Canada:
NORTH-HOLLAND PUBLISHING COMPANY
52 Vanderbilt Avenue
New York, N.Y. 10017

Printed in Hungary
ÁFÉSZ, VÁC
Sokszorosító üzeme

COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
23. TOPOLOGY, BUDAPEST (HUNGARY), 1978.

ON THE CATEGORY OF ČECH TOPOLOGICAL SPACES

C. CALUDE — M. MALITZA

A Čech (topological) space is a set endowed with an extensive and monotone operator. The Čech spaces (which generalize the usual topological spaces) are in many cases not operational because it is not yet known the form of the Čech product topology.

In the present paper we will show that this problem can be solved by categorical methods, by studying the category of Čech spaces (where morphisms are Čech continuous functions). We shall prove that this category has finite limits and colimits and exponentiation; however, it is not an elementary topos in the sense of Lawvere – Tierney [4] because it has no subobject classifier.

1. INTRODUCTION

Let X be a set and let $\varphi: \mathscr{P}(X) \to \mathscr{P}(X)$ be a function defined on the power set of X. Let us consider the following conditions:

(1) $\varphi(\phi) = \phi$,

(2) $A \subseteq \varphi(A)$ for every $A \subseteq X$,

(3) If $A \subseteq B$ then $\varphi(A) \subseteq \varphi(B)$ for all $A, B \subseteq X$,

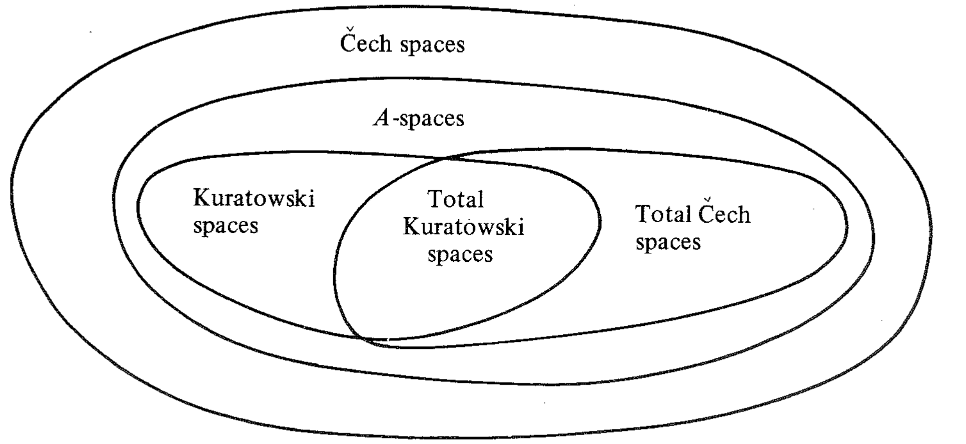
(4) $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$ for all $A, B \subseteq X$,

(5) For any family of subsets of X, $\{A_i\}_{i \in I}$, $\varphi\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} \varphi(A_i)$,

(6) $\varphi(\varphi(A)) = \varphi(A)$ for any $A \subseteq X$.

The couple (X, φ), where φ has the properties (1), (2) and (3) is called a Čech (topological) space [2]. A Čech space which verifies the condition (4) is called, following [2], an A-space (if φ verifies the conditions (1), (2) then (4) implies (3) but converse fails [5]). An A-space which has the property (6) is a Kuratowski (topological) space. A Čech space (or an A-space, or a Kuratowski space) is total if the condition (5) holds.

We have the following picture (all inclusion relations are proper):



A function $f\colon (X, \varphi_1) \to (Y, \varphi_2)$ from two Čech spaces is continuous if for every $A \subseteq X$, $f(\varphi_1(A)) \subseteq \varphi_2(f(A))$, [2]. We denote by Top $\check{C}$ (respectively, Top A, Top K, Top $\check{C}$, Top TK) the category of Čech spaces (respectively, the category of A-spaces, Kuratowski spaces, Total Čech or Kuratowski spaces), where the objects are just the spaces which give the name of the category and morphisms are continuous functions.

The category of Kuratowski spaces is well known [3]. The categories of Total Čech (Kuratowski) spaces are isomorphic with the category of reflexive (reflexive and transitive) relations, where the morphisms are the relation-preserving functions; these categories are studied in [1]. Most of our results are generalizations of those of [1].

2. FINITE LIMITS AND COLIMITS

We check the existence of finite limits and colimits in Top $\check{C}$. In view of an well known result [3] we study only the existence of products (coproducts), equalizers (coequalizers) and terminal (initial) objects.

Proposition 1. *The category* Top $\check{C}$ *has finite products.*

Proof. The product of Čech spaces (X_i, φ_i), $i = 1, 2$ is the Čech space $(X_1 \times X_2, \varphi_3)$, where for every $C \subseteq X_1 \times X_2$, $\varphi_3(C) = \varphi_1(\pi_1(C)) \times \varphi_2(\pi_2(C))$: π_i stand for the set-theoretic projections.

It is clear that φ_3 satisfies the conditions (1), (2) and (3). The projections are just the set-theoretic projections π_i (they are continuous because $\pi_i(\varphi_3(C)) = \varphi_i(\pi_i(C))$ for any $C \subseteq X_1 \times X_2$).

Let (Y, φ) be a Čech space and let p_i: $(Y, \varphi) \to (X_i, \varphi_i)$ be two arbitrary morphisms. Then, there exists an unique morphism p: $(Y, \varphi) \to (X_1 \times X_2, \varphi_3)$ such that $\pi_i p = p_i$. The morphism p is defined by the relation: $p(y) = (p_1(y), p_2(y))$. From the relations:

$$p(\varphi(C)) = \{(p_1(y), p_2(y)) \mid y \in \varphi(C)\},$$

$$\varphi_3(p(C)) = \varphi_1(\pi_1(p(C))) \times \varphi_2(\pi_2(p(C))) =$$

$$= \varphi_1(p_1(C)) \times \varphi_2(p_2(C))$$

we derive the inclusion $p(\varphi(C)) \subseteq \varphi_3(p(C))$, which shows that p is continuous.

Remark 1. The family of A-spaces (in particular, the class of Kuratowski spaces) is not closed under the construction from Proposition 1. Let us take $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$ and φ_i the identity of $\mathscr{P}(X_i)$. Let $M = \{(1, 3), (2, 4)\}$ and $N = \{(1, 4)\}$. We have:

$$\varphi_3(M \cup N) = M \cup N \cup \{(2, 4)\} \neq M \cup N = \varphi_3(M) \cup \varphi_3(N)$$

that is φ_3 does not satisfy the property (4).

Proposition 2. *The category* Top $\check{C}$ *has finite coproducts.*

Proof. Let (X_i, φ_i) be two Čech spaces. Their coproduct is the space $(\{1\} \times X_1 \cup \{2\} \times X_2, \varphi_4)$, where

$$\varphi_4(C) = \{1\} \times \varphi_1(X_1 \cap \pi_2(C)) \cup \{2\} \times \varphi_2(X_2 \cap \pi_2(C)).$$

The injections are $\mathrm{in}_i(x) = (i, x)$.

Proposition 3. *The category* Top $\check{C}$ *has equalizers.*

Proof. The equalizer of the pair

$$(X_1, \varphi_1) \underset{g}{\overset{f}{\rightrightarrows}} (X_2, \varphi_2)$$

is the Čech space (Y, φ_3), where $Y = \{x \mid x \in X_1,\ f(x) = g(x)\}$ and φ_3 is the restriction of φ_1 to Y, with the set-theoretic injection.

Proposition 4. *The category* Top $\check{C}$ *has coequalizers.*

Proof. The coequalizers of the pair of morphisms

$$(X_1, \varphi_1) \underset{g}{\overset{f}{\rightrightarrows}} (X_2, \varphi_2)$$

is the Čech space (Z, φ_4) defined as follows. Z is the quotient space $X_2/\bar{\rho}$, where $\bar{\rho}$ is the smallest equivalence relation which contains the relation $\rho = \{(f(x), g(x)) \mid x \in X_1\}$, and $\varphi_4(B) = \{[x] \mid x \in \varphi_2(\bar{B})\}$; $[x]$ is the $\bar{\rho}$-class of x and $\bar{B} = \{y \mid [y] \in B\}$. The surjection $p\colon (X_2, \varphi_2) \to$ $\to (Z, \varphi_4)$ is defined by $p(y) = [y]$.

First we shall prove that φ_4 satisfies the conditions (1), (2), (3). Obviously, $\varphi_4(\phi) = \phi$ and $B \subseteq \varphi_4(B)$ for every $B \subseteq Z$. Now let $B \subseteq C$ be two arbitrary subsets of Z. Then, we have $\bar{B} \subseteq \bar{C}$, $\varphi_2(\bar{B}) \subseteq \varphi_2(\bar{C})$ and finally, $\varphi_4(B) = \varphi_4(C)$.

Now we prove that p is continuous. In view of the relation $B = \overline{p(\bar{B})}$ we have:

$$p(\varphi_2(B)) = \{[x] \mid x \in \varphi_2(B)\} = \{[x] \mid x \in \varphi_2(\overline{p(B)})\} = \varphi_4(p(B)).$$

It is clear that $pf = pg$. Let $q: (X_2, \varphi_2) \to (V, \varphi)$ be such that $qf = qg$. Then, there exists an unique $h: (Z, \varphi_4) \to (V, \varphi)$ such that $hp = q$. As in the set-theoretic case $h([y]) = q(y)$. Let $w \in h(\varphi_4(M))$; then, there exists $[y] \in \varphi_4(M)$ such that $w = h([y]) = q(y)$. In view of the equality $\varphi(q(\bar{M})) = \varphi(h(M))$ and of the continuity of q it results $w \in \varphi(h(M))$, that is h is continuous.

Remark 2. In Top $\check{C}$ $(\{a\}, \varphi_a)$ and (ϕ, φ_ϕ) are terminal, respectively, initial objects.

Remark 3. The family of A-spaces is not closed under the constructions in Propositions 2-4.

We conclude with:

Theorem 1. *The category* Top $\check{C}$ *has finite limits and colimits.*

3. EXPONENTIATION

We prove the following result:

Theorem 2. *The category* Top $\check{C}$ *has exponentiation.*

Proof. Let (X_i, φ_i), $i = 1, 2, 3$ be Čech spaces. We shall prove the existence of a Čech space $(X_3, \varphi_3)^{(X_2, \varphi_2)}$ such that there is a bijection

$$\hom((X_1, \varphi_1) \times (X_2, \varphi_2), (X_3, \varphi_3)) \cong$$
$$\cong \hom((X_1, \varphi_1), (X_3, \varphi_3)^{(X_2, \varphi_2)})$$

natural in (X_1, φ_1) and (X_3, φ_3).

The object $(X_3, \varphi_3)^{(X_2, \varphi_2)}$ is defined as $(X_3^{X_2}, \varphi_4)$ where

$$X_3^{X_2} = \{f: (X_2, \varphi_2) \to (X_3, \varphi_3) \mid f \text{ continuous}\}$$

and

$$\varphi_4(M) = \{g: (X_2, \varphi_2) \to (X_3, \varphi_3) \mid g \text{ continuous},$$

there exists $f \in M$ such that for any

$$A \subseteq X_2, g(\varphi_2(A)) \subseteq \varphi_3(f(A))\}.$$

It is easy to check that φ_4 has the properties (1), (2) and (3) that is $(X_3^{X_2}, \varphi_4)$ is an object of Top $\check{C}$.

Let $f: (X_1, \varphi_1) \times (X_2, \varphi_2) \to (X_3, \varphi_3)$ be continuous. We define

$$\psi(f)(x)(y) = f(x, y) \text{ for all } x \in X_1, \ y \in X_2.$$

We shall prove that $\psi(f): (X_1, \varphi_1) \to (X_3^{X_2}, \varphi_4)$ is continuous, i.e. for any $M \subseteq X_1$, $\psi(f)(\varphi_1(M)) \subseteq \varphi_4(\psi(f)(M))$. Let $x \in \psi(f)(\varphi_1(M))$; x is of the form $x = \psi(f)(x')$, $x' \in \varphi_1(M)$. To prove that $x \in \varphi_4(\psi(f)(M))$ is to show that

(a) $x: (X_2, \varphi_2) \to (X_3, \varphi_3)$ is continuous,

(b) there exists $h: (X_2, \varphi_2) \to (X_3, \varphi_3)$, $h \in \psi(f)(M)$ for which $\psi(f)(x')(\varphi_2(A)) \subseteq \varphi_3(h(A))$ for any $A \subseteq X_2$.

We prove (a). Let $B \subseteq X_2$ and let $z \in \psi(f)(x')(\varphi_2(B))$. Clearly z is of the form $z = \psi(f)(x')(z') = f(x', z')$ with $z' \in \varphi_2(B)$. In view of the hypothesis f is continuous, that is for all $T \subseteq X_1 \times X_2$, $f(\varphi_*(T)) \subseteq \varphi_3(f(T))$; φ_* is the closure of the product space (Proposition 1). Now let $T = \{x'\} \times B$; we have $f(\varphi_*(\{x'\} \times B)) \subseteq \varphi_3(f(\{x'\} \times B))$, where $\varphi_*(\{x'\} \times B) = \varphi_1(x') \times \varphi_2(B)$.

Thus

$$\varphi_3(x(B)) = \varphi_3(\psi(f)(x')(B)) = \varphi_3(f(x', B))$$

and

$$z = \psi(f)(x')(z') \in \varphi_3(\psi(f)(x')(B)).$$

If in the proof of (a) we take $h = x$ we also have proved (b).

As in the set-theoretic case one proves that ψ is a bijection natural in the first and third argument.

Proposition 5. *Any exponential Čech space is total.*

Proof. Let (X_i, φ_i) be two Čech spaces and let $(X_2^{X_1}, \varphi_3)$ be the exponential object. We shall prove that for any family $\{A_j\}_{j\in I}$ of subsets of $X_2^{X_1}$ we have: $\varphi_3\left(\bigcup_{j\in I} A_j\right) = \bigcup_{j\in I} \varphi_3(A_j)$.

Let $g \in \varphi_3\left(\bigcup_{j\in I} A_j\right)$; $g\colon (X_1, \varphi_1) \to (X_2, \varphi_2)$ is continuous and there exists $f \in \bigcup_{j\in I} A_j$ such that for all $A \subseteq X_1$, $g(\varphi_1(A)) \subseteq \varphi_2(f(A))$. Thus, there exists f and $k \in I$, $f \in A_k$ such that $g(\varphi_1(A)) \subseteq \varphi_2(f(A))$, for any $A \subseteq X_1$. Hence $g \in \varphi_3(A_k) \subseteq \bigcup_{j\in I} \varphi_3(A_j)$.

Corollary. *Any exponential space is of the form* (X, φ_R), *where* R *is a reflexive relation on* X *and* $\varphi_R(A) = \{x \mid x \in X$ *there exists* $y \in A$, *with* $y\,R\,x\}$.

The proof is a consequence of Proposition 5 and Theorem 1 in [1].

Remark 4. Top $\check{C}$ is not an elementary topos in the sense of Lawvere – Tierney because in Top $\check{C}$ (as in Top *TK*) there is no sub-object classifier (it is only possible to classify the induced subspaces not all subspaces).

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