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ON THE TOPOLOGICAL SIZE OF SETS OF RANDOM STRINGS

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Introduction

KOLMOGOROV ([6]) has defined a string x to be random if, given its length, there is no string y sensibly shorter than x by means of which a partial recursive function could compute x. This remarkable intuition has been already validated in at least two ways. MARTIN-LÖF ([7]) has proved that random strings withstand all conceivable statistical tests, and CALUDE and CHIŢESCU ([1]) have shown that every infinite set of random strings is immune. This two results confirm KOLMOGOROV's definition from the point of view of the Probability Theory and of the Recursive Function Theory. Inspired by CALUDE's examination ([4]) of the topological size of some important sets of partial recursive functions, we study here various aspects concerning the topological size of sets of random strings. By showing that the set of random strings is rare, in contrast with the set of randomless (or non-random) strings which is not rare, we offer, in our opinion, a topological motivation of the definition of random strings given by KOLMOGOROV.

1. Basic notions

We recall here the basic notions involved in the paper. Let $X = \{0, 1\}$ be a binary alphabet and consider the following fixed enumeration of strings in X given by the lexicographical order: $a_1 = \lambda$, $a_2 = 0$, $a_3 = 1$, $a_4 = 00$, $a_5 = 01$, $a_6 = 10, \ldots$, where λ is the null word. For x in X^*, $l(x)$ will denote the length of x. Observe that $l(a_n) = [\log_2 n]$ for every n. For all a, x in X^* we write $a \leqq x$ in case $x = ay$, for some y in X^* (we say that a is a *prefix of* x).

For every partial recursive function $\varphi\colon X^* \times \mathsf{N} \to X$, the *Kolmogorov complexity* induced by φ is a function $K_\varphi\colon X^* \times \mathsf{N} \to \mathsf{N} \cup \{\infty\}$, defined by $K_\varphi(x \mid m) = \min\{l(y) \mid y \in X^*,\ \varphi(y, m) = x\}$ if $x = \varphi(y, m)$ for some y in X^* and $K_\varphi(x \mid m) = \infty$, otherwise. There exists a partial recursive function $\psi\colon X^* \times \mathsf{N} \to X^*$, which is called a *Kolmogorov universal algorithm*, having the property that for each partial recursive function $\varphi\colon X^* \times \mathsf{N} \to X^*$ there is a natural constant c such that $K_\psi(x \mid m) \leqq K_\varphi(x \mid m) + c$ for all x in X^* and all natural $m \geqq 1$ ([2], [6]). Denote by $K = K_\psi$ the complexity induced by a fixed universal algorithm. A string x in X^* is called *t-random* (t is in N) if $K(x \mid l(x)) \geqq l(x) - t$. The 0-random strings are also called *random strings*. For all naturals n and m with $n \geqq m$,

$$\mathrm{card}\{x \in X^* \mid l(x) = n, K(x \mid n) \geqq n - m\} > 2^n(1 - 2^{-m}) \geqq 0$$

(see Corollary 4 of [2]). Consequently for every n in N there exist random strings of length n and most strings are t-random if $t \geqq 1$.

[1]) I am indepted to the referee whose critical remarks have certainly improved the quality of this paper.

For every set $W \subseteq X^* \times \mathsf{N}$ and for every natural m we shall write

$$W_m = \{x \in X^* \mid (x, m) \in W\}.$$

A non-empty recursively enumerable set $V \subseteq X^* \times (\mathsf{N} \setminus \{0\})$ will be called a *Martin-Löf test* (see [7] and [2]) if it possesses the following properties:

1) For every natural $m \geqq 1$, $V_{m+1} \subseteq V_m$.

2) For all naturals $m, n, m \geqq 1$, $\operatorname{card}\{x \in X^* \mid l(x) = n,\ x \in V_m\} < 2^{n-m}$.

We agree upon the fact that the empty set is a Martin-Löf test.

For every partial recursive function φ: $X^* \times \mathsf{N} \to X^*$ the set

$$V(\varphi) = \{(x, m) \in X^* \times (\mathsf{N} \setminus \{0\}) \mid K_\varphi(x \mid l(x)) < l(x) - m\}$$

is a Martin-Löf test ([2]). For every Martin-Löf test W with W recursive, there exists a partial recursive function φ: $X^* \times \mathsf{N} \to X^*$ such that $W \subseteq V(\varphi)$ ([3]).

2. Results

One way of putting a topology on X^* is the following. For every a in X^* consider the set $U_a = \{x \in X^* \mid a \leqq x\}$. By a simple application of the definition, we can prove the following lemma.

Lemma 1. (1) *For every a, $a \in U_a$.* (2) *For all a_i, a_j such that $U_{a_i} \cap U_{a_j} \neq \emptyset$ there exists a_k such that $U_{a_i} \cap U_{a_j} = U_{a_k}$.* (3) *For every U_a there is a set $V \subseteq U_a$ such that for every y in V, $U_y \subseteq U_a$.*

From Lemma 1 we deduce that $(U_a)_{a \in X^*}$ is a system of basic neighborhoods in X^*. We shall work with the topology generated by this system. Our first aim is to see what a recursively rare set in X^* is. In a topological space, a set A is *rare* if its closure contains no nonempty open set. Observe that in the topological space constructed above the closure of a set A is the set $\bar{A} = \{x \in X^* \mid (\exists y \in A)\ y \geqq x\}$. Consequently A is rare if for every a_n we have $U_{a_n} \nsubseteq \bar{A}$. A set is recursively rare if for every a_n one can obtain a witness which certifies that $U_{a_n} \nsubseteq \bar{A}$, in a recursive way. Thus we arrive to the following formal definition which is very much in the spirit of the definition in [4].

Definition 1. A set $A \subseteq X^*$ is *recursively rare* if there is a recursive function r: $\mathsf{N} \to \mathsf{N}$ for which the following properties hold:

1) $a_n \leqq a_{r(n)}$ for all n in N,

2) there exists an i in N such that for every a_n for which $l(a_n) > i$, we have $A \cap U_{a_{r(n)}} = \emptyset$.

A set $A \subseteq X^*$ which is not recursively rare will be called a not recursively rare set.

The following lemma is an easy consequence of the Definition 1.

Lemma 2. (1) *If A is recursively rare and $B \subseteq A$, then B is recursively rare.* (2) *If A is a not recursively rare set and $B \supseteq A$, then B is a not recursively rare set.*

Lemma 2 states that the family of recursively rare sets is closed under subset and that the family of not recursively rare sets is closed under superset.

In order to show that the notion of recursively rare set is not trivial, we prove the following proposition.

Proposition 3. *There is a not recursively rare set* $A \subseteqq X^*$ *such that for every* w *in* X^*, *we have* $wX^* = \{y \in X^* \mid y = wz, z \in X^*\} \nsubseteq \bar{A}$.

Proof. Let $(\varphi_i)_{i \in \mathsf{N}}$ be an acceptable Gödelization of the set of partial recursive functions, $\varphi_i \colon \mathsf{N} \to \mathsf{N}$ ([8]). Take

$$A = \bigcup \{1^n 0 1^{k+1} 0 \{0, 1\}^{\varphi_n(k)} \mid n, k \geqq 0, \varphi_n(k)\downarrow\}.$$

a) One can easily prove that for every $w \in X^*$, $wX^* \nsubseteq \bar{A}$, by considering the different forms w can take.

b) We show that A is not recursively rare. Suppose that A is recursively rare, i.e. there exist a recursive function $r \colon \mathsf{N} \to \mathsf{N}$ and a natural constant i such that if $l(a_n) > i$, then $A \cap U_{a_{r(n)}} = \emptyset$. Consider the recursive functions $s \colon \mathsf{N} \to \mathsf{N}$ given by the relation $a_{s(n)} = 1^n 0 1^{n+1} 0$. Let $k \colon \mathsf{N} \to \mathsf{N}$ be the recursive function given by $k(n) = l(a_{r(s(n))}) - l(a_{s(n)})$. If $l(a_{s(n)}) = 2n + 3 > i$, then $a_{r(s(n))} \notin A$. Hence, keeping in mind that $1^n 0 1^{n+1} 0 = a_{s(n)} \leqq a_{r(s(n))}$, we have $\varphi_n(n) \neq k(n)$. Let j be an index for k such that $2j + 3 > i$. Then $\varphi_j(j) \neq \varphi_j(j)$, and φ_j is a recursive function. Contradiction.

We can now begin to study the topological size of various sets of random (or not random) strings. The first theorem we prove shows that the set of strings whose Kolmogorov complexity is bounded by a constant (we can call it the set of "strongly" non-random strings) is not recursively rare. Taking into account Lemma 2, it follows that the set of non-random strings is not recursively rare too.

Theorem 4. *There is a natural constant* c *such that the set* $A = \{x \in X^* \mid K(x \mid l(x)) \leqq d\}$ *is not recursively rare for all* $d \geqq c$.

Proof. We construct a subset B of A which is not recursively rare and then we interfere Lemma 2. For every n in N let $w_n = a_n 0^{n - l(a_n)}$. Thus $w_1 = 0$, $w_2 = 00$, $w_3 = 100$, $w_4 = 0000$, $w_5 = 01000$, $w_6 = 100000, \ldots$, a.s.o. Let $B = \{w_i \mid i > 0\}$. Consider the recursive function $\varphi \colon X^* \times \mathsf{N} \to X^*$, $\varphi(x, n) = w_n$ for every $x \in X^*$, $n \in \mathsf{N}$. We have $K_\varphi(w_n \mid n) = 0$ for all n, because $\varphi(\lambda, n) = w_n$. Consequently there is a constant c such that $K(w_n \mid n) \leqq c$ for all n. Hence $B \subseteqq A$ (where the constant appearing in the statement of Theorem 4 is the particular c defined above). We next show that B is not recursively rare. Suppose that there exist a recursive function $r \colon \mathsf{N} \to \mathsf{N}$ and a constant i such that $l(a_n) > i$ implies $B \cap U_{a_{r(n)}} = \emptyset$. Taking a_n in X^* with $l(a_n) > i$, we have that $w_{r(n)} \in B$ and $w_{r(n)} \in U_{a_{r(n)}}$, and consequently $w_{r(n)} \in B \cap U_{a_{r(n)}}$. Contradiction.

In Proposition 5 we shall need the Kolmogorov unconditioned complexity which is defined as follows: For every partial recursive function $\varphi \colon X^* \to X^*$ the *Kolmogorov unconditioned complexity* is a function $K_\varphi \colon X^* \to \mathsf{N} \cup \{\infty\}$ defined by

$$K_\varphi(x) = \min\{l(y) \mid y \in X^*, \varphi(y) = x\}$$

in case $x = \varphi(y)$ for some y in X^* and $K_\varphi(x) = \infty$, otherwise. For the Kolmogorov unconditioned complexity there exists a Kolmogorov universal algorithm ψ', too, i.e. there is a constant c in N such that $K_{\psi'}(x) \leqq K_\varphi(x) + c$ for all partial recursive functions $\varphi \colon X^* \to X^*$. Denote by $K(x) = K_{\psi'}(x)$ the complexity induced by some fixed Kolmogorov universal algorithm ψ'. There is a simple relation connecting the two variants of Kolmogorov complexity: there exists a constant d in N such that $K(x \mid l(x)) \leqq K(x) + d$ for all x in X^* ([6]).

Proposition 5. *Let $A \subseteqq X^*$ be a not recursively rare set. Then for every t in N there is an infinity of strings x in A such that $K(x \mid l(x)) \leqq l(x) - t$.*

Proof. We shall prove a bit stronger result, namely that for every t there is an infinity of strings x in A such that $K(x) \leqq l(x) - t$, and then we use the inequality between the two variants of Kolmogorov complexity.

Consider t in N and choose a recursive function $r\colon \mathsf{N} \to \mathsf{N}$ such that the function $u(n) = l(a_{r(n)}) - 2l(a_n) - 2$ is increasing and unbounded and $a_n \leqq a_{r(n)}$ (for example take $a_{r(n)} = w_n$, where w_n is from the proof of Theorem 4). Consider also the following three functions: $T\colon X^* \times X^* \to X^*$, $T(x, y) = x_1x_1x_2x_2 \ldots x_nx_n01y$, where $x = x_1x_2 \ldots x_n$ and x_i is in X for all i from 1 to n, $f\colon X^* \to X^*$,

$$f(z) = \begin{cases} x & \text{if } z = T(x, y) \text{ for some } y \text{ in } X^*, \\ \lambda & \text{otherwise,} \end{cases}$$

and $g\colon X^* \to X^*$,

$$g(z) = \begin{cases} y & \text{if } z = T(x, y) \text{ for some } x \text{ in } X^*, \\ \lambda & \text{otherwise.} \end{cases}$$

Observe that T, f and g are recursive. Finally define the recursive function $b\colon X^* \to X^*$, $b(z) = a_{r(n)}g(z)$, where $a_n = f(z)$. There is a natural constant q such that $K(x) \leqq K_b(x) + q$ for all x in X^*. Take $i \in \mathsf{N}$ such that $u([\log_2 i]) \geqq t + q$. Since A is not recursively rare, there exist a string a_n with $l(a_n) > i$ and a string $x \in X^*$ such that $a_{r(n)} \leqq x$. Let y be a string in X^* given by $x = a_{r(n)}y$. Then $b(T(a_n, y)) = a_{r(n)}y = x$ and $K_b(x) \leqq l(T(a_n, y)) = 2l(a_n) + 2 + l(x) - l(a_{r(n)}) = l(x) - (l(a_{r(n)}) - 2l(a_n) - 2) = l(x) - u(n) \leqq l(x) - u([\log_2 i]) \leqq l(x) - (t + q)$. Taking into account the way we have choosen q, we conclude that for every t there is some $x \in X^*$ with the property $K(x) \leqq l(x) - t$. Now for an $i > l(x)$ we find another x with the above property. The process can be iterated to find an infinity of strings x for which the inequality $K(x) \leqq l(x) - t$ holds.

Observe that as a corollary to Proposition 5 we can state that the set of t-random strings is recursively rare. However Theorem 7 provides a more constructive proof for this result.

The next proposition is somewhat similar to Proposition 5.

Proposition 6. *Let $A \subseteqq X^*$ be a not recursively rare set and $f\colon \mathsf{N} \to \mathsf{N}$ a recursive, increasing and unbounded function. Then there exists an infinity of strings in $\bar{A}$ such that $K(x \mid l(x)) \leqq f(l(x))$.*

Proof. We fix a recursive function $r\colon \mathsf{N} \to \mathsf{N}$ such that $a_n \leqq a_{r(n)}$ and $l(a_n) \leqq f(l(a_{r(n)}))$ for all n. Since A is not recursively rare, for every natural i there exist a_n with $l(a_n) > i$ and y in A such that $a_{r(n)} \leqq y$. We define the partial recursive function $b\colon X^* \times \mathsf{N} \to X^*$ as follows:

$$b(a_n, m) = \begin{cases} a_{r(n)} & \text{if } m = l(a_{r(n)}), \\ \lambda & \text{otherwise.} \end{cases}$$

Let i be an integer and y a string in A such that there exists a_n with $l(a_n) > i$ and $a_{r(n)} \leqq y$. Observe that $a_{r(n)} \in \bar{A}$, because $y \in A$. Also note that $b(a_n, l(a_{r(n)})) = a_{r(n)}$. Hence $K_b(a_{r(n)} \mid l(a_{r(n)})) \leqq l(a_n) \leqq f(l(a_{r(n)}))$. Now we take $x = a_{r(n)}$, and keeping in mind there is a constant q such that $K(x \mid l(x)) \leqq K_b(x \mid l(x)) + q$ for every x in X^*, we deduce that $K(x \mid l(x)) \leqq f(l(x)) + q$. (In order to obtain the exact statement of Proposition 6 we should have started with the function $f' = f - q$ instead of f.) Taking $i \geqq l(a_{r(n)})$, we obtain another string x in A with the demanded property and by repeating the same reasoning we get an infinity of strings x with the property $K(x \mid l(x)) \leqq f(l(x)) + q$.

Theorem 7. *For every t in* N *the set $A_t = \{x \in X^* \mid K(x \mid l(x)) \geqq l(x) - t\}$ is recursively rare.*

Proof. Consider a recursive function r: N $\to$ N satisfying the properties

1) $a_n \leqq a_{r(n)}$ for every n in N

and

2) there is a recursive function f: N $\to$ N such that $f(n) = m$ iff $l(a_{r(m)}) = n$ (take for example $a_{r(n)} = w_n$, where w_n is from Theorem 4).

Let $\varphi: X^* \times \mathrm{N} \to X^*$ be the recursive function defined by $\varphi(y, m) = a_{r(f(m \dot{-} l(y)))}y$. Since φ is recursive, there is a constant c such that for all strings x in X^* we have $K(x \mid l(x)) \leqq K_\varphi(x \mid l(x)) + c$. Take $i \geqq t + c$. We show that if $l(a_n) > i$, then $A_t \cap U_{a_{r(n)}} = \emptyset$. Indeed let $x = a_{r(n)}y$, where $l(a_n) > i$. Clearly $\varphi(y, l(x)) = x$. Consequently $K_\varphi(x \mid l(x)) \leqq l(y) = l(x) - l(a_{r(n)}) \leqq l(x) - l(a_n) < l(x) - i \leqq l(x) - (t + c)$. Then $K(x \mid l(x)) \leqq K_\varphi(x \mid l(x)) + c < l(x) - (t + c) + c = l(x) - t$. We conclude that $A_t \cap U_{a_{r(n)}} = \emptyset$ and the set A_t is recursively rare.

Remark. The comparison of Theorem 4 with Theorem 7 might produce a surprise at a first sight. Indeed the set of non-random strings, a small fragment of X^*, is not recursively rare, while the set of random strings, containing almost all strings in X^*, is recursively rare. However, recalling that the set of random strings is immune, we somewhat rediscover here the situation on the real line, where the transcedental numbers, while being majoritary, are much more difficult to capture than the algebraic numbers. In this way we feel that Theorem 4 and Theorem 7 represent a topological motivation of the definition of random strings given by Kolmogorov.

In what follows we try to reinforce the result in Theorem 7 by showing that the set of random strings is recursively rare in a great class of recursive sets. In fact we believe that the set of random strings is recursively rare in every recursive set.

In order to understand what means to be "recursively rare in a set A" we relativize our topology with the set A and reasoning as for Definition 1 we get the following definition.

Definition 2. Let $B, A \subseteq X^*$ be two sets. We say that B is (*primitive*) *recursively rare in A* if there is a (primitive) recursive function r: N $\to$ N such that

1) if $a_n \in A$, then $a_{r(n)} \in A$ and $a_n \leqq a_{r(n)}$,
2) there exists an integer i such that for every $a_n \in A$ with $l(a_n) > i$ we have $B \cap U^A_{a_{r(n)}} = \emptyset$ (where $U^A_{a_{r(n)}} = U_{a_{r(n)}} \cap A$).

We first show that the set of random strings is not primitive recursively rare in every recursive set. This will be an immediate consequence of the following lemma.

Lemma 8. *There is a recursive set A such that each infinite subset $B \subset A$ is not primitive recursively rare in A.*

Proof. Let f: $\mathsf{N} \to \mathsf{N}$ be a recursive function with $f(0) \neq 0$ which majorizes every primitive recursive function, i.e. for every primitive recursive function r: $\mathsf{N} \to \mathsf{N}$ there is a constant n_0 such that $f(n) > r(n)$ for all $n \geqq n_0$. (The existence of such a function is a well-known result of the Recursive Function Theory.) Consider the function $\bar{f}$: $\mathsf{N} \to \mathsf{N}$ defined by $\bar{f}(0) = 0$, $\bar{f}(n + 1) = f(\bar{f}(n)) + \bar{f}(n)$ for all n. Obviously $\bar{f}$ is recursive. The set A is defined by $A = \{x \in X^* \mid (\exists i \in \mathsf{N})\ l(x) = \bar{f}(i)\}$. Clearly A is a recursive set. Now take $B \subseteq A$, B infinite, and suppose that B is primitive recursively rare in A, i.e. there is a primitive recursive function r: $\mathsf{N} \to \mathsf{N}$ satisfying the properties

1) if $a_n \in A$, then $a_{r(n)} \in A$ and $a_{r(n)} \geqq a_n$,
2) there is an i in N such that if $a_n \in A$, $l(a_n) > i$, then $B \cap U^A_{a_{r(n)}} = \emptyset$.

Consider the function g: $\mathsf{N} \to \mathsf{N}$, $g(n) = \max\{l(a_{r(j)}) - l(a_j) \mid l(a_j) = n\}$. This function is primitive recursive because r is primitive recursive and the function $length(n) = l(a_n)$ is a primitive recursive function, too. Consequently there is a constant n_0 such that $f(n) > g(n)$ if $n \geqq n_0$. Take $j = \max(i, n_0)$. We show that if a_n is in A and $l(a_n) \geqq j$ then $a_{r(n)} = a_n$. Indeed suppose $a_{r(n)} < a_n$ and $l(a_n) = \bar{f}(k) \geqq j$. Then $l(a_{r(n)}) - l(a_n) \geqq \geqq \bar{f}(k + 1) - \bar{f}(k) = f(\bar{f}(k)) + \bar{f}(k) - \bar{f}(k) = f(\bar{f}(k))$. But we also have $g(\bar{f}(k)) \geqq \geqq l(a_{r(n)}) - l(a_n)$. Consequently we get $g(\bar{f}(k)) \geqq f(\bar{f}(k))$ and $\bar{f}(k) \geqq j$, which is a contradiction. Take now $a_n \in B$ with $l(a_n) \geqq j$ (recall that B is infinite). Since $a_{r(n)} = a_n$ we get that $a_n \in U^A_{a_{r(n)}}$. Hence $B \cap U^A_{a_{r(n)}} \neq \emptyset$, which is again a contradiction. It follows that B is not primitive recursively rare in A.

Proposition 9. *There is a recursive set A such that the set*

$$A_t = \{x \in A \mid K(x \mid l(x)) \geqq l(x) - t\}$$

is not primitive recursively rare in A for all t in N.

Proof. We consider the same set A as in Lemma 8. All it remains to prove is that the set $A_t \cap A$ is infinite. But this follows easily if we recall that for all $n, t \in \mathsf{N}$ there is a string $x \in X^*$ with $l(x) = n$ and $K(x \mid l(x)) \geqq l(x) - t$.

Thus the stronger conjecture that the set of random strings is primitive recursively rare in every recursive set is false. We shall prove that the set of random strings is recursively rare in every sparse or co-sparse set. We recall the necessary definition.

Definition 3 (see for example [5]). A set $A \subseteq X^*$ is *sparse* if there is an integer k such that $\mathrm{card}\{x \in A \mid l(x) = n\} \leqq n^k + k$ for all n in N.

Theorem 10. *Let A be a recursive and sparse set. Then the set*

$$A_t = \{x \in A \mid K(x \mid l(x)) \geqq l(x) - t\}$$

is finite.

Proof. Fix an integer k such that $\mathrm{card}\{x \in A \mid l(x) = n\} \leqq n^k + k$ for all n. Take n_0 in N such that $2^{n-n^{1/2}} - 1 > n^k + k$ for all $n \geqq n_0$, and construct the set $V \subseteq X^* \times (\mathsf{N} \setminus \{0\})$, $V = \{(x, m) \mid x \in A, l(x) \geqq n_0, 1 \leqq m \leqq l(x)^{1/2}\}$. We prove that V is a Martin-Löf test. Indeed V is recursively enumerable (in fact it is a recursive set), $V_{m+1} \subseteq V_m$ for all m in N, and $\mathrm{card}\{x \in X^* \mid l(x) = n, x \in V_m\} \leqq n^k + k < < 2^{n-n^{1/2}} - 1 \leqq 2^{n-m} - 1$. Consequently there is a partial recursive function φ such that $V \subseteq V(\varphi)$. It follows that for $(x, m) \in V$, $K_\varphi(x \mid l(x)) \leqq l(x) - m$. Using the

Kolmogorov theorem we get a constant c such that $K(x \mid l(x)) \leqq K_\varphi(x \mid l(x)) + c \leqq \leqq l(x) - (m - c)$. Now take n_1 such that $[n^{1/2}] - c > t$ for all $n \geqq n_1$. We show that A_t does not contain strings of length greater than $\max(n_0, n_1)$. Indeed take x in A with $l(x) \geqq \max(n_0, n_1)$. Then $(x, [l(x)^{1/2}]) \in V$ and it follows that $K(x \mid l(x)) \leqq \leqq l(x) - ([l(x)^{1/2}] - c) < l(x) - t$.

Theorem 10 which is important in its own (every recursive, sparse set contains only a finite number of random strings) has the following corollary.

Corollary 11. *Let A be a recursive and sparse set. Then the set A_t is recursively rare in A.*

Proof. Just observe that a finite set is recursively rare in every set.

Also note that the condition that A is sparse can be weakened by requiring only that $\lim_{n \to \infty} \operatorname{card}\{x \in A \mid l(x) = n\}/(2^{n-f(n)} - 1) = 0$, where f is a recursive, unbounded function.

The other result which sustains our conjecture is stated in the following theorem.

Theorem 12. *Let A be a recursive and co-sparse set. Then the set*

$$A_t = \{x \in A \mid K(x \mid l(x)) \geqq l(x) - t\}$$

is recursively rare in A.

Proof. Let $r\colon \mathsf{N} \to \mathsf{N}$ be a recursive function defined by

$$a_{r(n)} = \begin{cases} \text{the least (lexicographically) } y \in A \text{ such that } l(y) = n,\ y \geqq a_n \\ \quad \text{if such a } y \text{ exists,} \\ a_n \text{ otherwise.} \end{cases}$$

We show that only for a finite number of strings a_n, $a_{r(n)}$ is computed by using the second clause. Since A is co-sparse, we can fix a k such that $\operatorname{card}\{x \notin A \mid l(x) = n\} \leqq \leqq n^k + k$ for all n in N. Now take into account that $l(a_n) = s$ implies $2^s \leqq n < 2^{s+1} - 1$. It follows that there exist at least $2^{2^s - s}$ strings y with $l(y) = n$ and $y \geqq a_n$. If $2^{2^s - s} > > (2^{s+1} - 1)^k + k$, then at least one of the above y's belongs to A and consequently $a_{r(n)}$ is computed by the first clause.

This is why we take s_0 such that $2^{2^s - s} > (2^{s+1} - 1)^k + k$ for all $s \geqq s_0$. We get that if $a_n \in A$ and $l(a_n) \geqq s_0$, then $l(a_{r(n)}) = n$. Now consider the function $\varphi\colon X^* \times \mathsf{N} \to X^*$ defined by $\varphi(y, m) = a_{r(m \dot{-} l(y))}y$. There is a constant c such that $K(x \mid l(x)) \leqq \leqq K_\varphi(x \mid l(x)) + c$. We take $i = \max(s_0, t + c)$ and we show that if $a_n \in A$ and $l(a_n) > i$ then $A_t \cap U^A_{a_{r(n)}} = \emptyset$. Indeed take $x = a_{r(n)}y$ with $l(a_n) > i$. We get easily that $\varphi(y, l(x)) = x$. Consequently $K_\varphi(x \mid l(x)) \leqq l(y) = l(x) - l(a_{r(n)}) = l(x) - n$. Hence $K(x \mid l(x)) \leqq K_\varphi(x \mid l(x)) + c \leqq l(x) - (n - c) < l(x) - t$, which shows that $x \notin A_t$.

3. Open problems

Some open problems naturally arise. We display some of them.

1) Prove (or disprove) that if A is not recursively rare, then $\bar{A}$ contains random strings.
2) Prove (or disprove) that every immune set is recursively rare.
3) Prove (or disprove) that the set of random strings is recursively rare in every recursive set.

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