

PASSAGES OF PROOF

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1 To Prove or Not to Prove—That Is the Question!

*Whether 'tis nobler in the mind to suffer
The slings and arrows of outrageous fortune,
Or to take arms against a sea of troubles
And by opposing end them?
Hamlet 3/1, by W. Shakespeare*

In this paper we propose a new perspective on the evolution and history of the idea of mathematical proof. Proofs will be studied at three levels: syntactical, semantical and pragmatismal. Computer-assisted proofs will be given special attention. Finally, in a highly speculative part, we will anticipate the evolution of proofs under the assumption that the quantum computer will materialize. We will argue that there is little ‘intrinsic’ difference between traditional and ‘unconventional’ types of proofs.

2 Mathematical Proofs: An Evolution in Eight Stages

Theory is to practice as rigour is to vigour. D. E. Knuth

Reason and *experiment* are two ways to acquire knowledge. For a long time mathematical proofs required only reason; this might be no longer true. We can distinguish eight periods in the evolution of the idea of mathematical proof. The first period was that of pre-Greek mathematics, for instance the Babylonian one, dominated by observation, intuition and experience.

The second period was started by Greeks such as Pythagoras and is characterized by the discovery of deductive mathematics, based on theorems. Pythagoras proved his theorem, but the respective statement was discovered much earlier. Deductive mathematics saw a culminating moment in Euclid's geometry. The importance of abstract reasoning to ancient Greeks can be illustrated by citing Aristophanes's comedy *The Birds* which includes a cameo appearance of Meton, the astronomer, who claims that he had squared the circle. Knuth [44] rhetorically asked: "Where else on earth would a playwright think of including such a scene?" Examples would have been difficult to produce in 1985, but today the situation has changed. Take for example, the movie *Pi* written and directed by Darren Aronofsky Starring Sean Gullette or Auburn's play *Proof* [2] originally produced by the Manhattan Theatre Club on 23rd May 2000.

In a more careful description, we observe that deductive mathematics starts with Thales and Pythagoras, while the axiomatic approach begins with Eudoxus and especially with Aristotle, who shows that a demonstrative science should be based on some non-provable principles, some common to all sciences, others specific to some of them. Aristotle also used the expression "common notions" for axioms (one of them being the famous principle of non-contradiction). Deductive thinking and axiomatic thinking are combined in Euclid's *Elements* (who uses, like Aristotle, "common notions" for "axioms"). The great novelty brought by Euclid is the fact that, for the first time, mathematical proofs (and, through them, science in general) are built on a long distance perspective, in a step by step procedure, where you have to look permanently to previous steps and to fix your aim far away to the hypothetical subsequent steps. Euclid became, for about two thousands years, a term of reference for the axiomatic-deductive thinking, being considered the highest standard of rigour. Archimedes, in his treatise on static equilibrium, the physicists of the Middle Age (such as Jordanus de Nemore, in *Liber de ratione ponderis*, in the 13th century), B. Spinoza in *Ethics* (1677) and I. Newton in *Principia* (1687) follow Euclid's pattern. This tradition is continued in many more recent works, not only in the field of mathematics, but also in physics, computer science, biology, linguistics, etc.

However, some shortcomings of Euclid's approach were obstacles for the development of mathematical rigour. One of them was the fact that, until Galilei, the mathematical language was essentially the ordinary language, dominated by imprecision resulting from its predominantly spontaneous use, where emotional factors and lack of care have an impact. In order to diminish this imprecision and make the mathematical language capable to face the increasing need of precision and rigour, the ordinary language had to be supplemented by an artificial component of symbols, formulas and equations: with Galilei, Descartes, Newton and Leibniz, the mathematical language became more and more a mixed language, characterized by a balance between its natural and artificial components. In this

way, it was possible to pack in a convenient, heuristic way, previous concepts and results, and to refer to them in the subsequent development of mathematical inferences. To give only one example, one can imagine how difficult was to express the n th power of a binomial expression in the absence of a symbolic representation, i.e., using only words of the ordinary language. This was the third step in the development of mathematical proofs.

The fourth step is associated with the so-called epsilon rigour, so important in mathematical analysis; it occurred in the 19th century and it is associated with names such as A. Cauchy and K. Weierstrass. So, it became possible to renounce the predominantly intuitive approach via the infinitely small quantities of various orders, under the form of functions converging in the limit to zero (not to be confused with the Leibnizian infinitely small, elucidated in the second half of the 20th century, by A. Robinson's non-standard analysis). The epsilon rigour brought by the fourth step created the possibility to cope in a more accurate manner with processes with infinitely many steps such as limit, continuity, differentiability and integrability.

The fifth period begun with the end of the 19th century, when Aristotle's logic, underlining mathematical proofs for two thousands years, entered a crisis with the challenge of the principle of non-contradiction. This crisis was already announced by the discovery of non-Euclidean geometries, in the middle of the 19th century. Various therapies were proposed to free the mathematical proof of the dangerous effects of paradoxes (Russell-Whitehead, Hilbert, Brouwer, etc). This period covers the first three decades of the 20th century and is dominated by the optimistic view stating the possibility to arrange the whole mathematics as a formal system and to decide for any possible statement whether it is true or false. However, even during this period mathematicians were divided with respect to the acceptance of non-effective (non-constructive) entities and proofs (for example, Brouwer's intuitionism rejects the principle of excluded middle in the case of infinite sets). Intuitionism was a signal for the further development of constructive mathematics, culminating with the algorithmic approach leading to computer science.

The sixth period begins with Gödel's incompleteness theorem (1931), for many meaning the unavoidable failure of any attempt to formalise the whole of mathematics. Aristotle's requirement of complete lack of contradiction can be satisfied only by paying the price of incompleteness of the working formal system. Chaitin (1975) has continued this trend of results by proving that from N bits of axioms one cannot prove that a program is the smallest possible if it is more than N bits long; he suggested that complexity is a source of incompleteness because a formal system can capture only a tiny amount of the huge information contained in the world of mathematical truth. This principle has been proved in Calude and Jürgensen [19]. Hence, incompleteness is natural and inevitable rather than mys-

terious and esoteric. This raises the natural question (see Chaitin [26]): *How come that in spite of incompleteness, mathematicians are making so much progress?*

The seventh period belongs to the second half of the 20th century, when algorithmic proofs become acceptable only when their complexities were not too high. Constructiveness is no longer enough, a reasonable high complexity (cost) is mandatory. We are now living in this period. An important event of this period was the 1976 proof of the Four–Colour Problem (4CP): it marked the reconciliation of empirical–experimental mathematics with deductive mathematics, realized by the use of computer programs as pieces of a mathematical proof. Computer refers to classical von Neumann computer. At the horizon we can see the (now hypothetical) quantum computer which may modify radically the relation between empirical–experimental mathematics and deductive mathematics . . .

With the eighth stage, proofs are no longer exclusively based on logic and deduction, but also empirical and experimental. On the other hand, in the light of the important changes brought by authors like Hilbert, already at the beginning of the 20th century, primitive terms became to have an explicit status, axioms show their dependence on physical factors and the axiomatic–deductive method displays its ludic dimension, being a play with abstract symbols. Hilbert axiomatization of geometry is essentially different from Euclid’s geometry and this fact is well pointed out by Dijkstra in [31] where he considers that, by directing their attention towards provability, formalists circumvented the vague metaphysical notion of “truth”. Dijkstra qualifies as “philosophical pollution” the mentality which pushed Gauss not to publish his ideas related to non–Euclidean geometry. Contrary to appearances, believes Dijkstra, Euclidean geometry is not a prototype of a deductive system, because it is based to a large extent on pictures (so–called definitions of points and lines, for instance) motivated by the need of geometric intuition. For Dijkstra, the claim that the Euclidean geometry is a model of deductive thinking, is a big lie. As a matter of fact, the shortcomings to which Dijkstra refers were well-known, as can be seen in Morris Kline’s book [43], pp. 86–88. In contemporary mathematics we are facing a change of perspective, a change of scenario, replacing the old itinerary definition–theorem–proof by another one (see, for instance, W. Thurston), based on ideas, examples and motivations. The interesting fact is that the gap created between proof and intuition by Hilbert prepared the way for a new marriage between deduction and experiment, made possible by the computational revolution, as it was shown by the latest step in the evolution of proofs.

3 Proofs, Theorems and Truths

*Depuis les Grecs, qui dit Mathématique,
dit démonstration. Bourbaki*

What is a mathematical proof? At a first glance the answer seems obvious: a proof is a series of logical steps based on some axioms and deduction rules which reaches a desired conclusion. Every step in a proof can be checked for correctness by examining it to ensure that it is logically sound. In David Hilbert's words: "The rules should be so clear, that if somebody gives you what they claim is a proof, there is a mechanical procedure that will check whether the proof is correct or not, whether it obeys the rules or not." By making sure that every step is correct, one can tell once and for all whether a theorem has been proved. Simple! A moment of reflection shows that the problem may not be so simple. For example, what if the "agent" (human or computer) checking a proof for correctness makes a mistake (agents are fallible)? Obviously, another agent has to check that the agent doing the checking did not make any mistakes. Some other agent will need to check that agent, and so on. Eventually one runs out of agents who could check the proof and, in principle, they could all have made a mistake!

The mistake is the neighbour and the brother of proof, it is both an opponent and a stimulus. An interesting analysis, responding to Joseph L. Doob's challenge, of various possible mistakes in the proof of the 4CT can be found in the work of Schmidt [58]. In 1976, Kenneth Appel and Wolfgang Haken proved the 4CT. They used some of Alfred Kempe's ideas, but avoided his mistake.¹ They showed that if there is a map which needs five colours, then a contradiction follows. If there are several five-colour maps, they have chosen one with the smallest number of countries and proved that this map must contain one of 1,936 possible configurations; they also proved that every one of these possible configurations can be reduced into a smaller configuration which also needs five colours. This is a contradiction because we assumed that we already started with the smallest five-colour map. The reduction step, i.e., the step in which one shows that the 1,936 configurations could be reduced was actually done by brute force computer search through every configuration. No human being could ever actually read the entire proof to check its correctness. For Ron Graham, "The real question is this: If no human being can ever hope to check a proof, is it really a proof?"

In 1996 Robertson, Sanders, Seymour and Thomas [52] offered a simpler proof involving only 633 configurations. The paper [52] concludes with the following interesting comment (p. 24): "We should mention that both our programs use only integer arithmetic, and so we need not be concerned with round-off errors and similar dangers of floating point arithmetic. However, an argument can be made that our "proof" is not a proof in the traditional sense, because it contains steps that can never be verified by humans. In particular, we have not proved the

¹In 1879 Kempe announced his 'proof' of the 4CT in both the magazine *Nature* and the *American Journal of Mathematics*. Eleven years later, Percy Heawood found an error in the proof which nobody had spotted, despite careful checking.

correctness of the compiler we compiled our programs on, nor have we proved the infallibility of the hardware we ran our programs on. These have to be taken on faith, and are conceivably a source of error. However, from a practical point of view, the chance of a computer error that appears consistently in exactly the same way on all runs of our programs on all the compilers under all the operating systems that our programs run on is infinitesimally small compared to the chance of a human error during the same amount of case-checking. Apart from this hypothetical possibility of a computer consistently giving an incorrect answer, the rest of our proof can be verified in the same way as traditional mathematical proofs. We concede, however, that *verifying a computer program is much more difficult than checking a mathematical proof of the same length.*"²

According to Vladimir Arnold, "Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works do consist of proofs, just as poems do consist of characters." These analogies point out both the necessity and the insufficiency of proofs in the development of mathematics. Indeed, spelling is the way poetry takes expression, but it is equally the tool used by the common everyday language, in most cases devoid of any poetic effect. What should be added to spelling in order to get a piece of poetry remains a mystery. A poem consists of characters, but it is much more than a meaningful concatenation of characters.

Mathematics cannot be conceived in the absence of proofs. According to Foiş [37], "the theorem is the brick of mathematics". Obviously, "proof" and "theorem" go together; the object of a proof is to reach a theorem, while theorems are validated by proofs. Theorems are, for the construction of mathematics, what bricks are for the construction of a building. A building is an articulation of bricks and, analogically, a mathematical work is an articulation of theorems. Motivated by a similar view, Jean Dieudonné [34] defines a mathematician as a person who has proved at least one theorem. In contrast, Arnold's analogies point out the fact that mathematics is much more than a chain of theorems and proofs, so implicitly a mathematician should be much more than the author of a theorem. Probably the best example is offered by Bernhard Riemann whose lasting fame does not come (in the first instance) from his theorems or proofs, but from his conjectures, definitions, concepts and examples (see for example, the discussion in Hersh [42], pp. 50–51). Srinivasa Ramanujan is another famous example of a mathematician who produced more results than proofs. What the mathematical community seems to value most are "ideas". "The most respected mathematicians are those with strong 'intuition' " (Harris [39], p. 19).

²Our Italics.

4 Mathematical Proofs: The Syntactic Dimension

Cum Deus calculat, fit mundus. Leibniz

Of course, the first thing to be discussed is Gödel's incompleteness theorem (GIT) which says that *every formal system which is (1) finitely specified, (2) rich enough to include the arithmetic, and (3) consistent, is incomplete*. That is, there exists an arithmetical statement which (A) can be expressed in the formal system, (B) is true, but (C) is unprovable within the formal system. All conditions are necessary. Condition (1) says that there is an algorithm listing all axioms and inference rules (which could be infinite). Taking as axioms all true arithmetical statements will not do, as this set is not finitely listable. But what does it mean to be a "true arithmetical statement"? It is a statement about non-negative integers which cannot be invalidated by finding any combination of non-negative integers that contradicts it. In Alain Connes terminology (see [30], p. 6), a true arithmetical statement is a "primordial mathematical reality". Condition (2) says that the formal system has all the symbols and axioms used in arithmetic, the symbols for 0 (zero), S (successor), + (plus), \times (times), = (equality) and the axioms making them work (as for example, $x + S(y) = S(x + y)$). Condition (2) cannot be satisfied if you do not have individual terms for 0, 1, 2, ...; for example, Tarski proved that Euclidean geometry, which refers to points, circles and lines, is complete. Finally (3) means that the formal system is free of contradictions. The essence of GIT is to distinguish between truth and provability. A closer real life analogy is the distinction between truths and judicial decisions, between what is true and what can be proved in court.³ How large is the set of true and unprovable statements? If we fix a formal system satisfying all three conditions in GIT, then the set of true and unprovable statements is topologically "large" (constructively, a set of second Baire category, and in some cases even "larger"), cf. Calude, Jürgensen, Zimand [20]; because theorems proven in such a system have bounded complexity, the probability that an n -bit statement is provable tends to zero when n tends to infinity (see Calude and Jürgensen [19]).

There is a variety of reactions in interpreting GIT, ranging from pessimism to optimism or simple dismissal (as irrelevant for the practice of mathematics). For pessimists, this result can be interpreted as the final, definite failure of any attempt to formalise the whole of mathematics. For example, Hermann Weyl acknowledged that GIT has exercised a "constant drain on the enthusiasm" with which he has engaged himself in mathematics and for Stanley Jaki, GIT is a fundamental barrier in understanding the Universe. In contrast, scientists like Freeman Dyson acknowledge the limit placed by GIT on our ability to discover the truth in math-

³The Scottish judicial system which admits three forms of verdicts, guilty, not-guilty and not-proven, comes closer to the picture described by GIT.

ematics, but interpret this in an optimistic way, as a guarantee that mathematics will go on forever (see Barrow [6], pp. 218–221).

In modern times a penetrating insight into the incompleteness phenomenon has been obtained by an information–theoretic analysis pioneered by Chaitin in [24]. Striking results have been obtained by studying the Chaitin’s Omega Number, Ω , the halting probability of a self-delimiting universal Turing machine. This number is not only uncomputable, but also (algorithmically) random. Chaitin has proven the following important theorem: *If ZFC (Zermelo set theory with the Axiom of Choice) is arithmetically sound, that is, any theorem of arithmetic proved by ZFC is true, then, ZFC can determine the value of only finitely many bits of Ω , and one can give a bound on the number of bits of Ω which ZFC can determine.* Robert Solovay [56] (see more in [17, 15, 16, 14]) has constructed a self-delimiting universal Turing machine such that ZFC, if arithmetically sound, cannot determine any single bit of its halting probability (Ω). Re-phrased, the most powerful formal axiomatic system is powerless when dealing with the questions of the form “is the m th bit of Ω 0?” or “is the m th bit of Ω 1?”.

Chaitin has constructed an exponential Diophantine equation $F(t; x_1, \dots, x_n) = 0$ with the following property: the infinite binary sequence whose m th term is 0 or 1 depending whether the equation $F(m; x_1, \dots, x_n) = 0$ has finitely or infinitely many solutions is exactly the digits of Ω , hence it is random; its infinite amount of information is algorithmically incompressible. The importance of exponential Diophantine equations comes from the fact that most problems in mathematics can be formulated in terms of these type of equations; Riemann’s Conjecture is one such example. Manin [46], p. 158, noticed that “The epistemologically important point is the discovery that randomness can be defined without any recourse to physical reality ... in such a way that the necessity to make an infinite search to solve a parametric series of problems leads to the technically random answers. Some people find it difficult to imagine that a rigidly determined discipline like elementary arithmetic may produce such phenomena”.

Last but not least, is the truth achieved through a formal proof the ultimate expression of knowledge? Many (mathematicians) will give a positive answer, but perhaps not all. For the 13th century Oxford philosopher Roger Bacon, “Argument reaches a conclusion and compels us to admit it, but it neither makes us certain nor so it annihilates doubt that the mind rests calm in the intuition of truth, unless it finds this certitude by way of experience.” More recently, I. J. Schoenberg⁴ is cited by Epstein ([38]) as saying that Edmund Landau kept in his desk drawer for years a manuscript proving what is now called the two constants theorem: he had the complete proof but could not believe it until his intuition was ready to accept it. Then he published it. A “proof is only one step in the direction of confidence”

⁴Landau’s son-in-law.

argued De Millo, Lipton and Perlis in a classical paper on proofs, theorems and programs [33]. Written in the same spirit is Don Knuth's warning: "Beware of bugs in the above code: I have only proved it correct, not tried it."

5 Mathematical Proofs: The Semantic Dimension

*If one must choose between rigour and meaning,
I shall unhesitatingly choose the latter.* R. Thom

The above quotation turned slogan as "more rigour, less meaning", or better still, "less rigour, more meaning" (Chaitin [27]) points out the necessity to distinguish between the syntactic and the semantic aspects of proofs. Should proofs belong exclusively to logic, according to the tradition started by Greeks such as Pythagoras and Euclid? Or should they also be accepted as a cocktail of logical and empirical-experimental arguments, as in the proof of the 4CT (1976)? Mathematicians are now divided into those giving an affirmative answer to the first question and implicitly a negative answer to the second question and those giving a negative answer to the first question and an affirmative one to the second question. Computationally oriented mathematicians usually belong to the second category, while many other mathematicians (as, for instance, the Fields medalist William Thurston) belong to the first, so for them, the 4CT is not yet proved! Meaning is a key distinction. For mathematicians such as René Thom, Daniel Cohen and William Thurston, correctness by itself does not validate a proof; it is also necessary to "understand" it. "The mission of mathematics is understanding" says Cohen. Paul Halmos has also insisted on the "conceptual understanding". For him a "good" proof of a theorem is one that sheds light on why it is true. It is just the process of understanding which is in question with proofs like that given to the 4CT. Referring to the proof of the 4CT, Halmos says: "I do not find it easy to say what we learned from all that. . . . The present proof relies in effect on an Oracle, and I say down with Oracles! They are not mathematics!" In contrast with Halmos, who hopes that "100 years from now the map theorem will be . . . an exercise in a first-year graduate course, provable in a couple of pages by means of appropriate concepts, which will be completely familiar by then" (see [42], p. 54), R. Hersh thought that the problem itself might be responsible for the way it was solved: he is cited by saying dejectedly "So it just goes to show, it wasn't a good problem after all" (see [23] p. 73).

We will return later to these issues. For the moment we make the following two observations.

- A) Not only the hybrid proofs obtained as a combination of logical and empirical-experimental arguments might be hard/impossible to be understood in their

“globality”; this happens also for some pure deductive proofs. An example is the proof of the classification of finite simple groups called by Danny Gorenstein the “Thirty Years War” (for the classification battles were fought mostly in the decades 1950–1980), a work which comprises about 10,000–15,000 pages scattered in 500 journal articles by some 100 authors.⁵

According to Knuth [44] p. 18, “... program-writing is substantially more demanding than book-writing”. “Why is this so? I think the main reason is that a larger attention span is needed when working on a large computer program than when doing other intellectual tasks. ... Another reason is ... that programming demands a significantly higher standard of accuracy. Things don’t simply have to make sense to another human being, they must make sense to a computer.” Knuth compares his \TeX compiler (a document of about 500 pages) with Feit and Thompson [35] theorem that all simple groups of odd order are cyclic. He lucidly argues that the program might not incorporate as much creativity and “daring” as the proof of the theorem, but they come even when compared on depth of details, length and paradigms involved. What distinguishes the program from the proof is the “verification”: convincing a couple of (human) experts that the proof *works in principle* seems to be easier than making sure that the program *really works*. A demonstration that *there exists a way to compile \TeX* is not enough! Another example, which will be discussed later in this section, is the proof of Fermat’s Last Theorem (FLT).

- B)** Without diminishing in any way the “understanding” component of mathematics we note that the idea of distinguishing between “good” and “bad” proofs on the light they shed on their own truth seems to be, at least to some extent, relative and subjective.

Thom’s slogan ‘more rigour, less meaning’ was the main point in his controversy with Jean Dieudonné (as a representative of the Bourbaki group). Taking rigour as something that can be acquired only at the expense of meaning and conversely, taking meaning as something that can be obtained only at the expense of rigour, we oblige mathematical proof to have the status of what was called in physics a “conjugate (complimentary) pair”, i.e., a couple of requirements, each of them being satisfied only at the expense of the other (see [47]). Famous prototypes of conjugate pairs are (position, momentum) discovered by W. Heisenberg in quantum mechanics and (consistency, completeness) discovered by K. Gödel in logic. But similar warnings come from other directions. According to Einstein (see, for instance, [53] p. 195), “in so far as the propositions of mathematics

⁵Still, there is a controversy in the mathematical community on whether these articles provide a complete and correct proof. For a recent account see Aschbacher [1].

are certain, they do not refer to reality, and in so far as they refer to reality, they are not certain", hence (certainty, reality) is a conjugate pair. Obviously, reality is here understood as an empirical entity, hence mixed with all kinds of imprecision, ranging from obscurity and disorder to randomness, ambiguity and fuzziness [48]. Pythagoras' theorem is certain, but its most empirical tests will fail. There are some genuine obstacles in our attempts to eliminate or at least to diminish the action of various sources of imprecision. Einstein implicitly calls our attention on one of them. Proof, to the extent to which it wants to be rigorous, to give us the feeling of certainty, should be mathematical; but satisfying this condition, means failing to reach reality. In other words, the price we have to pay to obtain proofs giving us the feeling of total confidence is to renounce to be directly connected to reality. There is a genuine tension between certainty and reality, they form a conjugate pair, which is the equivalent of what in the field of humanities is an oxymoronic pair. However, there is an essential difference between Gödel's conjugate pair (consistency, completeness) and Einstein's conjugate pair (certainty, reality). While consistency and completeness are binary logical predicates, certainty and reality are a matter of degree, exactly like the terms occurring in Thom's conjugate pair: rigour and meaning. In last two situations there is room for manipulation and compromise.

Near to the above conjugate pairs is a third one: (rigour, reality), attributed to Socrates (see [51]). A price we have to pay in order to reach rigour is the replacement of the real world by a fictional one. There is no point and no line in the real world, if we take them according to their definitions in Euclid's *Elements*. Such entities belong to a fictional/virtual universe, in the same way in which the characters of a theatrical play are purely conventional, they don't exist as real persons. The rules of deduction used in a mathematical proof belong to a game in the style they are described in the scenario of a Hilbert formal system, which is, as a matter of fact, a machine producing demonstrative texts. A convention underlines the production of theorems and again a convention is accepted in a theatrical play. In the first case, the acceptance of the convention is required from both the author of the proof and its readers; in the second case all people involved, the author, the actors and spectators, have to agree the proposed convention. Since many proofs, if not most of them, are components of a modeling process, we have to add the unavoidable error of approximation involved in any cognitive model. The model should satisfy opposite requirements, to be as near as possible to the phenomenon modelled, in order to be relevant; to be as far as possible from the respective phenomenon, in order to be useful, to make possible the existence of at least one method or tool that can be applied to the model, but not to the original (see [49]). Theorems are discovered, models are invented. Their interaction leads to many problems of adequacy, relevance and correctness, i.e., of syntactic, semantic and pragmatic nature.

In the light of the situations pointed out above, we can understand some ironical comments about what a mathematician could be. It is somebody who can prove theorems, as Dieudonné claimed. But what kind of problems are solved in this way? "Any problem you want, . . . except those you need", said an engineer, disenchanted by his collaboration with a mathematician. Again, what is a mathematician? "It is a guy capable to give, after a long reflection, a precise, but useless answer", said another mathematician with a deep feeling of self irony. Remember the famous reflection by Goethe: "Mathematicians are like French people, they take your question, they translate it in their language and you no longer recognize it".

But things are controversial even when they concern syntactic correctness. In this respect, we should distinguish two types of syntactic mistakes: benign and malign. Benign mistakes have only a local, not global effect: they can be always corrected. Malign mistakes, on the contrary, contaminate the whole approach and invalidate the claim formulated by the theorem. When various authors (including the famous probabilist J. L. Doob, see [58]) found some mistakes in the proof of the 4CT, the authors of the proof succeeded in showing that all of them were benign and more than this, *any other possible mistake, not yet discovered, should be benign*. How can we accept such arguments, when the process of global understanding of the respective proof is in question? The problem remains open. A convenient, but fragile, solution is to accept Thom's pragmatic proposal: a theorem is validated if it has been accepted by a general agreement⁶ of the mathematical community (see [54, 55]).

The problems raised by the 4CT were discussed by many authors, starting with Tymoczko [57] and Swart [59] (more recent publications are D. MacKenzie [45], J. Casti [23], A.S. Calude [13]). Swart proposed the introduction of a new entity called *agnogram*, which is "a theorem-like statement that we have verified as best we could, but whose truth is not known with the kind of assurance we attach to theorems and about which we must thus remain, to some extent, agnostic." There is however the risk to give the status of agnogram to any property depending on a natural number n and verified only for a large, but finite number of values of n . This fact would be in conflict with Swart's desire to consider an agnogram less than a theorem, but more than a conjecture. Obviously, the 4CT is for Swart an agnogram, not a theorem. What is missing from an agnogram to be a theorem? A theorem is a statement which could be checked individually by a mathematician and confirmed also individually by at least two or three more mathematicians, each of them working independently. But already here we can observe the weakness of the criterion: how many mathematicians are to check individually and independently the status of an agnogram to give it the status of theorem?

⁶Perhaps "general" should be replaced here by "quasi-general".

The seriousness of this objection can be appreciated by examining the case of Andrew Wiles' proof of FLT—a challenge to mathematics since 1637 when Pierre de Fermat wrote it into the margin of one of his books. The proof is extremely intricate, quite long (over 100 printed pages⁷), and only a handful of people in the entire world can claim to understand it.⁸ To the rest of us, it is utterly incomprehensible, and yet we all feel entitled to say that “the FLT has been proved”. On which grounds? We say so because *we believe the experts* and *we cannot tell for ourselves*. Let us also note that in the first instance the original 1993 proof seemed accepted, then a gap was found, and finally it took Wiles and Richard Taylor another year to fix the error.⁹

According to Hunt [41], “In no other field of science would this be good enough. If a physicist told us that light rays are bent by gravity, as Einstein did, then we would insist on experiments to back up the theory. If some biologists told us that all living creatures contain DNA in their cells, as Watson and Crick did in 1953, we wouldn't believe them until lots of other biologists after looking into the idea agreed with them and did experiments to back it up. And if a modern biologist were to tell us that it were definitely possible to clone people, we won't really believe them until we saw solid evidence in the form of a cloned human being. Mathematics occupies a special place, where we believe anyone who claims to have proved a theorem on the say—so of just a few people—that is, until we hear otherwise.”

Suppose we loosely define a religion as any discipline whose foundations rest on an element of faith, irrespective of any element of reason which may be present. Quantum mechanics, for example, would qualify as a religion under this definition. Mathematics would hold the unique position of being a branch of theology possessing a “proof” of the fact that it should be so classified. “Where else do you have absolute truth? You have it in mathematics and you have it in religion, at least for some people. But in mathematics you can really argue that this is as close to absolute truth as you can get” says Joel Spencer.

⁷Probabilists would argue that very long proofs can at best be viewed as only probably correct, cf. [33], p. 273. In view of [19], the longer the statement, the lesser its chance is to be proved.

⁸Harris [39] believes that no more than 5% of mathematicians have made the effort to work through the proof. Does this have anything to do with what George Hardy has noted in his famous *Apology*: “All physicists and a good many quite respectable mathematicians are contemptuous about proof.”?

⁹According to Wiles, “It was an error in a crucial part of the argument, but it was something so subtle that I'd missed it completely until that point. The error is so abstract that it can't really be described in simple terms. Even explaining it to a mathematician would require the mathematician to spend two or three months studying that part of the manuscript in great detail.”

6 Mathematical Proofs: The Pragmatic Dimension

Truth is not where you find it, but where you put it. A. Perlis

In the second half of the 20th century, theorems together with their proofs occur with increasing frequency as components of some cognitive models, in various areas of knowledge. In such situations we are obliged to question the theorems not only with respect to their truth value, but also in respect to their adequacy and relevance within the framework of the models to which they belong. We have to evaluate the explanatory capacity of a theorem belonging to a model B, concerning the phenomenon A, to which B is referring. This is a very delicate and controversial matter, because adequacy, relevance and explanatory capacity are a matter of degree and quality, which cannot be settled by binary predicates. Moreover, there is no possibility of optimization of a cognitive model. Any model can be improved, no model is the best possible. This happens because, as we have explained before, a cognitive model B of an entity A has simultaneously the tendency to increase its similarity with A and stress its difference from A. To give only one example in this respect, we recall the famous result obtained by Chomsky [29], in the late 1950s, stating that context-free grammars are not able to generate the English language. This result was accepted by the linguistic and computer science communities until the eighties, when new arguments pointed out the weakness of Chomsky's argument; but this weakness was not of a logical nature, it was a weakness in the way we consider the entity called "natural language". As a matter of fact, the statement "English is a context-free language" is still controversial.

Mathematical proofs are "theoretical" and "practical". Theoretical proofs (formal, ideal, rigorous) are models for practical proofs (which are informal, imprecise, incomplete). "Logicians don't tell mathematicians what to do. They make a theory out of what mathematicians actually do", says Hersh [42], p. 50. According to the same author, logicians study what mathematicians do the way fluid dynamicists study water waves. Fluid dynamicists don't tell water how to wave, so logicians don't tell mathematicians what to do. The situation is not as simple as it appears. Logical restrictions and formal models (of proof) can play an important role in the practice of mathematics. For example, the key feature of constructive mathematics is the identification "existence = computability" (cf. Bridges [12]) and a whole variety of constructive mathematics, the so-called Bishop constructive mathematics, is mathematics with intuitionistic rather than classical underlying logic.

7 Quasi–Empirical Proofs: From Classical to Quantum

*Truth does not change because it is, or is not, believed
by a majority of the people.* Giordano Bruno

The use of large–scale programs, such as Mathematica, Maple or MathLab is now widespread for symbolical and numerical calculations as well as for graphics and simulations. To get a feeling of the extraordinary power of such programs one can visit, for example, the Mathematica website <http://www.wolfram.com>. New other systems are produced; “proofs as programs”, “proof animation” or “proof engineering” are just a few examples (see [40]). In some cases an experiment conveys an aesthetic appreciation of mathematics appealing to a much broader audience (cf. [7, 8, 21]). A significant, but simple example of the role an experiment may play in a proof is given by Beyer [9]. He refers to J. North who asked for a computer demonstration that the harmonic series diverges. We quote Beyer: “His example illustrates the following principle: Suppose that one has a computer algorithm alleged to provide an approximation to some mathematical quantity. Then the algorithm should be accompanied by a theorem giving a measure of the distance between the output of the algorithm and the mathematical quantity being approximated. For the harmonic series, one would soon find that the sum was infinite.” It is interesting to mention that in 1973 Beyer made together with Mike Waterman a similar attempt to compute Euler’s constant; their experiment failed, but the error was discovered later by Brent [11].

New types of proofs motivated by the experimental “ideology” have appeared. For example, rather than being a static object, the *interactive proof* (see Goldwasser, Micali, Rackoff [36], Blum [10]) is a two–party protocol in which the *prover* tries to prove a certain fact to the *verifier*. During the interactive proof the *prover* and the *verifier* exchange messages and at the end the *verifier* produces a verdict “accept” or “reject”. A holographic (or probabilistic checkable) proof (see Babai [4]) is still a static object but it is verified probabilistically. Errors become almost instantly apparent after a small part of the proof was checked.¹⁰ The transformation of a classical proof (which has to be self-contained and formal) into a holographic one requires super-linear time.

The blend of logical and empirical–experimental arguments (“quasi–empirical mathematics” for Tymoczko [57], Chaitin [25, 26, 28] or “experimental mathematics” for Bailey, Borwein [5], Borwein, Bailey [7], Borwein, Bailey, Girgensohn [8]) may lead to a new way to understand (and practice) mathematics. For example, Chaitin argued that we should introduce the Riemann hypothesis as an

¹⁰More precisely, a traditional proof of length l is checked in time a constant power of l while a holographic proof requires only constant power of $\log_2 l$. To appreciate the difference, the binary logarithm of the number of atoms in the known Universe is smaller than 300.

axiom: "I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles. I believe that Euclid's statement that an axiom is a self-evident truth is a big mistake. The Schrödinger equation certainly isn't a self-evident truth! And the Riemann hypothesis isn't self-evident either, but it's very useful. A physicist would say that there is ample experimental evidence for the Riemann hypothesis and would go ahead and take it as a working assumption." Classically, there are two equivalent ways to look at the mathematical notion of proof: *logical*, as a finite sequence of sentences strictly obeying some axioms and inference rules, and *computational*, as a specific type of computation. Indeed, from a proof given as a sequence of sentences one can easily construct a Turing machine producing that sequence as the result of some finite computation and, conversely, given a machine computing a proof we can just print all sentences produced during the computation and arrange them into a sequence.

This gives mathematics an immense advantage over any science: a proof is an explicit sequence of reasoning steps that can be inspected at *leisure*. *In theory*, if followed with care, such a sequence either reveals a gap or mistake, or can convince a sceptic of its conclusion, in which case the theorem *is considered proven*. The equivalence between the logical and computational proofs has stimulated the construction of programs which play the role of "*artificial*" mathematicians. The "theorem provers" have been very successful as "helpers" in proving many results, from simple theorems of Euclidean geometry to the computation of a few digits of a Chaitin Omega Number [18]. "Artificial" mathematicians are far less ingenious and subtle than human mathematicians, but they surpass their human counterparts by being infinitely more patient and diligent.

If a conventional proof is replaced by an "unconventional" one (that is a proof consisting of a sequence of reasoning steps obeying axioms and inference rules which depend not only on some logic, but also on the external physical medium), then the conversion from a computation to a sequence of sentences may be impossible, e.g. due to the size of the computation. An extreme, and for the time being hypothetical example, is the proof obtained as a result of a quantum computation (see Calude and Păun [22]). The quantum automaton would say "your conjecture is true", but (due to quantum interference) there will be no way to exhibit all trajectories followed by the quantum automaton in reaching that conclusion. The quantum automaton has the ability to check a proof, but it may fail to reveal any "trace" of the proof for the human being operating the quantum automaton. Even worse, any attempt to *watch* the inner working of the quantum automaton (e.g. by "looking" inside at any information concerning the state of the ongoing proof) may compromise forever the proof itself! We seem to go back to Bertrand Russell who said that "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true",

and even beyond by adding *and even when it's true we might not know why*.

Speculations about quantum proofs *may not affect* the essence of mathematical objects and constructions (which, many believe, have an autonomous reality quite independent of the physical reality), but they seem to *have an impact* on how we *learn/understand mathematics*, which is through the physical world. Indeed, our glimpses of mathematics are revealed only through physical objects, human brains, silicon computers, quantum automata, etc., hence, according to Deutsch [32], they have to obey not only the axioms and the inference rules of the theory, but the *laws of physics* as well. To complete the picture we need to take into account also the *biological* dimension. No matter how precise the rules (logical and physical) are, we need human consciousness to apply the rules and to understand them and their consequences. Mathematics is a human activity.

8 Knowledge Versus Proof

The object of mathematical rigour is to sanction and legitimize the conquests of intuition. J. Hadamard

Are there intrinsic differences between traditional and ‘unconventional’ types of proofs? To answer this question we will consider the following interrelated questions:

1. Do ‘unconventional’ methods supply us with a proof in some formal language?
2. Do ‘unconventional’ methods supply us with a mathematical proof?
3. Do ‘unconventional’ methods supply us with knowledge?
4. Does mathematics require knowledge or proof?

A blend of mathematical reasoning supported by some silicon or quantum computation or a classical proof of excessive length and complexity (for example, the classification of finite simple groups) are examples of “unconventional” proofs. The ultimate goal of the mathematical activity is the *advance human understanding of mathematics* (whatever this means!).

The answer to the first two questions is affirmative. Indeed, computations are represented in the programming language used by the computer (the ‘unconventional’ computer too), even if the whole proof cannot be globally ‘visualized’ by a human being. The proof can be checked by any other mathematician having the equipment used in the ‘unconventional’ proof. A proof provides knowledge only to the extent that its syntactic dimension is balanced by the semantic one; any gap between them makes the proof devoid of knowledge and paves the way for the proof to become a ritual without meaning. Proofs generating knowledge, quite

often produce much more, for example, 'insight' (think of the insight provided by understanding the algorithm used in the proof).

A misleading analogy would be to replace, in the above questions, '*unconventional*' methods with "*testimony from a respected and (relevantly) competent mathematician*". Certainly, such testimony provides knowledge; it does not qualify as a mathematical proof (even less as a formalized proof), but the result is a "mathematical activity" because it advances our knowledge of mathematics. The difference between 'unconventional' methods and 'relevant testimony' can be found in the mechanisms used to produce outputs: a 'relevant testimony' is the gut feeling of a respected, relevant, competent mathematician, by and large based on a considerable mathematical experience, while an 'unconventional' method produces an objective argument.

There is little 'intrinsic' difference between traditional and 'unconventional' types of proofs as i) first and foremost, *mathematical truth* cannot always be certified by proof, ii) correctness is not absolute, but almost certain, as mathematics advances by making mistakes and correcting and re-correcting them (mathematics fallibility was argued by Lakatos), iii) non-deterministic and probabilistic proofs do not allow mistakes in the applications of rules, they are just indirect forms of checking (see Pollack [50], p. 210) which correspond to various degrees of rigour, iv) the explanatory component, the understanding 'emerging' from proofs, while extremely important from a cognitive point of view, is subjective and has no bearing on formal correctness. As Hersh noticed, mathematics like music exists by some logical, physical and biological manifestation, but "it makes sense only as a mental and a cultural activity" ([42], p. 22).

How do we continue to produce rigorous mathematics when more research will be performed in large computational environments where we might or might not be able to determine what the system has done or why¹¹ is an open question. The blend of logical and empirical-experimental arguments are here to stay and develop. Of course, some will continue to reject this trend, but, we believe, they will have as much effect as King Canute's royal order to the tide. There are many reasons which support this prediction. They range from economical ones (powerful computers will be more and more accessible to more and more people), social ones (skeptical oldsters are replaced naturally by youngsters born with the new technology, results and success inspire emulation) to pure mathematical (new challenging problems, wider perspective) and philosophical ones (note that incompleteness is based on the analysis of the computer's behaviour). The picture holds marvelous promises and challenges; it does not eliminate the continued importance of extended personal interactions in training and research.

¹¹Metaphorically described as "relying on proof by 'Von Neumann says'".

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