INDUCTIVE COMPLEXITY OF THE P VERSUS NP PROBLEM*

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ABSTRACT

This paper does not propose a solution, not even a new possible attack, to the P versus NP problem. We are asking the simpler question: How "complex" is the P versus NP problem? Using the inductive complexity measure—a measure based on computations run by inductive register machines of various orders—developed in [2], we determine an upper bound on the inductive complexity of second order of the P versus NP problem. From this point of view, the P versus NP problem is significantly more complex than the Riemann hypothesis. To date, the P versus NP problem and the Goostein theorem (which is unprovable in Peano Arithmetic) are the most complex mathematical statements (theorems, conjectures and problems) studied in this framework [9, 5, 6, 2, 20].

1. A class of complexity measures

Mathematics is built upon theorems, conjectures and problems both open and resolved. Some problems intuitively seem highly complex, and have perhaps eluded solution for centuries. Others appear to be less complicated. We would like to be able to quantitatively capture this complexity, and thus be able to compare conjectures from vastly different fields of mathematics. One possible scale we can use has been developed in [9, 5, 6, 2, 8] and applied to different problems in [7, 10, 12, 13, 17]. This method considers the most intuitive way to solve a problem, a brute-force search for a counter-example to the claim. If the conjecture is false, a counter-example will eventually be found. But if a conjecture is true, the search will run on forever. If we could somehow determine ahead of time if the search will run forever, we would be able to prove the conjecture is true. Unfortunately, this equates to solving the halting problem, which is known to be undecidable. But not all is lost, since we are not actually trying to *solve* all mathematical conjectures, but rather to *compare* some of them: indeed, we wish to be able to compare conjectures regardless of their true/false or proven/unproven status.

For this aim we will use a more powerful model of computation than the Turing machine, the inductive computation. The search for a counter-example can be coded into a program, and the program can carefully be encoded into a string of ones and zeroes. Thus for some mathematical conjectures we can create a string of bits (along with an explanation of how to unambiguously read off the program) and say 'if this program halts, the conjecture is false; if it does not halt, the conjecture is true'. It naturally follows that some conjectures can be 'encoded' into bits more simply than others; these conjectures will be considered of low complexity. More complicated conjectures may take a large program and a huge number of bits; these programs are considered to have high complexity. Time complexity plays no role in this analysis.

Although the results we obtain may shed new light on the statements we analyse, they are not intended to solve the problems expressed by those statements, nor to predict how easy/difficult could be to find their solutions.

2. The P versus NP problem

The processing power of computers has grown—and continues to grow—incredibly quickly, and computer-users have become accustomed to newer and faster computers continually being released on the market. In such an environment, it might seem like there is no bound to the size and type of problems that computers can solve—and even if a program runs slowly on today's computers, surely in a few years it will be zipping along on the faster computers of the future. Unfortunately, this is not the case. The problem lies in the asymptotic behaviour of certain algorithms, i.e. their behaviour when the problem instance size gets very large. It makes sense that the larger a problem input size, the longer it takes to solve it, but in some cases the needed time grows faster than we will ever be able to account for with faster

computers. The usual solution is to simply find a faster, more efficient algorithm. But for a large class of problems, many of them of critical practical significance, *no* efficient algorithms have been found. This class is called NP (nondeterministic polynomial), while the class of problems that are known to have efficient algorithms is called P (deterministic polynomial, or, simply, polynomial).

Furthermore, there exists a set of NP problems, called NP-complete, which if one could figure out how to solve just one of them in polynomial-time then we could solve *all* of them in polynomial-time. In asking the question 'Does P=NP?' we are asking if it is possible to solve all NP problems in polynomial-time, or equivalently, if it is possible to solve an NP-complete problem in polynomial-time.

As a concrete example, we present the NP-complete problem used in our program: the subset-sum problem [16] (subsection 35.5: The subset-sum problem). This problem starts with a collection of numbers, and a target number—an instance of the problem—and asks the question: *Does some subset of our collection add up to equal the target?* In small instance sizes this is simple. For example, we can easily check that no subset of (1,2,5) adds up to 4, or that there is a subset of (1,2,5,8) that adds to 7 (namely, 2 and 5). But as the instance size gets larger, the number of possible subsets grows exponentially, and it takes exponential time to check every subset. The brute-force algorithm for solving the subset-sum problem cycles through all subsets of N numbers and, for every one of them, checks if the subset sums to the right number. The running-time is of order $O(N \cdot 2^N)$, since there are 2^N subsets and, to check each subset, we need to sum at most N elements. A faster algorithm proposed by Horowitz and Sahni [21] runs in time $O(2^{N/2})$. If one could show that there is some algorithm that solves every possible instance of subset-sum in polynomial-time, then we would show that P=NP.

The P versus NP problem, formulated independently by Cook [14] and Levin [24], is considered to be one of the most challenging open problems in mathematics. The Clay Mathematics Institute will award a prize of \$1.000.000 for its first correct solution, [28]. A constructive proof for P = NP based on an efficient simulation would have significant practical consequences; a proof for P $\neq$ NP (which is widely believed to be the case) would lack practical computational benefits, but would have important theoretical value.

With decades of research dedicated to its resolution, substantial insight was obtained: see more in the official Clay Mathematics Institute presentation of the problem by Cook [15], the papers by Fortnow [18] and Mulmuley [26], Moore-Mertens book [25] (Chapter 6, The Deep Question: P vs NP), and Wöginger's webpage [30].

The use of parallel computers instead of sequential computers does not help because if a problem requires exponential time to be solved on a sequential computer, every parallel computer with finite number of processors solving it runs also in exponential time since the speed-up can be only by a constant factor (e.g. the number of processors). The above result is not "absolute", i.e. it depends on the geometry of the space where the computation is run. More precisely, the above result is true in case the real time-space is Euclidean, and the reason is that the volume

of a sphere of radius r is a polynomial in r, $O(r^3)$. If the real time-space is hyperbolic, the volume of the sphere grows exponentially with r, and this growth can be exploited to dramatically speed-up parallel computations, hence the possibility to solve NP–hard problems in polynomial time. For more details see [23].

Is the polynomial-time algorithm the "correct" mathematical model for feasible computation? An affirmative answer is provided by Cobham's thesis which states that "P" means *easy* while "the complement of P" means *hard*. Is Cobham's thesis as "credible" as the Church-Turing thesis, which deals with computability in principle, i.e. by disregarding resources? According to Davis [22] (p. 568–569) the answer is negative[a]. In fact, we believe that the concept of feasible computation like the concept of randomness are paradoxical (blind spots in the terminology of [3]): they cannot be grasped in finite words. More precisely, *we conjecture that there is no good mathematical model for feasible computation*. From this perspective, the P versus NP problem is less a computer science problem than a mathematical one.

3. Goal

By measuring the complexity of the P versus NP problem we hope to shine a little more light on the problem; certainly, this is not an attempt to solve it. To do this, we have developed an inductive register machine program that searches for a counter-example to the claim that "P $\neq$ NP". This counter-example would be a program that runs in polynomial-time for all instances of the subset-sum problem, our choice of NP-complete problem. The register machine program has a prefix-free binary encoding, and the length of this string determines an upper bound of the complexity class of the P versus NP problem.

4. Method

The register machine language we use is a refinement, constructed in [6], of the language in [9]; see also [17]. The register machine language is simple and minimal (each instruction is essential; no instruction can be reduced to a combination of the other instructions). It consists of the following instructions:

= R1,R2,R3   If the content of R1 and R2 are equal, then the execution continues at the R3rd instruction of the program. If the contents of R1 and R2 are not equal, then execution continues with the next instruction in sequence.

& R1,R2   The content of register R1 is replaced by R2.

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[a]In the discussions following J. Hartmanis' invited lecture *Turing Machine Inspired Computer Science Results*, CiE2012, 22 June 2012, http://www.mathcomp.leeds.ac.uk/turing2012/WScie12/Content/abstracts/juris.html, M. Davis asked the question he posed the speaker about 30 years ago: "How would you feel if P=NP with a polynomial of degree 100?" Hartmanis' original answer was: "God cannot be so cruel!".

+ R1,R2   The content of register R1 is replaced by the sum of the contents of R1 and R2.

! R1   One bit is read into the register R1, so the content of R1 becomes either 0 or 1. Any attempt to read past the last data-bit results in a run-time error.

%   This is the last instruction for each register machine program before the input data. It halts the execution in two possible states: either successfully halts or it halts with an under-read error.

A register machine program is a finite list of these instructions. It is allowed access to an arbitrary number of registers, and each register can hold an arbitrarily large positive integer. The prefix free binary encoding of these instructions is discussed in detail in [5, 6], and briefly below.

Each instruction has its own binary op-code, registers names are encoded as the string $\text{code}_1 = \{0^{|x|}1x,\ x \in \{0,1\}^*\}$ and literals are encoded $\text{code}_2 = \{1^{|x|}0x,\ x \in \{0,1\}^*\}$. Some instructions can take registers or literals, but this encoding gives an unambiguous distinction between the two options. The encodings are summarised below:

(1) `& R1,R2` is coded in two different ways depending on R2: $01\text{code}_1(R1)\text{code}_i(R2)$, where $i = 1$ if R2 is a register and $i = 2$ if R2 is an integer.
(2) `+ R1,R2` is coded in two different ways depending on R2: $111\text{code}_1(R1)\text{code}_i(R2)$, where $i = 1$ if R2 is a register and $i = 2$ if R2 is an integer.
(3) `= R1,R2,R3` is coded in four different ways depending on the data types of R2 and R3: $00\text{code}_1(R1)\text{code}_i(R2)\text{code}_j(R3)$, where $i = 1$ if R2 is a register and $i = 2$ if R2 is an integer, $j = 1$ if R3 is a register and $j = 2$ if R3 is an integer.
(4) `!R1` is coded by $110\text{code}_1(R1)$.
(5) `%` is coded by 100.

Programs often need to execute the same operations many different times, and it is convenient to create routines for these operations. Routines that our program uses include MUL (multiply), POW (power/exponentiation), CMP (compare), SUBT (subtraction) and DIV2 (halves a number).

As a concrete example, the subtraction routine is given in Table 4. The routine SUBT uses registers `a` through `e`, computes $a - b$, stores the answer in `d`, then returns to the line number stored in `c`. It assumes that $a \geq b$.

The register names, `a` = 010, `b` = 00100, `c` = 00101, `d` = 00111, and `e` = 011, were chosen to minimise the overall number of bits used. In total, this routine is represented by the 70 bit string:

0100111100010110011110001100100101011010001011000011110110101001011010

Label	Instruction	Comments	Binary representation
SUBT1	& d, 0		01 00111 100
LS1	& e, d		01 011 00111
	+ e, b		100 011 00100
	= e, a, c	// d+b=a	101 011 010 00101
	+ d, 1		100 00111 101
	= a, a, LS1	// loop	101 010 010 11010

Table 1. SUBT

5. Register machine language implementation of arrays

We use the coding for the array data structure library developed by Dinneen [17] which represents arrays (lists) in a single register variable. An integer element a_i within an array A is represented as a sequence of a_i bits 0; the bit 1 is used as a (leading) separator or deliminator of the array elements. If there are no 1's (e.g. the register has value 0) then we have an array of size 0. For example, the array [1,4,0] is encoded 10100001 or 11000010, depending on chosen endianness (the encoding is from left to right or vice-versa).

A number of array operations are used frequently, and these have been packaged into subroutines. In our particular program we make use of SIZE (returns the size of the array), APPEND (appends one element), ELM (returns the element at a particular index) and RPL (replaces the element at a particular index).

6. From standard computation to inductive computation

The main program for the P versus NP problem consists of two nested loops. The outer loop tests every program-polynomial tuple. For each program and polynomial, the inner loop checks if the program can solve all instances of the subset-sum problem in polynomial steps or less. In the usual model of computation these nested loops have a serious pitfall: The program may run forever for two different reasons. It may run forever because it never finds a program that works (there are infinitely many programs), in which case P does not equal NP. The second reason it may run forever is because it *has* found a program that works, and since there are an infinite number of instances of the subset-sum problem, it loops forever testing all of them.

To resolve this issue, we chose to use a slightly modified version of computation: the *inductive computation* [1]. Under this model, a program is allowed to run forever but still be considered to give an answer if, after a finite number of steps, the *output stabilises*. To make our program suitable for an inductive register machine program, we must modify each loop in the following way: If the loop is successfully running, write a 1 into the output register, otherwise when the loop halts write a 0 into the output register (and stop looping). We thus ensure that the output register will not oscillate, and under the inductive computation model it will always return a result. Namely, the output will be 1 if the loop runs forever, and 0 if at some point it will

halt.

Finite standard Turing and inductive computations produce the same results; however, the inductive computation is more powerful than standard computation as in the former case some infinite computations can produce outputs.

In what follows we will use the above register machine language as a *universal prefix-free inductive machine* U^{ind} (see more in [2]). This type of computation gives rise to an *inductive complexity measure*.

7. An inductive register program for P versus NP

It is easy to note that P $\neq$ NP if and only if every polynomial-time program (i.e. a pair consisting of a program and a polynomial controlling the time of the execution of the program) cannot solve at least one instance of the subset-sum problem. Hence the P versus NP problem can be represented by a Π_2–sentence, i.e. a sentence of the form $\forall n \exists i R(n, i)$, where R is a computable predicate.[b] Starting from a representation $\forall n \exists i R(n, i)$ (where R is a computable predicate) of the P versus NP problem, we construct the inductive register machine program of first order $T_R^{ind,1}$ defined by

$$T_R^{ind,1}(n) = \begin{cases} 1, \text{ if } \exists i R(n, i), \\ 0, \text{ otherwise} . \end{cases}$$

Next we construct the inductive register machine $M_R^{ind,2}$ defined by

$$M_R^{ind,2} = \begin{cases} 0, \text{ if } \forall n \exists i R(n, i), \\ 1, \text{ otherwise} . \end{cases}$$

Clearly,

$$M_R^{ind,2} = \begin{cases} 0, \text{ if } \forall n \, (T_R^{ind,1}(n) = 1), \\ 1, \text{ otherwise} , \end{cases}$$

hence we say that $M_R^{ind,2}$ is an *inductive register machine of second order*.

Note that the predicate $T_R^{ind,1}(n) = 1$ is well-defined because the inductive register machine of first order $T_R^{ind,1}$ always produces an output. However, the inductive register machine $M_R^{ind,2}$ is of the *second order* because it uses an inductive register machine of the first order $T_R^{ind,1}$. This shows that the inductive register machine of second order $M_R^{ind,2}$ solves the P versus NP problem.

MAIN, the main algorithm for $M_R^{ind,2}$ that solves the P versus NP problem, is presented in the algorithm below. As we have already mentioned, the program consists of two nested loops; the outer loop goes through all possible program and polynomial pairs, and the inner loop runs the program with every possible instance of the subset-sum problem, letting it execute at most polynomial steps for each instance.

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[b]By now the reader has understood that the word "problem" has different meanings in Cook's theory and in our complexity analysis.

Our algorithm requires the testing of the correctness of every possible polynomial-time program for each instance of the subset-sum problem. Is it possible to achieve this testing given that the correctness problem is undecidable? The answer is affirmative because we are dealing with time-controlled computations, each running in a finite amount of time; the correctness test may take in some cases exponential time to complete.

It is important to note that the correctness of the polynomial-time program is established when it runs accurately on *all possible instances* of subset-sum problem. It is not enough for the program to run correctly in only some cases. Indeed, since we loop through all possibly instances, we will eventually come across the cases in which an invalid program fails. In particular, programs that randomly "guess", or that always give the same answer eventually fail.

The program-polynomial tuples are generated by incrementing through the natural numbers, treating each number as an array and asking if that array has three elements. Non-complying numbers are ignored; otherwise, we consider the first and second elements to be C and J respectively, which define the polynomial[c] $C*(x^J+1)$, and the third element to be the program P. To enumerate all instances of subset-sum problem, we similarly go through the natural numbers and interpret them as arrays with at least 2 elements. For each array we ask the question: Does some subset of its first $(N-1)$ elements sum to the Nth element, where N is the size of the array?

MAIN: RESULT IS 1 IF P≠NP, 0 IF P=NP

```
// Z is the output register, while the loop is running it is set to 1
Z ← 1
for all tuples (C,J,P) do
    // Now we run the simulation (also on an inductive Turing machine)
    run SIM
    // check the result register (Y)
    if Y = 1 then
        // found a polynomial-time algorithm, P=NP
        Z ← 0
        HALT
    else
        // that program didn't work, try the next one
        continue
    end if
end for
```

[c]Obviously, in this way we cover all possible run-time polynomials.

When simulating the program P we give it access to an unlimited number of registers which are stored in an array R. The unique coding of a register name is used to represent the index of that register in the array R, and the array dynamically grows to include all possible simulated registers. After running the program P we assume that its answer to the subset sum instance is in the register encoded as 010, which corresponds to R[2][d]. One can easily check if the proposed answer is correct.

As an example, consider the 672nd loop of the outer (main) program. In binary, this corresponds to the number 101010, which, interpreted as an array is $[1, 1, 1]$, meaning $C = 1$, $J = 1$ and $P = 1$. In binary, the program is simply 1.

SIM: RESULT IS 1 IF PROGRAM P SUCCEEDS IN POLYNOMIAL-TIME, 0 IF NOT

```
// Y is the output register, while the loop is running it is set to 1
Y ← 1
for all instances S of subset-sum do
    Simulate program P with input S for at most (C * (|S|^J + 1)) steps.
    if P executed without error and calculated the correct answer then
        continue to next instance
    else
        // This program doesn't work, stop looping
        Y ← 0
        return
    end if
end for
```

The outer program then executes the simulation on another inductive Turing machine. The first instance of subset-sum that it will test is 11, the two element array $[0, 0]$. It asks: Is there a subset of elements in $[0]$ that add up to 0? Clearly, the answer is positive. In preparation for simulation, the input for program P is made prefix free: 11 becomes 11011, so the program is expected to read five bits (which are read right to left) before halting. The simulation begins, but the parser quickly realises that 1 is not a valid program, and will branch to an error statement. In this statement, the inductive Turing machines writes 0 to its output register (named Y in the pseudocode) and halts. The main loop (the outer inductive Turing machine) queries the output register, reads the zero and continues on to the next loop.

In the simulation process every syntactical error is detected. The simulated program may fail because of compile or run time errors, e.g. invalid use of a register/literal encoding, the halt instruction appears in the middle of a program,

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[d]There is no register named 00 or 01, so R[2] is the first possible register that the simulated program could use. Of course, it may not ever use this register, in which case R[2] would always be 0, the default initial value.

branching out of bounds, not reading all input bits. The program is also disqualified if, after the prescribed polynomial number of steps, it has not naturally halted.

8. An upper bound for the inductive complexity of P versus NP

To every mathematical sentence of the form $\rho = \forall n \exists i R(n, i)$, where $R(n, i)$ is a computable predicate—called Π_2–sentence, we associate the inductive register machine of second order $M_R^{ind,2}$ as above.

Note that there are many programs for the universal prefix-free inductive machine U^{ind} which implement $M_R^{ind,2}$. For each of them we have:

$$\forall n \exists i R(n, i) \text{ is true if and only if } U^{ind}(M_R^{ind,2}) = 0.$$

The inductive complexity measure of second order is defined by:

$$C_U^{ind,2}(\rho) = \min\{|M_R^{ind,2}| \ : \ \rho = \forall n \exists i R(n, i)\},$$

and, correspondingly, the *inductive complexity class of second order* is:[e]

$$\mathfrak{C}_{U,n}^{ind,2} = \{\rho \ : \ \rho = \forall n \exists i R(n, i), C_U^{ind,2}(\rho) \leq 2^{10} \cdot n\}.$$

The complexity measure, as stated, is unfortunately *incomputable* (see [4]), so we resort to measuring upper bounds of the complexity. This is still a useful measurement and allows us to rank and compare Π_2–sentences [5].

The inductive register program for the P versus NP problem consists of 362 instructions. More details about the program, including flowcharts for the main program, principal routines, simulation, run and commands for the instructions &, +, =, !, as well as a fully commented version of the program are presented in Appendices A and B in [11]. The syntactical correctness of the program was checked using a tool developed in [19].

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[e]The threshold $2^{10}n$ is to some extent arbitrary; its main goal is only to provide a scale to compare/rank mathematical statements in a uniform way. An argument in favour of our choice is the following. If instead of U we use a different universal prefix-free Turing machine U' then one can compute a constant c (depending upon U and U') such that for every ρ one has $|C_U^{ind,2}(\rho) - C_{U'}^{ind,2}(\rho)| \leq c$. Experimental calculation shows that for minimal machines the constant c is smaller than 2^{10}.

1	= a a 265	41	= a a 26	81	= a a ce
2	& d 1	42	+ cs c	82	& ar a
3	& e 0	43	= a 0 45	83	& br b
4	= e b c	44	= a a 21	84	& cr c
5	+ e 1	45	& a as	85	& ir I
6	& f 1	46	& b bs	86	& b 88
7	& dp d	47	& c cs	87	= a a 37
8	= f a 4	48	= a a b	88	& nr c
9	+ d dp	49	& ap 0	89	& I 0
10	+ f 1	50	+ a a	90	& fr 0
11	= a a 8	51	+ a 1	91	= I nr 107
12	& d 0	52	= ap b 56	92	= I ir 103
13	= a b c	53	+ a a	93	& a ar
14	& e 0	54	+ ap 1	94	& c 96
15	& d 1	55	= a a 52	95	= a a 57
16	= a e c	56	= a a c	96	& b d
17	& d 2	57	& ae a	97	& a fr
18	= b e c	58	& be b	98	& c 100
19	+ e	59	& ce c	99	= a a 49
20	& d 0	60	& b 62	100	& fr a
21	& e d	61	= a a 37	101	+ I 1
22	+ e b	62	& a c	102	= a a 91
23	= e a c	63	& b I	103	& b br
24	+ d 1	64	& c 66	104	& a fr
25	= a a 21	65	= a a 20	105	& c 100
26	& ad a	66	& ie d	106	= a a 49
27	& a 0	67	& d 0	107	& a fr
28	& c 0	68	= ie d 73	108	& b br
29	& e a	69	& b 71	109	& c cr
30	+ e e	70	= a a 26	110	& I ir
31	= e ad b	71	+ d c	111	= a a c
32	& c 1	72	= a a 68	112	& bg b
33	+ e 1	73	& d 0	113	& ISLIT 0
34	= e ad b	74	& b 76	114	& b 116
35	+ a 1	75	= a a 26	115	= a a 26
36	= a a 28	76	= c 1 79	116	& b bg
37	& as a	77	+ d 1	117	+ a a
38	& bs b	78	= a a 74	118	= c 0 138
39	& cs 0	79	& a ae	119	& ISLIT 1
40	& b 42	80	& b be	120	+ a 1

```
121    & brl b         161    & a c          201    & b LIT2
122    & LIT 0         162    & b REG        202    & c 171
123    = a 0 321       163    & c 165        203    = a a 82
124    & irl 0         164    = a a 12       204    = e 1 207
125    & b 127         165    = d 1 168      205    & b 175
126    = a a 26        166    & a arr        206    = a a 138
127    + irl 1         167    = a a brr      207    + NEXT 1
128    = c 0 130       168    + R R          208    & b 210
129    = a a 125       169    + R 1          209    = a a 138
130    & jrl 0         170    = a a 158      210    = IN 0 321
131    + jrl 1         171    & bp b         211    & a IN
132    + LIT LIT       172    = T XT 321     212    & b 214
133    = c 0 135       173    & a P          213    = a a 26
134    + LIT 1         174    & ip 1         214    & b c
135    = irl jrl brl   175    = a 0 240      215    & a R
136    & b 131         176    = ip NEXT 179  216    & I REG
137    = a a 26        177    & e 0          217    & c 171
138    & brr b         178    = a a 181      218    = a a 82
139    & REG 0         179    & e 1          219    & b 221
140    = a 0 321       180    + T 1          220    = a a 26
141    & irr 0         181    + ip 1         221    = c 0 226
142    & b 144         182    & b 52         222    = e 1 224
143    = a a 26        183    = a a 26       223    = a a 191
144    = c 1 321       184    = c 0 56       224    & rn 199
145    + irr 1         185    & b 187        225    = a a 242
146    = c 1 149       186    = a a 26       226    = e 1 232
147    & b 145         187    = c 0 321      227    & b 229
148    = a a 26        188    & b 190        228    = a a 138
149    & jrr 0         189    = a a 26       229    & b 231
150    + jrr 1         190    = c 0 204      230    = a a 112
151    + REG REG       191    = e 1 196      231    = a a 194
152    = c 0 154       192    & b 194        232    & rn 234
153    + REG 1         193    = a a 138      233    = a a 242
154    = irr jrr 157   194    & b 175        234    = LIT1 LIT2 236
155    & b 150         195    = a a 112      235    = a a 171
156    = a a 26        196    & rn 198       236    & rn 238
157    & arr a         197    = a a 242      237    = a a 253
158    & a R           198    + LIT2 LIT1    238    & NEXT LIT2
159    & b 160         199    & I REG1       239    = a a 171
160    = a a 37        200    & a R          240    = ip NEXT bp
```

241	= a a 321	282	& c 284	323	= IN 0 325
242	+ NEXT 1	283	= a a 57	324	= a a 321
243	& b 245	284	& P d	325	& a 2
244	= a a 138	285	= a a 289	326	& b N
245	& REG1 REG	286	= Y 0 267	327	& c 329
246	& I REG1	287	& Z 1	328	= a a 2
247	& ax a	288	= a a 362	329	& da d
248	& a R	289	& S 0	330	& e 0
249	& c 251	290	& Y 1	331	& f 0
250	= a a 57	291	+ S 1	332	+ e 1
251	& LIT1 d	292	& a S	333	= e da 356
252	& a ax	293	& b 295	334	& a e
253	& b 255	294	= a a 37	335	& I 0
254	= a a 112	295	& N c	336	& s 0
255	& ax a	296	= N 1 291	337	= a 0 332
256	= ISLIT 1 262	297	& a N	338	+ I 1
257	& I REG	298	& b J	339	& b 341
258	& a R	299	& c 301	340	= a a 26
259	& c 261	300	= a a 2	341	= c 0 337
260	= a a 57	301	+ d 1	342	& aa a
261	& LIT d	302	& e 0	343	& a S
262	& LIT2 LIT	303	& XT 0	344	& c 346
263	& a ax	304	= e C 308	345	= a a 57
264	= a a rn	305	+ XT d	346	+ s d
265	& PP 0	306	+ e 1	347	& a aa
266	& Z 0	307	= a a 304	348	= a a 337
267	+ PP 1	308	& IN S	349	& a S
268	& a PP	309	+ IN IN	350	& I N
269	& b 271	310	& i 0	351	& c 353
270	= a a 37	311	= i N 316	352	= a a 57
271	= c 3 273	312	+ i 1	353	= s d 355
272	= a a 267	313	+ IN IN	354	= a a 332
273	& I 1	314	+ IN 1	355	& f 1
274	& c 276	315	= a a 311	356	& I 2
275	= a a 57	316	& T 0	357	& a R
276	& C d	317	& R 3	358	& c 291
277	& I 2	318	& NEXT 1	359	= a a 57
278	& c 280	319	& b 323	360	= d f 291
279	= a a 57	320	= a a 171	361	= a a 321
280	& J d	321	& Y	362	%
281	& I 3	322	= a a 286		

The first ten instructions in the program for the P versus NP problem are presented in machine code followed by the length of the corresponding binary encoding (accruing to the scheme presented in Section 4.).

```
Instruction number=1              Instruction number=2
= 00                              & 01
a 010                             d 00110
a 010                             1 101
265 11111111000001011             The number of bits 10
The number of bits 25

Instruction number=3
& 01                              Instruction number=7
e 00101                           & 01
0 100                             dp 00000101100
The number of bits 10            d 00110
                                  The number of bits 18

Instruction number=4
= 00                              Instruction number=8
e 00101                           = 00
b 011                             f 0001011
c 00100                           a 010
The number of bits 15            4 11010
                                  The number of bits 17

Instruction number=5
+ 111                             Instruction number=9
e 00101                           + 111
1 101                             d 00110
The number of bits 11            dp 00000101100
                                  The number of bits 19

Instruction number=6
& 01                              Instruction number=10
f 0001011                         + 111
1 101                             f 0001011
The number of bits 12            1 101
                                  The number of bits 13
```

which leads to the first 150 bits of the code for the P versus NP program:

$$0001001011111111000001011010011010101001011000001$$
$$0101100100111001011010100010111010100000101100011$$
$$00000010110101101011100110000001011001110001011101$$

After optimising the coding according to the frequency of registers, the code of

the inductive register machine program consists of 6,495 bits,[f] putting the P versus NP problem into the *inductive complexity class of second order 7*. The full binary encoding of the program is in Appendix E of [11].

The Riemann hypothesis, another problem on the list of the Clay Mathematics Institute millennium open problems [29] and arguably the most important open problem in mathematics, is in the *inductive complexity class of first order 3*, a significantly lower complexity class. Goodstein theorem—which is unprovable in Peano Arithmetic—is also in *the inductive complexity class of second order 7*. The Collatz conjecture and the twin-prime conjecture fall into the first inductive complexity class of second order, cf.[2]; all other problems studied till now fall into inductive complexity classes of first order smaller than 4, cf. [7, 10, 12, 13, 17].

There are probably further modifications that can be made to shorten the program, possibly by improving the simulation potential polynomial-time programs and/or by using a different NP–complete problem. Our analysis has used the representation of the P versus NP problem as a Π_2–sentence; it is an open question whether *the P versus NP problem is a Π_1–sentence*, i.e. of the form $\forall n P(n)$, where P is a unary predicated.

We reiterate that with the inductive complexity we compare problems from one point of view only, and the upper-bounds obtained, reflecting the current knowledge of those problems, are time-dependent. In this context it would be interesting to find lower bounds of the inductive complexity for the mathematical problems for which upper bounds have been estimated.

Acknowledgment

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[f]This code is with 296 bits shorter than the one announced in the *UCNC 2012* extended abstract.

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