Journal of Automata, Languages and Combinatorics **23** (2018) 1–3, 103 – 109

A SIMPLE CONSTRUCTION OF ABSOLUTELY DISJUNCTIVE LIOUVILLE NUMBERS

Cristian S. Calude[(A)] Ludwig Staiger[(B)]

[(A)] *Department of Computer Science, University of Auckland, Auckland, New Zealand*
`cristian@cs.auckland.ac.nz`

[(B)] *Martin-Luther-Universität Halle-Wittenberg, Institut für Informatik Halle, Germany*
`staiger@informatik.uni-halle.de`

ABSTRACT

A disjunctive sequence is an infinite sequence in which every finite string appears as a substring. An absolutely disjunctive number (or lexicon) is a real whose expansion with respect to every base is disjunctive.

In this note we give a simple construction of absolutely disjunctive Liouville numbers (reals which can be "quite closely" approximated by sequences of rationals).

Keywords: disjunctive sequences, Liouville numbers, computability

1. Introduction

Disjunctivity is a qualitative form of (Borel) normality: normal sequences are disjunctive, but the converse is false. Like normality [7, 15], disjunctivity is not base-invariant (for more details see [9]).

Jürgensen and Thierrin [11] gave a construction of Liouville numbers disjunctive in base b. Highly incomputable Liouville numbers disjunctive to every base have been presented in [19, Theorem 15].

The recent construction of a computable absolutely normal Liouville number in [1] yields also computable, absolutely disjunctive Liouville numbers. This construction, however, is based on rather complicated measure-theoretic arguments from [2]. The aim of this note is to present a simple algorithm producing weaker examples, that is, computable Liouville numbers disjunctive to every base.

1.1. Notation

In this section, we introduce the notation used throughout the paper. By

$$\mathbb{N} = \{0, 1, 2, \ldots\},$$

we denote the set of natural numbers. Its elements will be usually denoted by letters $i, \ldots, n$. The set $A_b = \{0, 1, \ldots, b-1\}$, where $b \geq 2$ is a positive integer, is called the *b-base*; the elements of A_b are called b-digits. By A_b^* we denote the set of all finite strings (words) with ε denoting the empty string; A_b^ω is the set of all infinite sequences (ω-words) over A_b; ω-words are usually denoted by $\mathbf{x}, \mathbf{y}$. The length of a finite or infinite string η over A_b is denoted by $|\eta|$.

For $w \in A_b^*$ and $\eta \in A_b^* \cup A_b^\omega$, $w \cdot \eta$ is their *concatenation*. This concatenation product extends in an obvious way to subsets $L \subseteq A_b^*$ and $B \subseteq A_b^* \cup A_b^\omega$. If $w \in A_b^*$ and $i \geq 0$ is an integer, then w^i is the concatenation $ww \cdots w$ (i times) and w^ω is the infinite concatenation $ww \cdots w \cdots$. The $\cdot$ operator can be omitted when the meaning is clear, as in $w\eta$.

By $w \sqsubseteq u$ and $w \sqsubset \mathbf{y}$, we denote that w is a prefix of u and $\mathbf{y}$, respectively. Further, let

$$\mathbf{pref}(\mathbf{y}) = \{ w \mid w \sqsubset \mathbf{y} \}$$

and

$$\mathbf{infix}(\mathbf{y}) = \{ w \mid \exists v (v \cdot w \sqsubset \mathbf{y}) \}$$

be the set of prefixes and infixes of $\mathbf{y}$, respectively.

1.2. Preliminary Definitions and Results

In this section, we define the classes of real numbers studied in the paper.

A real number α is called a *Liouville number* if it is irrational and for every positive integer k, there exist integers p_k and q_k with $q_k > 1$ such that

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^k} \cdot$$

A real $\alpha \in [0, 1]$ is called *computable* if for some $b \geq 2$ it has a b-ary computable expansion $\alpha = 0.x_1x_2\ldots$, that is, there is a computable function f_α such that $f_\alpha(n) = x_n$, for all $n \geq 1$. This condition is equivalent to the requirement that there is a computable sequence of rationals $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}}$ such that

$$|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{2^n},$$

for all $n \in \mathbb{N}$. This shows that if α is computable, then its expansions in any base b are computable.

Originally, ω-words $\mathbf{x}$ were called disjunctive because the syntactic monoid of the set $\{\mathbf{x}\}$ is disjunctive, that is, its syntactic congruence is the identity (see [10]). Equivalently, disjunctive ω-words are those which have every finite word as subword.[1] In fact, in a disjunctive ω-word every word appears infinitely many times.

[1]In view of this latter property, they are also called *rich ω-words*.

Disjunctivity is also related to randomness: disjunctive ω-words are exactly the ω-words not contained in any null-set definable by finite automata [16, 17]. For more properties of disjunctive sequences see [4].

A real number $\alpha \in [0,1]$ is *disjunctive* (or *rich*) *in base* b if its b-ary expansion is disjunctive. For example, Champernowne's number $0.0123456789101112\ldots$ is computable and disjunctive in base 10 [8]. No rational number is disjunctive in any base.

An *absolutely disjunctive* number (or *lexicon*) is a real which is disjunctive in every base. Every Martin-Löf random real is a lexicon, but the converse is false [3].

In the sequel, we denote by $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{D}$ the set of all Liouville numbers, computable numbers and absolutely disjunctive numbers in $[0,1]$, respectively.

1.3. Co-meagre and Dense Sets

It is useful to consider the unit interval $[0,1]$ and the spaces of infinite sequences A_b^ω as metric spaces. Suitable metrics are the usual distance $|\alpha-\beta|$ in $[0,1]$ and

$$\rho(\mathbf{x},\mathbf{y}) = b^{-\inf\{\, i\in\mathbb{N} \mid i\geq 1, x_i\neq y_i \,\}},$$

for infinite words $\mathbf{x} = x_1\cdots x_i\cdots, \mathbf{y} = y_1\cdots y_i\cdots$ with $x_i, y_i \in A_b$. With these metrics $[0,1]$ and A_b^ω become complete metric spaces.

Let $\mathcal{X}$ be a complete metric space. A set $M \subseteq \mathcal{X}$ is *nowhere dense* if its closure (smallest closed set containing M) does not contain a non-empty open subset. A set $M \subseteq \mathcal{X}$ is *meagre* (or of *first Baire category*) if it is a countable union of nowhere dense sets. A complement of a meagre set is called *co-meagre* (or *residual*).

The following closure property of co-meagre sets is well-known (see [12]).

Fact 1. *In a complete metric space the family of co-meagre sets is closed under countable intersection.*

A set $M \subseteq \mathcal{X}$ is *dense* if $M \cap M' \neq \emptyset$ for every non-empty open set $M' \subseteq \mathcal{X}$. Note that in a complete metric space every co-meagre set is dense, but a dense set might be meagre, even countable.

The following relations hold for subsets $F \subseteq A_b^\omega$ and their counterparts in $[0,1]$.

Lemma 2 [18]. *Let* $F \subseteq A_b^\omega$ *and* $M_F = \{\, 0.\mathbf{x} \mid \mathbf{x} \in F \,\} \subseteq [0,1]$. *Then*

(i) *F is nowhere dense if and only if M_F is nowhere dense.*

(ii) *F is co-meagre if and only if M_F is co-meagre.*

(iii) *F is dense if and only if M_F is dense.*

Fact 3 [14].

(i) *The set of Liouville numbers $\mathcal{L}$ is co-meagre.*

(ii) *The set of computable numbers $\mathcal{C}$ is countable, meagre and dense.*

2. Disjunctive ω-words

As mentioned above, disjunctive ω-words are infinite words $\mathbf{x} \in A_b^\omega$ having

$$\mathbf{infix}(\mathbf{x}) = A_b^*.$$

By

$$D_b = \{ \mathbf{x} \mid \mathbf{x} \in A_b^\omega \wedge \mathbf{infix}(\mathbf{x}) = A_b^* \}$$

we denote the set of all disjunctive ω-words in A_b^ω. Then the set of all absolutely disjunctive numbers in $[0,1]$ is

$$\mathcal{D} = \{ \alpha \mid \alpha \in [0,1] \wedge \forall b(b \geq 2 \to \exists \mathbf{x}(\mathbf{x} \in D_b \wedge \alpha = 0.\mathbf{x})) \}.$$

The set $\mathcal{D}$ has the following topological property:

Lemma 4 [6, 18]. *The set $\mathcal{D}$ is co-meagre in* $[0,1]$.

Then from Fact 1 and Lemma 2, it follows that the set of absolutely disjunctive Liouville numbers is "topologically" large:

Corollary 5. *The intersection $\mathcal{L} \cap \mathcal{D}$ is co-meagre in* $[0,1]$.

Corollary 5 gives only an existence proof, not a constructive one. Furthermore, since the set of computable reals $\mathcal{C}$ is countable, it does not even show that $\mathcal{L} \cap \mathcal{D} \cap \mathcal{C}$ is not empty.

To show the existence of computable absolutely disjunctive Liouville numbers, we use a representation of the b-ary counterparts

$$\{ \mathbf{x} \in A_b^\omega \mid 0.\mathbf{x} \in \mathcal{D} \}$$

of $\mathcal{D}$ via computable languages. In Section 4, we then show how this description can be transformed into an algorithm computing an absolutely disjunctive Liouville number.

Theorem 6 [18]. *For every base b, there effectively exists a computable language $W_b \subseteq A_b^*$ such that the ω-language*

$$\{ \mathbf{x} \in A_b^\omega \mid \textit{the set } \mathbf{pref}(\mathbf{x}) \cap W_b \textit{ is infinite} \}$$

is the set of all b-ary expansions of absolutely disjunctive reals in $[0,1]$.

More explicitly, Theorem 6 ([18, Theorem 21]) provides, for every base b, an increasing computable function $g : \mathbb{N} \to A_b^*$ such that $g(\mathbb{N}) = W_b$. This function g naturally induces a computable order on W_b.

Since $\mathcal{D}$ is dense in $[0,1]$, from Lemma 2.III, we deduce that the ω-language

$$\{ \mathbf{x} \in A_b^\omega \mid \text{the set } \mathbf{pref}(\mathbf{x}) \cap W_b \text{ is infinite} \}$$

is dense in A_b^ω. This yields the following.

Corollary 7. *For every $u \in A_b^*$ there is a $v \in W_b$ such that $u \sqsubset v$.*

Proof. As the ω-language

$$\{ \mathbf{x} \in A_b^\omega \mid \text{the set } \mathbf{pref}(\mathbf{x}) \cap W_b \text{ is infinite} \}$$

is dense, every open subset of A_b^ω contains an $\mathbf{x}$ such that $\mathbf{pref}(\mathbf{x}) \cap W_b$ is infinite.

Consider the open ω-language $u \cdot A_b^\omega$ (see e. g., [18]). Then there is an $\mathbf{x}$ for which $\mathbf{pref}(\mathbf{x}) \cap W_b$ is infinite. Consequently, there is a $v \in \mathbf{pref}(\mathbf{x}) \cap W_b$ such that $u \sqsubset v$. □

3. Expansions of Liouville Numbers

For our purposes, it is useful to have the following property of b-ary expansions $\mathbf{x}$ of reals which guarantees that $0.\mathbf{x}$ is a Liouville number. A similar criterion was sketched, without proof, by Maillet in [13].

Using finitely or infinitely many strings $w_i \in A_b^*$ and a function $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, we construct b-ary expansions of real numbers in the following way.

Define $\Lambda_{j=0}^{\infty} w_j^{f(j)}$ as the concatenation of w_0 ($f(0)$ times), w_1 ($f(1)$ times),w_2 ($f(2)$ times). . . .

Lemma 8 [5]. *Let*

- $(w_i)_{i\in\mathbb{N}}$ *be a family of non-empty strings* $w_i \in A_b^*$,
- $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, *and*
- $n_i = \sum\limits_{j=0}^{i} f(j) \cdot |w_j|$.

If

$$\liminf_{i\to\infty} \frac{n_{i-1} + |w_i|}{n_{i-1} + f(i) \cdot |w_i|} = 0\,, \tag{1}$$

then $\mathbf{x} = \Lambda_{j=0}^{\infty} w_j^{f(j)}$ *is the b-ary expansion of a rational or a Liouville number.*

4. The Algorithm

The following algorithm computes the b-ary expansion $\mathbf{x} = \Lambda_{j=0}^{\infty} w_j^{f(j)}$ of an absolutely disjunctive Liouville number whose b-ary expansion starts with a given word $w_0 \in A_b^*$. It uses the computable injective ordering $g : \mathbb{N} \to W_b$ of the computable language W_b given by Theorem 6.

Algorithm Liouville-disjunctive

0 **initialise** $w_0 = u_0 = v_0, \quad f(0) = 1$

1 **for** $i = 1$ **to** ∞ **do**

2 $v_i =$ first word in $\big(W_b \cap u_{i-1} \cdot A_b^*\big) \setminus \{u_{i-1}\}$

3 **calculate** w_i **from** $v_i = u_{i-1} \cdot w_i$

4 **calculate** $f(i) = \min\Big\{ k \mid \dfrac{|u_{i-1}| + |w_i|}{|u_{i-1}| + k \cdot |w_i|} < \dfrac{1}{i} \Big\}$

5 $u_i = u_{i-1} \cdot w_i^{f(i)}$

6 **endfor**

The algorithm computes three families of words $(u_i)_{i\in\mathbb{N}}$, $(v_i)_{i\in\mathbb{N}}$, and $(w_i)_{i\in\mathbb{N}}$ and a function $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$. Note that at each step the set $(W_b \cap u_{i-1} \cdot A_b^*) \setminus \{u_{i-1}\}$ is effectively ordered according to g.

First, Step 2 implies $v_i \in W_b$ and together with Step 5, by induction,

$$u_{i-1} \sqsubset v_i \sqsubseteq u_i \sqsubset v_{i+1}.$$

From the Step 3 and $u_{i-1} \sqsubset v_i$, we have $|w_i| > 0$. Then, again using Step 5, by induction one verifies that

$$u_i = \Lambda_{j=0}^{i} w_j^{f(j)}. \tag{2}$$

It remains to show that the algorithm will produce an infinite computable ω-word, that is, it never stops. To this end it suffices to show that the choice in Step 2 is always possible. From Corollary 7 we know that for every $u \in A_b^*$ there is a $v \in W_b$ such that $u \sqsubset v$. This makes it possible to choose the first element in W_b w. r. t. g which has u as a proper prefix.

Thus, the algorithm computes two computable approximations of an ω-word

$$\mathbf{x} = \Lambda_{j=0}^{\infty} w_j^{f(j)}$$

via the families of prefixes $(u_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$. From $v_i \in W_b$, we obtain $0.\mathbf{x} \in \mathcal{D}$ via Theorem 6, and, because of (2), Step 4 shows that the words u_i and w_i satisfy Eq. (1). Thus, Lemma 8 verifies that $0.\mathbf{x}$ is also a Liouville number. The computability of $\mathbf{x}$ follows directly from the algorithm.

Acknowledgement

The authors are grateful to Helmut Jürgensen for introducing them (long time ago) to Liouville numbers and disjunctive ω-words.

References

[1] V. Becher, P. A. Heiber, T. A. Slaman, A computable absolutely normal Liouville number. *Math. Comp.* **84** (2015) 296, 2939–2952.

[2] Y. Bugeaud, Nombres de Liouville et nombres normaux. *C. R. Math. Acad. Sci. Paris* **335** (2002) 2, 117–120.

[3] C. S. Calude, *Information and Randomness: An Algorithmic Perspective.* Springer, Berlin, 2002.

[4] C. S. Calude, L. Priese, L. Staiger, Disjunctive Sequences: An Overview. *CDMTCS Research Report* 63, 1997.

[5] C. S. Calude, L. Staiger, Liouville, Computable, Borel normal and Martin-Löf random numbers. *Theory Comput. Syst.*, First online 27 April 2017, `https://doi.org/10.1007/S00224-017-9767-8`.

[6] C. S. Calude, T. Zamfirescu, Most numbers obey no probability laws. *Publ. Math. Debrecen* **54** (1999) suppl., 619–623. Automata and formal languages, VIII (Salgótarján, 1996).

[7] J. W. S. Cassels, On a problem of Steinhaus about normal numbers. *Colloq. Math.* **7** (1959), 95–101.

[8] D. G. Champernowne, The construction of decimals normal in the Scale of Ten. *J. London Math. Soc.* **S1-8**, 4 (1933), 254.

[9] P. Hertling, Disjunctive ω-words and real numbers. *J.UCS* **2** (1996) 7, 549–568.

[10] H. Jürgensen, H. J. Shyr, G. Thierrin, Disjunctive ω-languages. *Elektron. Informationsverarb. Kybernet.* **19** (1983) 6, 267–278.

[11] H. Jürgensen, G. Thierrin, Some structural properties of ω-languages. In: *Proc. 13th Nat. School with Internat. Participation, Applications of Math. in Technology*, Sofia, 1988, 56–63.

[12] K. Kuratowski, *Topology. Vol. I.* New edition, revised and augmented. Translated from the French by J. Jaworowski, Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw, 1966.

[13] E. Maillet, Sur les nombres quasi-rationnels et les fractions arithmétiques ordinaires ou continues quasi-périodiques. *C. R. Acad. Sci., Paris* **138** (1904), 410–411.

[14] J. C. Oxtoby, *Measure and Category.* Springer-Verlag, New York-Berlin, 1980.

[15] W. M. Schmidt, On normal numbers. *Pacific J. Math.* **10** (1960), 661–672.

[16] L. Staiger, Reguläre Nullmengen. *Elektron. Informationsverarbeit. Kybernetik* **12** (1976) 6, 307–311.

[17] L. Staiger, Rich ω-words and monadic second-order arithmetic. In: M. Nielsen, W. Thomas (eds.), *Computer Science Logic (Aarhus, 1997).* Lecture Notes in Computer Science 1414, Springer-Verlag, Berlin, 1998, 478–490.

[18] L. Staiger, How large is the set of disjunctive sequences? *J.UCS* **8** (2002) 2, 348–362.

[19] L. Staiger, The Kolmogorov complexity of real numbers. *Theoret. Comput. Sci.* **284** (2002) 2, 455–466.

(Received: March 5, 2017; revised: November 24, 2017)