Indeterminism and Randomness

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Abstract. Quantum randomness is postulated and generally reduced to the indeterminism of quantum measurements. The connection to randomness is often made via unpredictability: because the outcome is indeterministic there is no way to predict it, hence it is random. Here we argue that indeterminism and randomness are theoretical, means relative, concepts which don't imply each other.

1 Indeterminism

Broadly speaking, indeterminism occurs when the state of a system at one time does not uniquely fix the state of the system at some future time [12]. More precisely, indeterminism is a failure in one or more forms of determinism. Classifying indeterminism is more difficult because simply negating determinism does not give a unique notion. A means-relative classification for functions can be based on the fact that a deterministic function can be i) incomputable—in an infinity of stronger and stronger forms—ii) computable, but not feasibly computable, iii) feasibly computable—again in an infinity of stronger and stronger forms.

2 Randomness

In contrast with notions such as rain or water, some concepts like number or velocity are only theoretical with no *direct* counterpart in nature: they are used to model some "reality", they can be measured, but are not directly observed in the natural world. Randomness is also such a concept. As noted in [18],

> *... randomness is not in the world, it is in the interface between our theoretical descriptions and 'reality' as accessed by measurement. Randomness is unpredictability with respect to the intended theory and measurement.*

So, what is randomness? Where does it come from?

Intuitively, randomness is identified with unpredictability (see [11]), lack of correlations (irregularity) and typicality. These characteristics of randomness can be tested in concrete examples of "random" events, like coin-tossing. For example, a sequence of coin tosses looks very irregular, and no matter how many times we have tossed the coin, even thousands and thousands of times, predicting the outcome of the next toss seems impossible. We used the formulation "seems impossible" because, in principle, coin-tossing is as predictable as the motion of the planets once the initial conditions are given. However, we "believe" that prediction is impossible—and this feeling is confirmed by experiment—because of the peculiar combination of circumstances of coin-tossing, more precisely, the sensitive dependence on (some set of) initial conditions coupled with the inability to know these conditions with infinite precision.[1]

Probability theory assumes "randomness" and develops a very successful calculus with random events/processes, but remains silent with respect to the randomness of individual outcomes. Two theories, Ramsey theory [14] and algorithmic information theory [5,9], deal with randomness in "its individuality", i.e. not only as a statistical (global) phenomenon. Each of them has a strong message:

– Ramsey theory: pure/absolute/true randomness (for finite or infinite objects) does not exist because it is mathematically vacuous [5,14].[2] For example, *in every infinite sequence of zeros and ones there are infinitely many correlations*, [14]. In particular, even algorithmically incompressible infinite sequences have infinitely many correlations.
– Algorithmic information theory: disproving randomness is a resource-based process, so there exist degrees of randomness with no upper limit. The more resources we have, the more patterns we discover. For example, quantum randomness obtained by measuring a value indefinite observable [1,2] is "more random" than coin-tossing [8] or software-generated randomness (pseudo-randomness): the first is highly incomputable, the last two are computable. The sequence of quantum random bits obtained by the process described above has significantly less patterns and correlations then pseudo-random sequences of bits; however, it cannot avoid itself infinitely many correlations.

There is another way to define randomness: via an indeterministic process. This is the standard argument in favour of quantum randomness: because the outcome of a measurement is indeterministic, there is no way to predict it, hence it is random. The form of randomness obtained in this way is sometimes called *process randomness* (see, for example, [11]) because its "certification" comes from the properties of the process producing it.[3] Process randomness is different from *product randomness*, discussed in algorithmic information theory, which ignores the process generating the bits and studies just the result (product). Process ran-

[1] Technically, this is expressed by the phenomenon of deterministic chaos [16].

[2] This was anticipated by philosophers long time ago in connection to chance, a different but closely related concept. The 5th century BCE philosopher Leucippus was probably the first to note (see [10, p. 133]) that *Nothing occurs by chance, but there is a reason and necessity in everything.* Under the influence of mathematician A. de Moivre, Hume [15, p. 56] called chance a mere word: *. . . there be no such thing as Chance in the world.* See also [20].

[3] The name "process randomness" is also used for a form of algorithmic randomness: see [19,21,9].

domness has no mathematical formalisation and can be certified/validated only through a theory (difficult, if not impossible, to test) or product randomness.

3 Indeterminism vs. Randomness

Does indeterminism imply randomness? Does randomness imply indeterminism?

Coin-tossing randomness—discussed in the previous section—is a simple example that shows that determinism is compatible with a form of randomness.

Is indeterminism necessary for randomness? The answer is negative. Software-generated randomness is a (weak, still useful) form of randomness which is algorithmically produced—no form of indeterminism is required. In fact, both algorithmic information theory [5,9] and the practice of generating randomness show that *randomness (of any form) is*:

- *defined by avoiding algorithmically defined classes of patterns* and
- *produced algorithmically.*

The halting probability of a universal self-delimiting Turing machine U is called the Omega number of U and denoted by Ω_U (see [6] and more in [5,9]). The infinite binary expansion of Ω_U is uniquely determined by U. However, this sequence is "highly random"—technically, Martin-Löf random—because it passes all Martin-Löf tests of randomness, i.e. it avoids all patterns defined by the Martin-Löf algorithmic model of randomness. Randomness appears when we "don't know" that the sequence *is the Omega number of* U. But, as Ramsey theory proves, in every Martin-Löf random sequence there are infinitely many patterns and correlations.

In the same way, the Schrödinger equation describes how the quantum state of some physical system changes with time: this evolution is deterministic. Quantum randomness appears when we observe/measure certain individual quantum observables, for example, value indefinite observables [2].

There is a loose analogy between the above two examples. The description of Ω_U by U corresponds to the Schrödinger equation. A first similarity appears when we compare the processes of computing the bits of Ω_U and measuring value indefinite observables in the quantum experiment $\mathfrak{E}$ described in [1]. One can prove that the sequence of bits of Ω_U cannot be algorithmically computed (given U) and, similarly, the sequence of bits obtained by performing *ad infinitum* the quantum experiment $\mathfrak{E}$ cannot be algorithmically computed (given full information about the experiment [1,3]).

A second, more interesting similarity, appears when we compare the unpredictability of individual bits. An individual bit of Omega is Martin-Löf unpredictable, but only some are maximally unpredictable:[4] a bit obtained in the

[4] The Omega numbers of Solovay machines [7].

quantum experiment $\mathfrak{E}$ is maximally unpredictable in the sense of [3].[5] These results are provable in both cases.[6]

Is randomness necessary for indeterminism? As discussed in Section 1, indeterminism, as a negation of determinism, has a variety of forms. Here we will provide again a negative answer to the above question by looking at indeterminism in computing science.

Of course, randomness is a very useful ingredient for computing, but a plethora of computing machines work in an indeterministic way without any use of randomness. The simplest example is the indeterministic computation[7] of finite automata. A deterministic finite automaton computation uses finitely many rules/transitions, but at each step one single rule can be applied; this guarantees the uniqueness of both the dynamics of the computation and the result (accept or reject). Replicating a computation (with the same input and deterministic automation) will always produce the same output. In contrast, a non-deterministic finite automaton provides more choices at each step of the computation. The indeterminism here consists in the multiple branches of computation: no one is pre-imposed, all are equally possible, the results may be different. Computationally, this type of indeterminism is a form of parallel computation. To obtain a unique result and invariance under replication we need to adopt an acceptance rule. For example, to determine if the automaton accepts or rejects the input, one has to compute on all possible branches and accept if one branch accepts. Of course, one can modify this rule in various ways: instead of "accept if one branch accepts" one can use "accept if a single branch accepts" or "accept if the majority of branches accept" or "accept if more than half of the branches accept". We have the same situation for the more powerful computations executed by Turing machines. Another form of indeterminism is to allow the machine (automaton or Turing machine) to choose between available rules according to some probability distribution. These types of computations can make mistakes, but the errors are "controlled" (e.g. less than 50% mistakes). Working with probability distributions may introduce strong correlations/biases which can be algorithmically detected. The mathematical result which is typically true for many forms of indeterministic computation is the following: the computation of the non-deterministic machine (e.g. finite automaton or Turing machine) can be simulated by the computation of a deterministic machine of the same type. The difference is in computational complexity: deterministic machines have to work harder to simulate their non-deterministic counterparts, see [17].

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[5] The Schrödinger equation cannot give the exact result of an individual measurement: it can predict only the probability distributions.

[6] The unpredictability/randomness of *every* individual quantum outcome was conjectured/postulated by Born [4].

[7] The term used in computing science is non-determinism. However, there is a tendency to change the terminology: the negation of computable, which used to be called noncomputable, is now referred as incomputable.

4 Instead of a Conclusion

Identifying indeterminism with randomness—as in [13, p. 31]—is misleading and renders problematic any analysis which is based on this assumption (see [10] for a historical review). In particular, this means that much more is required to understand quantum randomness.

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