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TOPOLOGICAL SIZE OF SETS OF PARTIAL RECURSIVE FUNCTIONS

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1. Introduction

Notations

N = the set of non-negative integers,

P = the set of unary partial recursive functions,

(a_i) = an acceptable Gödelization of P (ROGERS [16]),

(A_i) = a Blum measure of computational complexity (BLUM [1]),

R^i = the set of recursive functions of i arguments,

$\mathsf{R} = \mathsf{R}^1$,

F = the set of unary recursive functions of finite support,

$\mathrm{supp}(f) = \{x \mid f(x) \neq 0, \infty\}$ $(f \in \mathsf{P})$,

$l(f) = \mathrm{card}(\mathrm{supp}(f))$ $(f \in \mathsf{F})$,

$f \sqsubseteq g$ iff $\mathrm{supp}(f) \subseteq \mathrm{dom}(g)$ and $g|_{\mathrm{supp}(f)} = f|_{\mathrm{supp}(f)}$ $(f, g \in \mathsf{P})$,

$f \sqsubset g$ iff $f \sqsubseteq g$ and $f \neq g$,

$\mathsf{N} \times \mathsf{N} \underset{K,L}{\overset{J}{\rightleftarrows}} \mathsf{N}$, $\mathsf{N}^3 \underset{I_i^{(3)}}{\overset{J^{(3)}}{\rightleftarrows}} \mathsf{N}$ = Cantor pairing functions,

a.e. = almost everywhere,

r.e. = recursively enumerable,

$\mathsf{C}_t = \{f \in \mathsf{R} \mid$ there exists i, $a_i = f$ and $A_i(x) \leq t(x)$ a.e.$\}$,
(the complexity class of $t \in \mathsf{R}$),

$A \subseteq \mathsf{P}$ is a *measured* set iff $A = \{f_i(x) \mid i \geq 0\}$
and the predicate "$f_i(x) = y$" is recursive (BLUM [1]),

f is *g-honest* iff there exists i such that $a_i = f$ and $A_i(x) \leq g(f(x))$ a.e.
$(f \in \mathsf{P}, g \in \mathsf{R})$ (BLUM [1]),

$\mathsf{E}(f)$ = the elementary recursive class of f $(f \in \mathsf{R})$ (MEYER-RITCHIE [12]),

$\mathsf{Pol}(f)$ = the polynomial class of f $(f \in \mathsf{R})$ (MELHORN [11]),

$\mathsf{Pr}(f)$ = the primitive recursive class of f $(f \in \mathsf{R})$ (MACHTEY [6]).

The category-theoretic methods (in Baire sense) were used in the theory of degrees (MYHILL [13]; MELHORN [10]), in the theory of recursive operators (ROGERS [16]), in α-recursion theory (LOWENTHAL [5]). We shall define a natural recursive topology on the set of partial recursive functions and recursive variants of the notions of no-

1) I wish to thank Prof. G. WECHSUNG for helpful comments on this work.

where dense set and meagre set. These instruments will be used to analyse the topological size of various sets of partial recursive functions. All results are compatible with those obtained by MELHORN [10] for classes of recursive functions. Topological refinements of the Honesty and Gap Theorems will be equally obtained.

We shall work in a fixed *Blum space*, i.e. a couple $A = ((a_i), (A_i))$, where (a_i) is an acceptable Gödelization of P and (A_i) are Blum step counting functions, satisfying the axioms:

1) For every $i \geqq 0$, $\mathrm{dom}(a_i) = \mathrm{dom}(A_i)$.

2) The predicate $M(i, x, y) = \begin{cases} 1, & \text{if } A_i(x) = y \\ 0, & \text{otherwise} \end{cases}$ is recursive.

The set of functions of finite support is r.e. We shall adopt the following fixed enumeration of F. By the *s-m-n*-Theorem there exists a recursive function $s(i, y, z)$ such that

$$a_{s(i,y,z)}(x) = \begin{cases} a_i(x), & \text{if } x \leq y \text{ and } A_i(x) \leq z \\ 0, & \text{otherwise.} \end{cases}$$

Let $h(i) = s(I_1^{(3)}(i), I_2^{(3)}(i), I_3^{(3)}(i))$; h is recursive and $\mathsf{F} = \{a_{h(i)}(x) \mid i \geqq 0\}$. From the above construction it follows that for every $x \geqq I_2^{(3)}(n)$, $a_{h(n)}(x) = 0$ and $\mathrm{supp}(a_{h(n)}) \subseteqq \subseteqq \{0, 1, \ldots, I_2^{(3)}(n)\}$.

Finally, let us observe that the relation $\sqsubseteq$ is a quasi-order in F and P.

2. Basic Topological Constructions

We define a system of basic open neighborhoods which induces a topology in the set of (unary) partial recursive functions. For every $t \in \mathsf{F}$ we put $U_t = \{f \mid f \in \mathsf{P}, t \sqsubseteq f\}$.

Lemma 1. *For every $t_i \in \mathsf{F}$ such that $U_{t_1} \cap U_{t_2} \neq \emptyset$ there exists $t_3 \in \mathsf{F}$ such that $U_{t_1} \cap U_{t_2} = U_{t_3}$.*

Proof. Let $X = \mathrm{supp}(t_1) \cap \mathrm{supp}(t_2)$. Since $U_{t_1} \cap U_{t_2} \neq \emptyset$ it follows that if $x \in X$, then $t_1(x) = t_2(x)$. The required function t_3 is defined by

$$t_3(x) = \begin{cases} t_1(x) \ (= t_2(x)), & \text{if } x \in X, \\ t_1(x), & \text{if } x \in \mathrm{supp}(t_1) - X, \\ t_2(x), & \text{if } x \in \mathrm{supp}(t_2) - X, \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 1 we deduce that $(U_t)_{t \in \mathsf{F}}$ is a system of basic neighborhoods in P. We shall work with the topology generated by this system.

Proposition 1. *Let X be a subset of P. Then, the following statements are equivalent:*

(i) *X is open.*

(ii) *For every $f \in X$ and for every $g \in \mathsf{P}$, if $f \sqsubseteq g$, then $g \in X$ (i.e. X is solid), and for every $f \in X$ there exists $t \in X \cap \mathsf{F}$ such that $t \sqsubseteq f$.*

(iii) $X = \bigcup_{t \in X \cap \mathsf{F}} U_t$.

The proof is obvious.

Remark. The topology defined on P is not separated; it is quasi-compact and it has an enumerable base (for every $f \in \mathsf{P}$, $(U_t)_{t \sqsubseteq f}$).

We give the crucial definition, i.e. the definition of the recursive nowhere dense set. A set $X \subseteqq \mathsf{P}$ is *nowhere dense* (*under f and g*) if $f, g \in \mathsf{R}$ and the following four conditions hold:

1) For all n, $a_{f(n)} \in \mathsf{F}$.

2) For all m and n, $m > g(n)$ implies $a_{f(n)}(m) = 0$.

3) For all n, $a_{h(n)} \sqsubseteq a_{f(n)}$.

4) There exists a number i such that for every n for which $l(a_{h(n)}) > i$, we have $X \cap U_{a_{f(n)}} = \emptyset$.

Remark. Usually, a nowhere dense set is a set X together a function D which maps non-empty open sets into non-empty open sets such that for every (non-empty open) set U, $D(U) \cap X = \emptyset$ (Oxtoby [14]). It is clear that we can work with the restriction of D to the family of basic open neighborhoods. These facts motivate the general principle of the above construction. The additional restrictions are imposed by the constructive nature of our concept. Particularly, the condition 2) is motivated by the necessity that the support of every function $a_{f(n)}$ could be recursively determined.

In order to show the compatibility with Melhorn definition we prove the following equivalence.

Proposition 2. *Let X be a subset of* P. *Then, the following statements are equivalent:*

(1) *X is nowhere dense under f and g.*

(2) *There exists a recursive function r for which the following two conditions hold:*

(a) *For every n, $a_{h(n)} \sqsubseteq a_{h(r(n))}$.*

(b) *There exists j such that for all n with $l(a_{h(n)}) > j$, we have $X \cap U_{a_{h(r(n))}} = \emptyset$.*

Proof. (1) $\Rightarrow$ (2). We define the function r by the formula

$$r(i) = \mu j[(\forall x)\,(x \leqq g(i)\ \&\ x \in \operatorname{supp}(a_{f(i)}) \Rightarrow a_{h(j)}(x) = a_{f(i)}(x))$$
$$\&\ (\forall x)\,(x \leqq I_2^{(3)}(j)\ \&\ x \in \operatorname{supp}(a_{h(j)}) \Rightarrow a_{f(i)}(x) = a_{h(j)}(x))].$$

From this formula it is clear that $r \in \mathsf{R}$. Moreover, for all x, $x \in \operatorname{supp}(a_{f(i)})$ is equivalent to $0 \neq a_{f(i)}(x) = a_{h(r(i))}(x)$, i.e. it is equivalent to $x \in \operatorname{supp}(a_{h(r(i))})$; hence $a_{f(i)}(x) = a_{h(r(i))}(x)$. The condition (a) follows from the property 3), and the last condition is a consequence of the property 4) for $j = i$.

(2) $\Rightarrow$ (1). We put $f(i) = h(r(i))$, $g(i) = I_2^{(3)}(r(i))$. The conditions 1)$-$4) are obviously verified (for the condition 2) we use the relation $\operatorname{supp}(a_{h(i)}) \subseteqq \{0, 1, \ldots, I_2^{(3)}(i)\}$).

The following result will be useful in the proof of the main theorem.

Lemma 2. *Let $X \subseteqq$* P. *Then, the following statements are equivalent:*

(1) *A is nowhere dense under f and g.*

(2) *There exist two functions $f', g' \in$* R *such that X together f' and g' satisfy the conditions* 1), 2), 4) *in the definition of the nowhere dense set, and the condition*

3′) *For every n, $a_{h(n)} \sqsubset a_{f'(n)}$.*

Proof. We must prove only the implication (1) $\Rightarrow$ (2). Thus, let A be a nowhere dense set under f and g. First let us construct the auxiliary function

$$p(i, x) = \begin{cases} a_{f(i)}(x), & \text{if } x \leqq g(i), \\ 1, & \text{if } x = g(i) + 1, \\ 0, & \text{if } x > g(i) + 1. \end{cases}$$

It is obvious that $p \in \mathrm{R}^2$. By the *s-m-n*-Theorem we get a function $s \in \mathrm{R}$ such that $p(i, x) = a_{s(i)}(x)$. We put $f'(i) = s(i)$, $g'(i) = g(i) + 1$. Clearly, $f', g' \in \mathrm{R}$. By construction, for every i, $p(i, x) = 0$ a.e., hence $a_{f'(i)} \in \mathrm{F}$. Moreover, if m and n are arbitrary such that $m > g'(n)$, then $a_{f'(n)}(m) = a_{s(n)}(m) = p(n, m) = 0$. For all n, $a_{f'(n)} = a_{s(n)} \sqsupset \sqsupset a_{f(n)} \sqsupseteq a_{h(n)}$, so $a_{f'(n)} \sqsupset a_{h(n)}$. Finally, in view of the condition 4) for the triple X, f, g there exists a number i such that for all n with $l(a_{h(n)}) > i$, we have $X \cap U_{a_{f(n)}} = \emptyset$. But, from the relation $a_{f'(n)} \sqsupset a_{f(n)}$ we derive the inclusion $U_{a_{f'(n)}} \subset U_{a_{f(n)}}$, hence $X \cap U_{a_{f'(n)}} = \emptyset$.

Thus, by Lemma 2, in the definition of the nowhere dense set we may equally use the condition 3) or the condition 3′), i.e. $\sqsubseteq$ or $\sqsubset$.

A set $X \subseteq \mathrm{P}$ is *meagre* (or a *set of the first Baire category*) if there exist a sequence $(X_i)_{i \geqq 0}$, $X_i \subseteq \mathrm{P}$, and two r.e. sets $(f_i)_{i \geqq 0}$, $(g_i)_{i \geqq 0}$, $f_i, g_i \in \mathrm{R}$ such that the following conditions are fulfiled:

(1) $X = \bigcup\limits_{i \geqq 0} X_i$.

(2) For every $i \geqq 0$, X_i is nowhere dense under f_i and g_i.

If $X \subseteq \mathrm{P}$ is not meagre, then X is called a *set of the second Baire category*.

Remarks. a) Intuitively, the meagre sets are "recursively small" sets, in opposition to the sets of the second Baire category which are "recursively big" ones. b) Every nowhere dense set is meagre but the converse fails.

Proposition 3. *The family of meagre sets is closed under subset.*

Proof. Let Y be a subset of a meagre set X. Thus $X = \bigcup\limits_{i \geqq 0} X_i$, and for all $i \geqq 0$, X_i is nowhere dense under f_i and g_i, where (f_i) and (g_i) are r.e. sets of recursive functions. If $Y_i = Y \cap X_i$, then Y becomes meagre under the decomposition $Y = \bigcup\limits_{i \geqq 0} Y_i$.

Corollary 1. *The family of sets of the second Baire category is closed under superset.*

Proposition 4. *Let $X \subseteq \mathrm{P}$ be a set which can be written as $X = \bigcup\limits_{i \geqq 0} X_i$, and for which there exist two functions $f, g \in \mathrm{R}^2$ such that the following two conditions hold:*

(1) *For all $i \geqq 0$, $X_i = \bigcup\limits_{j \geqq 0} Y_{i,j}$, $Y_{i,j} \subseteq \mathrm{P}$.*

(2) *For all $i, j \geqq 0$, $Y_{i,j}$ is nowhere dense under $a_{f(i,j)}$ and $a_{g(i,j)}$.*

Then, X is meagre.

Proof. From the hypothesis it follows that every set X_i is meagre. For every $m \geqq 0$ we set $C_m = Y_{K(m),L(m)}$, $r_m(x) = a_{f(K(m),L(m))}(x)$, and $p_m(x) = a_{g(K(m),L(m))}(x)$. In view of the fact that K and L are pairing functions we have: $X = \bigcup\limits_{i \geqq 0} X_i = \bigcup\limits_{i \geqq 0} (\bigcup\limits_{j \geqq 0} Y_{i,j}) = \bigcup\limits_{m \geqq 0} C_m$. Now it is obvious that A is meagre under the above decomposition (i.e. C_m is nowhere dense under r_m and p_m).

Corollary 2. *The family of meagre sets is closed under union.*

Theorem 1 (Main Theorem). *For every meagre set $X \subseteq \mathrm{P}$ and every $t \in \mathrm{P}$, there is a recursive function $f \in U_t - X$.*

Proof. Since X is meagre, it follows that it can be written as $X = \bigcup\limits_{i \geqq 0} X_i$, where

X_i is nowhere dense under f_i and g_i; the sets (f_i) and (g_i) are r.e. In view of the Lemma 2 we can suppose that $a_{h(n)} \sqsubset a_{f_i(n)}$, for all n and i. Let us observe that for fixed n and i the equality $a_{f_i(n)}(x) = a_{h(j)}(x)$, for all x, is equivalent to the following two conditions:

(a) for every $x \leqq g_i(n)$, $x \in \mathrm{supp}(a_{f_i(n)})$, $a_{h(j)}(x) = a_{f_i(n)}(x)$;
(b) for every $x \leqq I_2^{(3)}(n)$, $x \in \mathrm{supp}(a_{h(j)})$, $a_{h(j)}(x) = a_{f_i(n)}(x)$.

Hence, the predicate

$$Q(i, j, n) = \begin{cases} 1, & \text{if } a_{f_i(n)}(x) = a_{h(j)}(x), \text{ for all } x \\ 0, & \text{otherwise} \end{cases}$$

is recursive. Moreover, since $a_{f_i(n)} \in \mathsf{F}$, for all numbers i and n, there is a number j such that $Q(i, j, n) = 1$. By means of the predicate Q and of the number q (with $t = a_{h(q)}$) we construct the recursive function r: $r(0) = q$, $r(x+1) = \mu_j[Q(K(x), j, r(x)) = 1]$ and a sequence (t_m) of functions of finite support:

$$t_0(x) = t(x), \qquad t_m(x) = a_{f_{K(m)}(r(m))}(x) \quad (m > 0).$$

Let us note that from the above construction we deduce the following useful relation:

$$a_{h(r(m+1))}(x) = a_{f_{K(m)}(r(m))}(x), \quad \text{for all } m \text{ and } x.$$

For every $m \geqq 0$, $t_m \sqsubset t_{m+1}$. Indeed,

$$t_1(x) = a_{f_0(q)}(x) \sqsupset a_{h(q)}(x) = t(x),$$
$$t_{m+1}(x) = a_{f_{K(m+1)}(r(m+1))}(x) \sqsupset a_{h(r(m+1))}(x) = a_{f_{K(m)}(r(m))}(x) = t_m(x), \text{for } m > 0.$$

Now we can define the function f. Set $f(x) = t_m(x)$, if $x \leqq g_{K(m)}(r(m))$. Since $t_i \in \mathsf{F}$ and $t_i \sqsubset t_{i+1}$, it follows that the definition is correct; moreover, $f \in \mathsf{R}$.

By construction, for every $m \geqq 0$, $t_m \sqsubset f$; in particular, $t = t_0 \sqsubseteq f$, i.e. $f \in U_t$. We must prove that $f \notin X$. Suppose, by contrary, that $f \in X$. There must exist a number i such that $f \in X_i$. But, in view of the hypothesis, X_i is nowhere dense under f_i and g_i. We use the property 4): there exists a number n_i such that for all n, $l(a_{h(n)}) > n_i$ implies $X_i \cap U_{a_{f_i(n)}} = \emptyset$. We choose a number m such that $K(m+1) = i$ and $l(t_m) > n_i$. The existence of such a number follows from the fact that for every j the equation $K(x) = j$ has an infinity of solutions and from the monotonicity of sequence (t_m). Set $n = r(m+1)$. We have $t_m(x) = a_{h(r(m+1))}(x) = a_{h(n)}(x)$; $l(t_m) = l(a_{h(n)}) > n_i$. We obtain $a_{f_i(n)}(x) = a_{f_{K(m+1)}(r(m+1))}(x) = t_{m+1}(x) \sqsubseteq f(x)$; hence, $f \in U_{a_{f_i(n)}}$. We arrived to a contradiction.

Corollary 3 (Baire Category Theorem). *The set of recursive functions is a set of the second Baire category.*

Proof. Suppose, by contrary, that R is meagre. By Theorem 1 we get the following contradictory relation: for every $t \in \mathsf{F}$, there exists a recursive function f with $t \sqsubseteq f$, but $f \notin \mathsf{R}$.

Remark. The relation between Corollary 3 and the classical Baire Category Theorem is more profund. Every real $0 < \alpha < 1$ has a continued fraction expansion $n_0, n_1, \ldots$ defined by the relations

$$n_i = \left[\frac{1}{r_i}\right], \quad \text{if } r_i \neq 0, \text{ where } r_0 = \alpha,\ r_{i+1} = \frac{1}{r_i} - n_i.$$

Hence, every number-theoretic function f can be identified with the expansion $f(0)+1, f(1)+1, \ldots$. This yields a one-to-one and onto correspondence between the set of number-theoretic functions and the irrationals between 0 and 1, which preserve the correspondence between recursive functions and recursive irrationals in [0, 1] (see RICE [15]). This correspondence allows to reformulate Corollary 3 as follows: *The set of recursive real numbers between* 0 *and* 1 *is a set of the second Baire category.*

Corollary 4. *The set of partial recursive functions is a set of the second Baire category.*

Proof. It follows from Corollaries 1 and 3.

Corollary 5. *Every non-empty open set is a set of the second Baire category.*

Proof. From Theorem 1 it follows that every basic open neighborhood U_t is of the second Baire category. We apply Corollary 1 and we obtain the Corollary 5.

3. Applications in Computational Complexity

In this paragraph we analyse some well-known results in computational complexity from the point of view of the Baire Category Theorem.

Theorem 2. *Every measured set is meagre.*

Proof. Let $X \subseteqq \mathsf{P}$ be a measured set. Then $X = \{f_i(x) \mid i \geqq 0\}$, and the predicate

$$M(i, x, y) = \begin{cases} 1, & \text{if } f_i(x) = y \\ 0, & \text{otherwise} \end{cases}$$

is recursive. We shall prove the existence of two r.e. sets of recursive functions (d_i) and (g_i) such that for all i, (f_i) is nowhere dense. In order to construct the function d_i we apply two fold the s-m-n-Theorem to the following recursive function:

$$p(i, n, x) = \begin{cases} a_{I_1^{(3)}(n)}(x), & \text{if } x \leqq I_2^{(3)}(n),\ A_{I_1^{(3)}(n)}(x) \leqq I_2^{(3)}(n) \\ 1, & \text{if } x = I_2^{(3)}(n) + 1,\ \sum_{y=0}^{x+2} M(i, x, y) = 0 \\ x + 3, & \text{if } x = I_2^{(3)}(x) + 1,\ \sum_{y=0}^{x+2} M(i, x, y) \geqq 1 \\ 0, & \text{otherwise.} \end{cases}$$

We obtain a function $s(i) \in \mathsf{R}$ such that $p(i, n, x) = a_{a_{s(i)}(n)}(x)$. Put $d_i(n) = a_{s(i)}(n)$ and $g_i(n) = I_2^{(3)}(n) + 1$, for all i. The sets $\{d_i\}$, $\{g_i\}$ are obviously r.e. We verify the four conditions in the definition of the nowhere dense set. The first two conditions are obviously fulfiled: $a_{d_i(n)}(x) = p(i, n, x) = 0$ a.e., and $a_{d_i(n)}(m) = 0$ for all n and m with $m > g_i(n) = I_2^{(3)}(n) + 1$. For all n, $a_{d_i(n)}(x) = a_{a_{s(i)}(n)}(x) \sqsupset a_{h(n)}(x)$ because $a_{h(n)}(x) = a_{I_1^{(3)}(n)}(x)$ for all $x \in \operatorname{supp}(a_{h(n)}) \cap \{0, 1, \ldots, I_2^{(3)}(n)\}$. To prove the last condition we set $n_i = 0$ and we show that for all n with $l(a_{h(n)}) > 0$, $f_i(x) \notin U_{a_{d_i(n)}}$. Suppose, by contrary, that for some i, $f_i(x) \in U_{a_{d_i(n)}}$, i.e. $a_{d_i(n)} \sqsubseteq f_i$. This means that for every $x \in \operatorname{supp}(a_{d_i(n)})$, $f_i(x) = a_{d_i(n)}(x)$. Let $x_0 = I_2^{(3)}(n) + 1$. We must analyse two cases. In the first case, $\sum_{y=0}^{x_0+2} M(i, x_0, y) = 0$, that is, for all $0 \leqq y \leqq x_0 + 2$, $M(i, x_0, y) = 0$, i.e. $f_i(x_0) \neq y$. We have $a_{d_i(n)}(x_0) = 1 \leqq x_0 + 2$, and $f_i(x_0) > x_0 + 2$ or $f_i(x_0)$ is undefined. Hence, in this case $f_i(x_0) \neq a_{d_i(n)}(x_0)$. In the second case $\sum_{y=0}^{x_0+2} M(i, x_0, y) \geqq 1$, that is, $0 \leqq f_i(x_0) \leqq x_0 + 2$. But, $a_{d_i(n)}(x_0) = x_0 + 3$; in conclusion, $f_i(x_0) \neq a_{d_i(n)}(x_0)$. In both cases we arrived to a contradiction.

Remarks. a) The proof of Theorem 2 shows that if the graph of f is recursive, then $\{f\}$ is nowhere dense. b) For every $f \in \mathsf{R}$, $\{f\}$ is nowhere dense.

Corollary 6. *Every r.e. set of recursive functions is meagre.*

Proof. Every r.e. set of recursive functions is a measured set. By Theorem 2 it follows that the set is meagre.

Corollary 7. *The following sets are meagre:*

1) *The set of primitive recursive functions.*

2) *Every subset of the set of primitive recursive functions (in particular, every class in the Grzegorczyk hierarchy, the set of Kalmár elementary functions, the set of context-sensitive languages).*

3) *The family* $(f_i(x))$ *defined by*

$$f_i(x) = \begin{cases} a_i(x), & \text{if } a_i(x) \text{ is defined and } A_i(x) \leqq i \cdot a_i(x) \\ 0, & \text{otherwise.} \end{cases}$$

4) *The set of real-time computable functions.*

5) *Every r.e. complexity class.*

6) $\mathsf{E}(f)$, $\mathsf{Pol}(f)$ *and* $\mathsf{Pr}(f)$, *for every* $f \in \mathsf{R}$.

If we combine Corollary 6 and the Honesty Theorem (McCreight-Meyer [9]) we obtain

Theorem 3. *There exists a meagre set* $S \subset \mathsf{R}$ *such that for every* $t \in \mathsf{R}$ *we can find effectively a function* $t' \in S$ *for which* $C_t = C_{t'}$.

We may ask whether all complexity classes (not only those r.e.) are meagre.

Corollary 8. *Every complexity class is meagre.*

Proof. Let $t \in \mathsf{R}$. Then, there exists a recursive function t' such that $C_t \cup \mathsf{F} \subseteqq C_{t'}$. But, every complexity class which contains the set of functions of finite support is r.e. By Corollary 7, 5) $C_{t'}$ is meagre. Now we apply the Proposition 3, $C_t \subseteqq C_{t'}$, and we deduce that C_t is meagre.

Remark. The existence of complexity classes which are not r.e. (Landweber and Robertson [4]) shows that the converse of Theorem 2 fails.

Corollary 9. *The sets of algebraic numbers (and, in particular, the set of rationals) is meagre.*

Proof. By a well-known result of Hartmanis and Stearns [3] the set of algebraic numbers is contained in a complexity class. Hence, by Corollary 8, it is meagre.

Remark. Corollary 3 and Corollary 9 reinforce the classical result on the real line: in the set of recursive reals only a few numbers are algebraic (rationals).

Since the set of step-counting functions is obviously measured we obtain

Theorem 4. *In any Blum space the set of step-counting functions is meagre.*

Remarks. a) Theorems 3 and 4 show that the sets occuring in the Honesty and Gap Theorems are small not only in algebraic sense, but also in a topological sense. b) Theorem 3 is based on the Honesty Theorem. Theorem 4 is independent of the Gap Theorem (Borodin [2]). Moreover, Theorem 4 shows that the Gap Theorem is not a consequence of the distribution of step-counting functions between all partial recursive functions.

4. Concluding Remarks

S. Marcus [8] pointed out that in Real Function Theory the proofs of category theorems show that the property of a set of functions to be meagre is, in some way, conditioned to some degree of effectiveness of the definitions of the functions. Our results reinforces this remark: the distinction between meagre and nonmeagre sets of partial recursive functions is based on the difference on the effectiveness of the definitions of these sets.

Many results in Computational Complexity can be studied from a topological point of view. Hence, a great number of open problems naturally arise. We display some of them.

1. (Conjecture) The set of (partial) recursive $0-1$ valued functions is a set of the second Baire category.
2. (Conjecture) The set of strictly partial recursive functions is a set of the second Baire category, i.e. the set of recursive functions is not a residual.
3. Find topological versions of Speed-up and Compression Theorems (Blum [1]).
4. The set of (optimal) Gödel Numberings is meagre? (Schnorr [17])
5. Find analogue results from the point of view of a recursive measure.

References

[1] Blum, M., A machine-independent theory of complexity of recursive functions. Journal of ACM **14** (1967), 322–330.
[2] Borodin, A., Complexity classes of recursive functions and the existence of complexity gaps. In: ACM Symposium on theory of computing. Marina del Rey, Ca. (1969), pp. 67–78.
[3] Hartmanis, J., and R. E. Stearns, On the computational complexity of algorithms. Trans. Amer. Math. Soc. **117** (1965), 285–306.
[4] Landweber, H. H., and E. L. Robertson, Recursive properties of abstract complexity classes. Journal of ACM **19** (1972), 296–308.
[5] Lowenthal, F. D., Some results on measure and category in α-recursion theory. Notices AMS **20** (1973), A–350.
[6] Machtey, M., Augmented loop languages and classes of computable functions. J. Comp. Syst. Sc. **6** (1972), 603–624.
[7] Machtey, M., and P. Young, An introduction to the general theory of algorithms. North-Holland Publ. Comp., Amsterdam 1978.
[8] Marcus, S., Oral communication. July 1979.
[9] McCreight, E. M., and A. R. Meyer, Classes of computable functions defined by bounds on computation. In: ACM Symposium on theory of computing. Marina del Ray, Ca. (1969), pp. 79–88.
[10] Melhorn, K., On the size of computable functions. In: Annual IEEE Symposium on switching and automata theory, University of Iowa, Ca. (1973), pp. 190–196.
[11] Melhorn, K., Polynomial and abstract subrecursive classes. J. Comp. Syst. Sc. **12** (1976), 147–177.
[12] Meyer, A. R., and D. M. Ritchie, A classification of recursive functions. This Zeitschr. **18** (1972), 71–80.
[13] Myhill, J., Category methods in recursion theory. Pacific J. of Math. **11** (1961), 1479–1486.
[14] Oxtoby, J. C., Measure and category. Springer-Verlag, Berlin–Heidelberg–New York 1971.
[15] Rice, H. G., Recursive real numbers. Proc. Amer. Math. Soc. **5** (1954), 784–791.
[16] Rogers Jr., H., Theory of recursive functions and effective computability. McGraw-Hill Book, New York 1967.
[17] Schnorr, C. P., Optimal enumerations and optimal Gödel numberings. Math. Systems Theory **8** (1975), 182–191.

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