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# RECURSIVE BAIRE CLASSIFICATION AND SPEEDABLE FUNCTIONS

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**Abstract**

Using recursive variants of Baire notions of nowhere dense and meagre sets we study the topological size of speedable and infinitely often speedable functions in a machine-independent framework. We show that the set of speedable functions is not "small" whereas the set of infinitely often speedable functions is "large". In this way we offer partial answers to a question in [4].
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## 1. Introduction

Let N be the set of natural numbers and let $\{\varphi_i : i \geqq 0\}$ be an acceptable gödelization of PR, the set of unary partial recursive (p.r.) functions. For $\varphi \in \mathrm{PR}$ we put

$$\mathrm{dom}(\varphi) = \{x \in \mathrm{N} \mid \varphi(x) \text{ is defined}\}, \qquad \mathrm{range}(\varphi) = \{\varphi(x) \mid x \in \mathrm{dom}(\varphi)\},$$
$$\mathrm{supp}(\varphi) = \{x \in \mathrm{dom}(\varphi) \mid \varphi(x) \neq 0\}.$$

In case $x \in \mathrm{dom}(\varphi)$ we write $\varphi(x) < \infty$, otherwise $\varphi(x) = \infty$. A recursively enumerable (r.e.) set is the domain of a p.r. function. By min we denote the minimization operator and by $\langle\_,\_\rangle : \mathrm{N}^2 \to \mathrm{N}$ a fixed pairing function.

By $\boldsymbol{R}$ and $\boldsymbol{R}(0)$ we denote, respectively, the sets of unary recursive functions and unary recursive functions of finite support. The set $\boldsymbol{R}(0)$ is r.e., i.e. $\boldsymbol{R}(0) = \{\varphi_{h(n)} \mid n \geqq 0\}$, for some $h \in \boldsymbol{R}$. The function $h$ will be fixed throughout the paper as well as the recursive function $l(n) = \mathrm{card}(\mathrm{supp}(\varphi_{h(n)}))$.

For $\varphi, \theta \in \mathrm{PR}$ we put $\varphi \subseteqq \theta$ in case $\mathrm{supp}(\varphi) \subseteqq \mathrm{supp}(\theta)$ and $\varphi(x) = \theta(x)$, for every $x \in \mathrm{supp}(\varphi)$. If, in addition, $\varphi \neq \theta$, we write $\varphi \subset \theta$. For $X \subseteqq \mathrm{PR}$ we define the *finite trace of* $X$ to be set $X_* = \{t \in \boldsymbol{R}(0) \mid t \subseteqq \varphi \text{ for some } \varphi \in X\}$. For every $t \in \boldsymbol{R}(0)$, we put $U_t = \{\varphi \in \mathrm{PR} \mid t \subseteqq \varphi\}$. The family $(U_t)$ is a system of basic neighborhoods in PR; we shall work with the topology generated by this system [4], [5].

Following [11], [4], [5] we say that a set $X \subseteqq \mathrm{PR}$ is *recursively nowhere dense* with respect to $f, g \in \boldsymbol{R}$ in case the following four conditions hold: (i) $\varphi_{f(n)} \in \boldsymbol{R}(0)$, (ii) $m > g(n)$ implies $\varphi_{f(n)}(m) = 0$, (iii) $\varphi_{h(n)} \subset \varphi_{f(n)}$, (iv) there exists a natural $i$ such that $X \cap U_{\varphi_{f(n)}} = \emptyset$ whenever $l(\varphi_{f(n)}) > i$. A set $X \subseteqq \mathrm{PR}$ is *recursively meagre* if there exist a sequence $(X_i)_{i \geqq 0}$, $X_i \subseteqq \mathrm{PR}$ and two r.e. sets $(f_i)_{i \geqq 0}$, $(g_i)_{i \geqq 0}$, $f_i, g_i \in \boldsymbol{R}$, such that $X = \bigcup X_i$ and each $X_i$ is recursively nowhere dense with respect to $f_i$ and $g_i$. A set which is not recursively meagre is called a *set of recur-*

*sively second Baire category*. The above notions are recursive variants of the classical notions introduced by BAIRE [12].

A *Blum space* is a couple $\Phi = ((\varphi_i), (\Phi_i))$, where $(\varphi_i)$ is an acceptable gödelization and $(\Phi_i)$ is a sequence of p.r. functions (called the *step-counting functions*) satisfying the following two axioms (called *Blum axioms*): (i) $\mathrm{dom}(\varphi_i) = \mathrm{dom}(\Phi_i)$, (ii) the ternary predicate $M(i, x, y) = 1$ if $\Phi_i(x) \leq y$ and $M(i, x, y) = 0$ otherwise is recursive [1], [10], [7]. A set $(\Phi_i)$ satisfying only the second Blum axiom is called a *measured set*.

Following [10] we shall fix a Blum space $B = ((b_i), (B_i))$ satisfying the following two additional properties:

(1) There exists a recursive function $s$: $\mathrm{N}^2 \to \mathrm{N}$ such that $b_{s(i,x)}(y) = b_i(\langle x, y\rangle)$ and $B_{s(i,x)}(y) \leq B_i(\langle x, y\rangle)$, for all natural $i, x, y$.

(2) There exists a recursive function $k$: $\mathrm{N}^2 \to \mathrm{N}$ such that

$$b_{k(i,x)}(y) = \begin{cases} \psi_x(y) & \text{if } y \in \mathrm{dom}(\psi_x), \\ b_i(y) & \text{otherwise}, \end{cases}$$

$B_{k(i,x)}(y) \leq B_i(y)$, for every $y \notin \mathrm{dom}(\psi_x)$; here $(\psi_x)$ is an enumeration of the finite functions whose domains are the nonempty initial segments of the natural numbers. For example, the time/space RAM complexities [10] and in fact virtual all "natural" Blum spaces satisfy (1) and (2).

In what follows, when considering the Blum spaces $\Phi$ and $B$ we shall fix the recursive bijection $v$: $\mathrm{N} \to \mathrm{N}$ such that $\varphi_i = b_{v(i)}$, for all natural $i$ (ROGERS' Isomorphism Theorem) and the recursive, increasing in the second argument, function $r$: $\mathrm{N}^2 \to \mathrm{N}$ satisfying for all natural $i$ the inequalities $\Phi_i(x) \leq r(x, B_{v(i)}(x))$ a.e. and $B_{v(i)}(x) \leq r(x, \Phi_i(x))$ a.e. (Recursive Relatedness Theorem); here a.e. refers to "for all but finitely many" [10], [5]. Also in Section 3, $R(0) = \{b_{h(n)} \mid n \geq 0\}$, with $h \in R$.

The *complexity class named by $f \in R$ with respect to the Blum space $\Phi$* is the set $C_f^{\Phi} = \{\varphi_i \mid \Phi_i(x) \leq f(x) \text{ a.e.}\}$.

A Blum space $\Phi$ has the *parallel computation property* [9] if there is a recursive function $g$: $\mathrm{N}^2 \to \mathrm{N}$ such that

$$\varphi_{g(i,j)}(x) = \begin{cases} \varphi_i(x) & \text{if } \Phi_i(x) \leq \Phi_j(x), \\ \varphi_j(x) & \text{otherwise}, \end{cases}$$

and $\Phi_{g(i,j)}(x) = \min(\Phi_i(x), \Phi_j(x))$. This property is not satisfied by all Blum spaces; however, (i) the model of many-tape, many-heads Turing machines, with a read-only head on their input tape offers an example of a Blum space having the parallel computation property and (ii) each set of step-counting functions can be extended to a set of step-counting functions satisfying the above property (by introducing sufficiently many new programs) [16].

## 2. Recursively nowhere dense sets

We present a simple characterization of recursively nowhere dense sets and some examples of such sets occuring in BLUM's abstract complexity theory. They provide interesting refinements of some results in [5].

The first result is an easy consequence of definitions.

Lemma 1. *Let $t$ be in $R(0)$ and $X \subsetneqq \mathrm{PR}$. The following assertions are equivalent:*
(a) $X \cap U_t = \varnothing$, (b) $X_* \cap U_t = \varnothing$, (c) $t \notin X_*$. □

Next we offer an algebraic characterization of recursively nowhere dense sets.

Proposition 2. *Let $X \subsetneqq \mathrm{PR}$. The following assertions are equivalent:*

(1) *the set $X$ is recursively nowhere dense,*

(2) *the set $X_*$ is recursively nowhere dense,*

(3) *there exists a recursive function $p$: $\mathrm{N} \to \mathrm{N}$ such that for all naturals $n$, $\varphi_{h(n)} \subsetneqq \varphi_{h(p(n))}$ and $\varphi_{h(p(n))} \notin X_*$.*

Proof. In view of Lemma 1 the statements (1) and (2) are equivalent.

(1)$\Rightarrow$(3): Suppose $X$ is recursively nowhere dense via the recursive functions $f$ and $g$. By a standard use of the s-m-n Theorem (valid for the numbering $\{\varphi_{h(n)}\}$ of $R(0)$) we define a recursive function $s$: $\mathrm{N} \to \mathrm{N}$ such that

$$\varphi_{h(s(n))}(x) = \begin{cases} \varphi_{h(n)}(x) & \text{if } x \leqq l(n), \\ 1 & \text{if } l(n)+1 \leqq X \leqq l(n)+i+1, \\ 0 & \text{otherwise}, \end{cases}$$

where $i \geqq 1$ is the natural such that $X \cap U_{\varphi_{f(n)}} = \varnothing$ whenever $l(n) > i$. We construct the recursive function $\tau$: $\mathrm{N} \to \mathrm{N}$,

$$\begin{aligned} \tau(y) = \min_j [&(x \leqq g(y) \;\&\; x \in \operatorname{supp}(\varphi_{f(y)}) \Rightarrow \varphi_{h(j)}(x) = \varphi_{f(y)}(x)) \\ &\& \; (x \leqq l(j) \;\&\; x \in \operatorname{supp}(\varphi_{h(j)}) \Rightarrow \varphi_{h(j)}(x) = \varphi_{f(y)}(x))]. \end{aligned}$$

and we notice that $\varphi_{f(y)} = \varphi_{h(\tau(y))}$. Finally, we define the recursive function $p$: $\mathrm{N} \to \mathrm{N}$ by $p(n) = \tau(s(n))$. If follows that $\varphi_{h(n)} \subsetneqq \varphi_{h(p(n))} = \varphi_{f(s(n))}$; consequently, if $l(s(n)) \geqq i+1$, then $U_{\varphi_{h(p(n))}} \cap X = \varnothing$, which by Lemma 1 means that $\varphi_{h(p(n))} \notin X_*$.

(3)$\Rightarrow$(1): Put $f(n) = h(p(n))$ and $g(n) = l(p(n))$; using Lemma 1 one can easily verify that $X$ is recursively nowhere dense under $f$ and $g$. □

Example 3. In every Blum space each measured set (and in particular the set of step-counting functions) and every complexity class are recursively meagre ([4], [5]). Sometimes, these sets can be recursively nowhere dense (Example 3); in other cases, they are not recursively nowhere dense (Example 7).

(i) Let $\Phi$ be a Blum space with the property that for all natural $i$ and $x$, $\Phi_i(x) \geqq x$ (e.g. the time complexity for Turing machines). The set of step-counting functions $\{\Phi_i \mid i \geqq 0\}$ is recursively nowhere dense. Indeed, define the recursive function $r$: $\mathrm{N} \to \mathrm{N}$ such that

$$\varphi_{h(r(n))}(x) = \begin{cases} \varphi_{h(n)}(x) & \text{if } x \leqq l(n), \\ 0 & \text{if } x = l(n)+1, \\ 1 & \text{if } x = l(n)+2, \\ 0 & \text{otherwise}, \end{cases}$$

and notice that $\varphi_{h(r(n))} \notin \{\Phi_i \mid i \geqq 0\}_*$, for all natural numbers $n$ (reason: $\Phi_i(l(n)+2) \geqq l(n)+2 > 1 = \varphi_{h(r(n))}(l(n)+2)$). The conclusion follows from Proposition 2 (3).

(ii) There exists an infinite, r.e. complexity class which is nowhere dense. To prove this consider a recursive function $f$: $\mathrm{N} \to \mathrm{N}$, $f(x) \geqq x+1$, for all natural $x$, and a injective recursive function $g$: $\mathrm{N} \to \mathrm{N}$ such that

$$\varphi_{g(i)}(x) = \begin{cases} f(x) & \text{if } x \leqq i, \\ 0 & \text{otherwise.} \end{cases}$$

Next define the recursive function $H$: $\mathrm{N} \to \mathrm{N}$ by

$$H(0) = 0 \qquad \text{and} \qquad H(x+1) = g(\min_i [g(i) > H(x)]).$$

It follows that $I = \text{range}(H)$ is an infinite recursive set. Consider now an arbitrary Blum space $\Phi$ with $\Phi_i(x) \geqq x+1$, for all natural $i$ and $x$, and modify its step-counting functions as follows:

$$\Phi'_i(x) = \begin{cases} \Phi_i(x) & \text{if } i \notin I, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $\Phi' = ((\varphi_i), (\Phi'_i))$ is a Blum space. Construct the recursive function $p$: $\mathrm{N} \to \mathrm{N}$ by

$$\varphi_{h(p(n))}(x) = \begin{cases} \varphi_{h(n)}(x) & \text{if } x \leqq l(n), \\ f(x)+1 & \text{if } x = l(n)+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $t$ be the zero function and notice that the complexity class $C_t^{\Phi'}$ is exactly the set $\{\varphi_{H(n)} \mid n \geqq 0\}$. Further, $\varphi_{h(p(n))} \notin (C_t^{\Phi'})_*$, for all natural $n$, which proves the assertion. □

The following three corollaries provide useful sufficient conditions for a set to be or not to be recursively nowhere dense.

Corollary 4. *If $X \subseteq \mathrm{PR}$ and $X_* = R(0)$, then $X$ is not recursively nowhere dense.*

Proof. Directly from Proposition 2 (3). □

Corollary 5. *Let $Y \subseteq \mathrm{N}$ be a set which contains a non-zero element. Then the set $X = \{\varphi \in \mathrm{PR} \mid range(\varphi) \subseteq Y\}$ is a recursively nowhere dense set.*

Proof. Let $p$: $\mathrm{N} \to \mathrm{N}$ be the recursive function given by

$$\varphi_{h(p(n))}(x) = \begin{cases} \varphi_{h(n)}(x) & \text{if } x \leqq l(n), \\ a & \text{if } x = l(n)+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $a \in \mathrm{N} \setminus (Y \setminus \{0\})$. Clearly, $\varphi_{h(n)} \subseteq \varphi_{h(p(n))}$ and $\varphi_{h(p(n)} \notin X_*$. □

Comment. If $Y = \{0, 1\}$ or $Y = \{0, 1, \ldots, k\}$, we obtain results proved in [13] and [18]. Another example is $Y = \mathrm{N} \setminus \{k\}$, where $k \neq 0$.

A set $X \subseteq \mathrm{PR}$ is closed *under finite variations* in case for every $f \in X$ and $g \in \mathrm{PR}$, if $f = g$ a.e., then $g \in X$.

Corollary 6. *Let $X \subseteq \mathrm{PR}$ be a non-empty set closed under finite variations. Then $X$ is not a recursively nowhere dense set.*

Proof. Take $f$ in $X$ and $t$ in $R(0)$. The p.r. function $g(x) = t(x)$ if $x \in \text{supp}(t)$, and

$g(x) = f(x)$ otherwise, belongs to $X$ since $f = g$ a.e. Consequently, $X_* = \boldsymbol{R}(0)$ and the conclusion follows from Corollary 4. □

Comment. Many natural and important classes of p.r. functions are closed under finite variations; for example, the class of unary primitive recursive functions, each unary Grzegorczyk class, the complexity classes P, NP, PSPACE etc.

In contrast with Example 3 we present

Example 7. (i) Let $f$: $\mathrm{N} \to \mathrm{N}$ be a recursive function and $I \subseteq \mathrm{N}$ an infinite recursive set such that $f = \varphi_i$, for all $i \in I$. Let $g$: $\mathrm{N} \to \mathrm{N}$ be a strictly increasing recursive function such that $\mathrm{range}(g) = I$. The p.r. function $p$: $\mathrm{N} \to \mathrm{N}$ defined by $p(i) = 0$ if $i \notin I$, and $p(i) = \min_n [g(n) = i]$ otherwise, is recursive and surjective. Modify an arbitrary Blum space $\Phi$ as follows:

$$\Phi'_i(x) = \begin{cases} \Phi_i(x) & \text{if } i \notin I, \\ \varphi_{h(p(i))}(x) & \text{otherwise}. \end{cases}$$

It is seen that $\Phi' = ((\varphi_i), (\Phi'_i))$ is a Blum space, $\{\Phi'_i \mid i \geqq 0\}_* = \boldsymbol{R}(0)$, so the set of step-counting functions in $\Phi'$ is not recursively nowhere dense.

(ii) Take a Blum space $\Phi$ with the *property of finite variation*, i.e. for every recursive functions $f$, $g$ with $f = g$ a.e. and for every recursive function $w$ one has $f \in C_w^\Phi$ iff $g \in C_w^\Phi$. Such a complexity class $C_v^\Phi$ is r.e. ([9], [7]) and closed under finite variations, so it is not recursively nowhere dense, by Corollary 6. □

Comment. In contrast with the case of complexity classes, the *honesty-classes* of [17] can never be recursively nowhere dense. Indeed, given a p.r. function $f$: $\mathrm{N}^2 \to \mathrm{N}$, the honesty-class of $f$ is

$$H_f = \{\varphi \in \mathrm{PR} \mid \text{there exists an index } i \text{ such that for almost all } x, \text{ if } \varphi_i(x) < \infty \text{ and } f(x, \varphi_i(x)) < \infty, \text{ then } \Phi_i(x) \leqq f(x, \varphi_i(x))\}$$

and $(H_f)_* = \boldsymbol{R}(0)$.

## 3. Speedable functions

This section presents the main results on this paper, the topological size of speedable functions, thus offering partial answers to a question in [4].

Fix a Blum space $\Phi = ((\varphi_i), (\Phi_i))$. For a recursive function $g$: $\mathrm{N}^2 \to \mathrm{N}$, increasing in the second argument, we define the set of *g-speedable functions*

$$\begin{aligned} &\mathrm{SPEED}(g; \Phi; \text{a.e.}) \\ &= \{f: \mathrm{N} \to \mathrm{N} \mid f \text{ recursive and for every } i \text{ with } \varphi_i = f \text{ there exists a } j \\ &\qquad \text{with } \varphi_j = f \text{ and } \Phi_i(x) \geqq g(x, \Phi_j(x)) \text{ a.e.}\}; \end{aligned}$$

replacing the condition a.e. by i.o. (infinitely often) we get the set of *g-i.o. speedable functions* SPEED$(g; \Phi;$ i.o.$)$; see [5], [16], [17].

The sets SPEED$(g; \Phi;$ a.e.$)$ and SPEED$(g; \Phi;$ i.o.$)$ are machine-dependent (one can easily construct a new Blum space for which infinitely many functions in SPEED$(g; \Phi;$ a.e.$)$

have zero complexity). In what follows we shall offer machine-independent results concerning the size of these sets of functions.

Theorem 8. *The sets* SPEED$(g; \Phi;$ a.e.$)$ *and* SPEED$(g; \Phi;$ i.o.$)$ *are not recursively nowhere dense.*

Proof. Recall that $B = ((b_i), (B_i))$ is the particular Blum space introduced in Section 1. First we construct a recursive function $q$: $\mathbb{N}^2 \to \mathbb{N}$, increasing in the second argument such that SPEED$(q; B;$ a.e.$) \subsetneq$ SPEED$(g; \Phi;$ a.e.$)$. To this aim we shall use the recursive functions $r$ and $v$ satisfying (1), (2) (see § 1) and we set $q(x, y) = r(x, g(r(x, y)))$. Take $f$ in SPEED$(q; B;$ a.e.$)$ and pick an index $i$ with $\varphi_i = f$. Then $B_{v(i)}(x) \leqq r(x, \Phi_i(x))$ a.e. and there exists an index $j$ with $f = \varphi_i = b_{v(i)} = b_j$ and

$$B_{v(i)}(x) \geqq q(x, B_j(x)) \text{ a.e.} = r(x, g(x, r(x, B_j(x)))).$$

Since $r$ is increasing in the second argument it follows that

$$\Phi_i(x) \geqq g(x, r(x, B_j(x))) \text{ a.e.} \geqq g(x, B_j(x)) \text{ a.e.},$$

i.e. $f \in$ SPEED$(g; \Phi;$ a.e.$)$.

Next we prove that SPEED$(q; B;$ a.e.$)$ is closed under finite variations and thus is not recursively nowhere dense (by Corollary 6). Accordingly, from the relations SPEED$(q; B;$ a.e.$) \subsetneq$ SPEED$(g; \Phi;$ a.e.$) \subsetneq$ SPEED$(g; \Phi;$ i.o.$)$ and the fact that the sets not recursively nowhere dense are closed under superset [4], [5], it follows that the last sets are not recursively nowhere dense.

To finish the proof let $f_1 \in$ SPEED$(q; B;$ a.e.$)$ and let $f_2$: $\mathbb{N} \to \mathbb{N}$ be a recursive function such that $f_1(x) = f_2(x)$ for all $x \geqq m$. Let $f_2 = b_i$ and let $y \in \mathbb{N}$ such that $\psi_y(x) = f_1(x)$ if $x \leqq m$, and $\psi_y(x) = \infty$ otherwise. Then $b_{k(i,y)}, = f_1$ and there exists a natural $j$ such that $f_1 = b_j$ and $B_{k(i,j)}(x) \geqq q(x, B_j(x))$ a.e. Take a natural $z$ with $\psi_z(x) = f_2(x)$ if $x \leqq m$, and $\psi_z(x) = \infty$ otherwise; one has $b_{k(j,z)} = f_2$ and $B_j(x) \geqq B_{k(j,z)}(x)$ for $x \geqq m$. We conclude with

$$B_i(x) \geqq B_{k(i,y)}(x) \text{ a.e.} \geqq q(x, B_j(x)) \text{ a.e.} \geqq q(x, B_{k(j,z)}(x)) \text{ a.e.},$$

i.e. $f_2 \in$ SPEED$(q; B;$ a.e.$)$. □

For the set SPEED$(g; \Phi;$ i.o.$)$ the above result will be significantly improved in Theorem 9. The importance of the set SPEED$(g; \Phi;$ i.o.$)$ comes from the fact that the process of constructing a faster program for a $g$-i.o. speedable function is *algorithmic* [2], [16], in contrast with the case of speedable functions [5], [16]. Furthermore, from the fact that $\varphi_j$ is i.o. faster than $\varphi_i$ the possibility that $\varphi_j$ is slower than $\varphi_i$ on the remainder points does not exclude. However, as pointed out in [16], by requiring the parallel computation property to be valid one can replace $\varphi_j$ by a new program which runs $\varphi_i$ and $\varphi_j$ in parallel, thus ensuring that the i.o. speed-up of $\varphi_i$ is nowhere slower than the original one.

Theorem 9. *The set* SPEED$(g; \Phi;$ i.o.$)$ *is of the recursively second Baire category.*

Proof. For $t$ in $R(0)$ and natural $n$ we define $t^{(n)}$ in $R(0)$ by

$$t^{(n)}(x) = \begin{cases} t(x) & \text{if } t(x) \neq 0 \text{ or } x > n, \\ 1 & \text{if } t(x) = 0 \text{ and } x \leqq n, \end{cases}$$

and we notice that for every set $X \subseteqq \mathrm{PR}$, $X \cap U_{t^{(n)}} = \emptyset$ whenever $X \cap U_t = \emptyset$, and $t^{(n)} \subseteqq f$ implies $t \subseteqq f$.

Fix a recursive function $g$: $\mathrm{N}^2 \to \mathrm{N}$, increasing in the second argument, and suppose, by absurdity, that SPEED$(g; \Phi;$ i.o.) is recursively meagre. Proceeding as in the proof of Theorem 8 we can find a recursive function $q$: $\mathrm{N}^2 \to \mathrm{N}$, increasing in the second argument, such that SPEED$(q; B;$ i.o.) $\subseteqq$ SPEED$(g; \Phi;$ i.o.) and consequently recursively meagre. Working from now on in the Blum space $B$ we fix an enumeration $h$ of $\boldsymbol{R}(0)$, i.e. $\boldsymbol{R}(0) = \{b_{h(n)} \mid n \geqq 0\}$.

In view of our assumption it follows that there exist a sequence of sets $(\mathrm{SPEED}_j)_{j \geqq 0}$ and a r.e. family of unary recursive functions $(f_i)$, $(g_i)$ such that

(i) SPEED = SPEED$(q; B;$ i.o.) $= \bigcup_{j \geqq 0} \mathrm{SPEED}_j$,

(ii) $b_{h(n)} \subseteqq b_{f_j(n)}$ for all natural $n, j$,

(iii) $b_{f_j(n)}(m) = 0$ whenever $m > g_j(n)$,

(iv) $\mathrm{SPEED}_j \cap U_{b_{f_j(n)}} = \emptyset$, for all $j$ and sufficiently large $n$.

Replacing $t \in \boldsymbol{R}(0)$ by some $t^{(m)} \in \boldsymbol{R}(0)$, as mentioned in the beginning of the proof, we may suppose in (iii) that $b_{f_j(n)}(m) \neq 0$ for $m \leqq g_j(n)$.

The idea of the proof is to construct a function $z \in$ SPEED such that for every natural $j$ there exist infinitely many $n$ with $b_{f_j(n)} \subseteqq z$. The construction of $z$ is done by stages. At stage $s+1$, with $s = \langle j, k \rangle$, we try to force $z$ to extend $b_{f_j(n)}$, for some $n$ with $l(n) > k$ in such a way that $z$ will be speedable in some new point. The speedable process is performed in the standard way (see [5], [10], [19]).

Define a family of p.r. functions $(z_s)_{s \geqq 0}$, $z_s$: $\mathrm{N}^3 \to \mathrm{N}$, in a construction by stages. At stage $s$ we define the p.r. function $z_s$ and the sequence of integers $(l_n(s))_{n \geqq 0}$ with the following property: for every natural $w$, $z_s(n, w, x) = \infty$ whenever $x > l_n(s)$. Simultaneously we construct, for each natural $n$, a recursive set called CRITICAL-SET$(n)$, in which, at each stage, we try to introduce some natural number (i.e. the point in which $z$ will be speedable). Let CRITICAL-SET$(n, s)$ be the set of elements introduced in CRITICAL-SET$(n)$ by stage $s$, inclusively.

We display now the construction.

Stage $s = 0$. Set $z_0(n, w, x) = \infty$, $l_n(0) = -1$ and CRITICAL-SET$(0, 0) = \emptyset$, for all natural $n, w, x$.

Stage $s+1$. Let $s = \langle j, k \rangle$ and compute an index $k(n, s+1)$ such that

$$b_{h(k(n,s+1))}(x) = \begin{cases} z_s(n, 0, x) & \text{if } x \leqq l_n(s), \\ 1 & \text{if } l_n(s) + 1 \leqq x \leqq k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $z_s(n, 0, x) = \infty$ for some $x \leqq l_n(s)$, then $k(n, s+1)$ is not defined. Let $t_{s+1}^{(n)} = g_j(k(n, s+1)) + 1$ and call it *the critical element computed at stage* $s+1$ (in case it exists!). Construct then the set

$$\begin{aligned} &C(n, w, t_{s+1}^{(n)}) \\ &= \{i \in \mathrm{N} \mid w \leqq i < t_{s+1}^{(n)},\ i \notin C(n, w, y) \text{ for all } y < t_{s+1}^{(n)} \text{ and } y \in \text{CRITICAL-SET}(n,s), \\ &\quad B_i(t_{s+1}^{(n)}) < q(x, B_n(\langle i+1, t_{s+1}^{(n)} \rangle))\}. \end{aligned}$$

Note that if $t_{s+1}^{(n)}$ is defined and $w \geqq t_{s+1}^{(n)}$, then $C(n, w, t_{s+1}^{(n)}) = \emptyset$.

Further we construct:

$$z_{s+1}(n, w, x) = \begin{cases} z_s(n, w, x) & \text{if } x \leqq l_n(s), \\ b_{f_j}(k(n, s+1))(x) & \text{if } l_n(s) < x \leqq g_j(k(n, s+1)), \\ \min_y [y > 0, y \neq b_i(x) \text{ for all } i \in C(n, w, t_{s+1}^{(n)})] & \text{if } x = t_{s+1}^{(n)}, \\ \infty & \text{otherwise}, \end{cases}$$

$\text{CRITICAL-SET}(n, s+1) = \text{CRITICAL-SET}(n, s) \cup \{t_{s+1}^{(n)}\}$, $l_n(s+1) = g_j(k(n, s+1)) + 1$.

End of the construction.

Note that $z_{s+1}$ extends $z_s$ for each natural $s$, so there exists a p.r. function $\tilde{z}$: $\mathsf{N}^3 \to \mathsf{N}$ given by $\tilde{z}(n, w, x) = \lim_{s\to\infty} z_s(n, w, x)$. Let $H$: $\mathsf{N} \to \mathsf{N}$ be a recursive function such that $\tilde{z}(n, w, x) = b_{H(n)}(w, x)$ and let $n$ be a fixed-point of function $H$, i.e. $b_n(w, x) = b_{H(n)}(w, x) = \lim_{s\to\infty} z_s(n, w, x)$. This $n$ will be fixed throughout the rest of the proof.

We continue our proof with the following facts.

Fact 1. *For all natural $x$, $s$ and $w$, $z_s(n, w, x) < \infty$, whenever $x \leqq l_n(s)$.*

We proceed by induction on $s$. For $s = 0$, $l_n(0) = -1$, so the assertion holds. Assume now that the statement is true for $s$. In view of the induction hypothesis, $k(n, s+1)$ is defined; all it remains to prove reduces to the convergence of the computation of $z_{s+1}(n, w, x)$ for $x = t_{s+1}^{(n)}$. First we determine the set $C(n, w, x)$, which means the computation of the sets $C(n, w, y)$ for all $y < x$, $y \in \text{CRITICAL-SET}(n, s)$, and of the values $b_n(\langle u, x\rangle)$ for all $w < u \leqq x$. The set $C(n, w, y)$ is constructed during the computation of $b_n(\langle w, y\rangle)$:

$$\begin{array}{l} b_n(\langle x, x\rangle), b_n(\langle x, x-1\rangle), \ldots, b_n(\langle x, 0\rangle), \\ \vdots \\ b_n(\langle w+1, x\rangle), b_n(\langle w+1, x-1\rangle), \ldots, b_n(\langle w+1, 0\rangle), \\ b_n(\langle w, x\rangle), b_n(\langle w, x-1\rangle), \ldots, b_n(\langle w, 0\rangle). \end{array}$$

We notice that $b_n(\langle w, x\rangle) < \infty$, whenever $b_n(\langle u, x\rangle) < \infty$ for $w < u \leqq x$ and $b_n(\langle w, y\rangle) < \infty$ for $0 \leqq y < x$; denote this statement the "rectangle rule". For $0 \leqq y < x$, $b_n(\langle w, y\rangle) < \infty$ for all $w$, i.e. in the above table, the entries in all columns but the first one are defined. Indeed, two cases may occur: (i) if $y \leqq l_n(s)$, then $z_s(n, w, y) < \infty$ (the induction hypothesis), and consequently, $b_n(\langle w, y\rangle) = z_s(n, w, y) < \infty$; (ii) if $l_n(s) + 1 \leqq y \leqq x - 1$, then $z_{s+1}(n, w, y) = b_{f_j}(k(n, s+1))(y) < \infty$ and $b_n(\langle w, y\rangle) = z_{s+1}(n, w, y) < \infty$. It remains to show that all entries in the first column converge. Since $C(n, x, x) = \emptyset$, $z_{s+1}(n, x, x) < \infty$ and consequently $b_n(\langle x, x\rangle) < \infty$. In view of the "rectangle rule" $b_n(\langle x-1, x\rangle) < \infty$, stepping down in the first column we get the desired conclusion; $b_n(\langle u, x\rangle) < \infty$ for $w < u \leqq x$, i.e. $z_{s+1}(n, w, x) < \infty$, thus ending the induction step. (Notice that we have proved a bit more than the statement of Fact 1.)

Fact 2. *For every natural $w$ there exists a natural $m_w$ such that $b_n(\langle 0, x\rangle) = b_n(\langle w, x\rangle)$, whenever $x \geqq m_w$.*

An induction on $x$ shows that for all natural $w$ and $x \in \text{CRITICAL-SET}(n)$,

$$C(n, w, x) = C(n, 0, x) - \{0, 1, \ldots, w-1\}.$$

Since an index $i$ appears at most once in some $C(n, 0, x)$, $x \geqq 0$, there exists an integer $m'_w$

such that $C(n, w, x) = C(n, 0, x)$ for all $x \geqq m'_w$. The sequence $\{t_s^{(n)}\}_s$ is strictly increasing, so there is some $s$ with $t_s^{(n)} \geqq m'_w$. Consequently, for every $x \geqq m_w = t_s^{(n)}$, $b_n(\langle 0, x\rangle) = b_n(\langle w, x\rangle)$. Let $z: \mathrm{N} \to \mathrm{N}$ be the function defined by $z(x) = b_n(\langle 0, x\rangle)$. In view of Fact 1 it follows that $z$ is a recursive function.

Fact 3. *Let $i$ be an index such that $b_i = z$. For infinitely many $x$, $B_i(x) \geqq q(x, B_n(\langle i+1, x\rangle))$.*

Suppose, by absurdity, that $B_i(x) < q(x, B_n(\langle i+1, x\rangle))$ for almost all $x$. Take $x$ the minimal natural such that $x \geqq i$, $x \in$ CRITICAL-SET$(n)$ and $B_i(x) < q(x, B_n(\langle i+1, x\rangle))$; it follows that $i \in C(n, 0, x)$, which implies $b_i(x) = b_n(\langle 0, x\rangle) = z(x)$, a contradiction.

Fact 4. *The function $z$ is in* SPEED$(q; B$; i.o.).

Let $b_i = z$. In view of Fact 3, for infinitely many $x$, $B_i(x) \geqq q(x, B_n(\langle i+1, x\rangle))$. By Fact 2, there exists a natural $m_{i+1}$ such that $b_n(\langle i+1, x\rangle) = b_n(\langle 0, x\rangle)$, whenever $x \geqq m_{i+1}$. Take $y$ such that

$$\psi_y(x) = \begin{cases} b_n(\langle 0, x\rangle) & \text{if } x < m_{i+1}, \\ \infty & \text{otherwise}. \end{cases}$$

In view of (1) and (2), for infinitely many $x$

$$B_i(x) \geqq q(x, B_n(\langle i+1, x\rangle)) \geqq q(x, B_{s(n,i+1)}(x)) \geqq q(x, B_{k(s(n,i+1)),y)}(x))$$

and $b_{k(s(n,i+1)),y)}(x) = z(x)$ for all $x$.

We are now in a position to conclude our proof. From $z \in$ SPEED$(q; B$; i.o.) it follows that $z \in \mathrm{SPEED}_j$ for some $j$. There exists also a natural $c$ with $\mathrm{SPEED}_j \cap U_{b_{f(n)}} = \emptyset$ for $n \geqq c$. Take $s = \langle j, c\rangle$ and consider the operations performed during the stage $s+1$. We get

$$\lambda x.z_s(n, 0, x) \sqsubseteq b_{h(k(n,s+1))} \sqsubseteq b_{f(k(n,s+1))}.$$

For all $m \leqq g_j(n)$ we have $b_{f(n)}(m) \neq 0$; so $0 \notin \mathrm{range}(\lambda x.z_s(n, 0, x))$ and thereafter $\lambda x.z_{s+1}(n, 0, x) \sqsupseteq b_{f_j(k(n,s+1))}$. This implies that $z \sqsupset b_{f(k(n,s+1))}$. Keeping in mind that $k(n, s+1) \geqq c$ we get that $z \in \mathrm{SPEED}_j \cap U_{b_{f(n)}}$ for some $n \geqq c$, a contradiction. □

Comment. For every recursive functions $f$, $g$ we can find $z \in$ SPEED$(g; \Phi$; i.o.) such that $z \notin C_f^{\Phi}$ (use Theorem 10 and the fact that every complexity class is recursively meagre). In other words, there exist arbitrarily complex $g$-i.o. speedable functions, a statement in accord with our intuition.

Finally, let us briefly consider the case of p.r. functions. Clearly, Theorem 8 and 9 remain true when we replace recursive functions by p.r. functions. In the spirit of [14], [15] and [3] one may define the set

$X = \{f \in \mathrm{PR} \mid$ for every recursive increasing in the second argument function $g: \mathrm{N}^2 \to \mathrm{N}$ and for every $\varphi_i = f$ there exists $\varphi_j = f$ with $\Phi_i(x) \geqq g(x, \Phi_j(x))$ i.o.$\}$.

This set is non-empty [3].

Proposition 11. *The set $X$ is recursively meagre.*

Proof. First notice that $X \cap R = \emptyset$. Indeed, take $\varphi_i \in R$ and put $g(x, y) = \Phi_i(x) + 1$ for all $x$ and $y$. Consequently, $X$ is included in the set of strictly partial, p.r. functions, which is recursively meagre [13], [18], [6]. □

Open problems. Determine the exact position of the sets SPEED($g$; $\Phi$; a.e.), PR $\setminus$ SPEED($g$; $\Phi$; a.e.), PR $\setminus$ SPEED($g$; $\Phi$; i.o.) according to the above Baire classification.

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