Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 36, S. 143–147 (1990)

ON A THEOREM OF GÜNTER ASSER

by CRISTIAN CALUDE and LILA SÂNTEAN in Bucharest (Romania)

1. Introduction

Recently, G. ASSER [2] has obtained two interesting characterizations of the class of unary primitive recursive string-functions over a fixed alphabet as Robinson algebras. Both characterizations use a somewhat artificial string-function, namely the string-function lexicographically associated with the number-theoretical excess-over-a-square function. Our aim is to offer two new and natural Robinson algebras which are equivalent to ASSER's algebras.

Let N denote the set of naturals, i.e. $\mathsf{N} = \{0, 1, 2, \dots\}$, and $\mathsf{N}_+ = \mathsf{N} \setminus \{0\}$. We consider a fixed alphabet $\mathsf{A} = \{a_1, a_2, \dots, a_r\}$, $r \geqq 2$, and denote by A^* the free monoid generated by A under concatenation (with e as the null-string). By length(w) we denote the length of the string w (length$(e) = 0$). For every $w \in \mathsf{A}^*$ and $m \in \mathsf{N}$ let $w^m = ww\dots w$ (m times), in case $m > 0$, and $w^0 = e$. By *Fnc* (respectively, Fnc_{A}) we denote the set of all unary number-theoretical (respectively, string) functions. By I, $Succ$, E, C_m, Pd we denote the following number-theoretical functions: $I(x) = x$; $Succ(x) = x + 1$; $E(x) = x \dot{-} [\sqrt{x}]^2$; $C_m(x) = m$; $Pd(x) = x \dot{-} 1$, where $x \dot{-} y = \max(x - y, 0)$. By I^{A}, $Succ_i^{\mathsf{A}}$, C_u^{A}, σ, δ, π we denote the following string-functions: $I^{\mathsf{A}}(w) = w$; $Succ_i^{\mathsf{A}}(w) = wa_i$ $(1 \leqq i \leqq r)$; $C_u^{\mathsf{A}}(w) = u$; $\sigma(e) = a_1$, $\sigma(wa_i) = wa_{i+1}$, if $1 \leqq i < r$, and $\sigma(wa_r) = \sigma(w)\, a_1$; $\delta(e) = e$, $\delta(wa_i) = w$ $(1 \leqq i \leqq r)$; $\pi(e) = e$, $\pi(\sigma(w)) = w$. Furthermore we use the bijections c: $\mathsf{A}^* \to \mathsf{N}$, $\bar{c}$: $\mathsf{N} \to \mathsf{A}^*$ given by $c(e) = 0$, $c(wa_i) = r \cdot c(w) + i$, $1 \leqq i \leqq r$, and $\bar{c}(0) = e$, $\bar{c}(m + 1) = \sigma(\bar{c}(m))$; obviously $c(\bar{c}(m)) = m$ and $\bar{c}(c(w)) = w$.

To each $f \in Fnc$ we associate the string-function $s(f) \in Fnc_{\mathsf{A}}$ defined by $s(f)(w) = \bar{c}(f(c(w)))$; conversely, to each string-function g we associate the number-theoretical function $n(g) \in Fnc$ defined by $n(g)(x) = c(g(\bar{c}(x)))$. It is easily seen that for every $f \in Fnc$ and $g \in Fnc_{\mathsf{A}}$ one has $n(s(f)) = f$ and $s(n(g)) = g$. For example, $s(C_m) = C^{\mathsf{A}}_{\bar{c}(m)}$, $s(Succ) = \sigma$, $s(I) = I^{\mathsf{A}}$, $s(Pd) = \pi$.

For every $F \subseteqq Fnc$ and $G \subseteqq Fnc_{\mathsf{A}}$ we put $s(F) = \{s(f) \mid f \in F\}$ and $n(G) = \{n(g) \mid g \in G\}$. A mapping from Fnc^n in *Fnc* ($n \in \mathsf{N}_+$) is called an *n-ary operator in Fnc*, and analogous for Fnc_{A}. We consider the following operators in *Fnc* and Fnc_{A}:

$sub(f, g) = h$ iff $f, g, h \in Fnc$ and $h(x) = f(g(x))$;

$it_x(f) = h$ iff $f, h \in Fnc$ and $h(0) = x$, $h(y + 1) = f(h(y))$;

$add(f, g) = h$ iff $f, g, h \in Fnc$ and $h(x) = f(x) + g(x)$;

$diff(f, g) = h$ iff $f, g, h \in Fnc$ and $h(x) = f(x) \dot{-} g(x)$;

$sub_{\mathsf{A}}(f, g) = h$ iff $f, g, h \in Fnc_{\mathsf{A}}$ and $h(w) = f(g(w))$;

$\sigma\text{-}it_{\mathsf{A},u}(f) = h$ iff $f, h \in Fnc_{\mathsf{A}}$ and $h(e) = u$, $h(\sigma(w)) = f(h(w))$;

$it_{\mathsf{A},u}(f_1, \dots, f_r) = h$ iff $f_1, \dots, f_r$, $h \in Fnc_{\mathsf{A}}$ and
$h(e) = u$, $h(wa_i) = f_i(h(w))$, $1 \leqq i \leqq r$;

$con_{\mathsf{A}}(f, g) = h$ iff $f, g, h \in Fnc_{\mathsf{A}}$ and $h(w) = f(w)\, g(w)$.

For every operator φ in *Fnc*,

$$s(\varphi)(f) = s(\varphi(n(f))), \text{ for every } f \in Fnc;$$

analogously, for every operator θ in *Fnc*,

$$n(\theta)(g) = n(\theta(s(g))), \text{ for every } g \in Fnc.$$

For example, $s(it_x) = \sigma\text{-}it_{A,c(x)}$, $n(\sigma - it_{A,w}) = it_{\bar{c}(w)}$.

Finally, for every subset $F \subseteq Fnc$ and every set X of operators in *Fnc*, $[F; X]$ denotes the smallest subset of *Fnc* which contains F and is closed under the operators belonging to X, and analogously for Fnc_A.

A simple, but useful, result in ASSER [2] establishes the following relations:

(1) *For every $F \subseteq Fnc$ and for every set X of operators in Fnc,* $s([F; X]) = [s(F); s(X)]$.

(2) *For every $G \subseteq Fnc_A$ and for every set Y of operators in Fnc_A,* $n([G; Y]) = [n(G); n(Y)]$.

2. Main results

The primitive recursive string-functions were introduced by ASSER [1] and studied by various authors (see EILENBERG and ELGOT [6], BRAINERD and LANDWEBER [4], CALUDE [5]). A famous result of R. M. ROBINSON [9] gives the following characterisation of the class $Prim^1$ of unary primitive recursive number-theoretical functions:

$$Prim^1 = [\{Succ, E\}; \{sub, it_0, add\}].$$

In ASSER [2] the following characterizations of the class $Prim_A^1$ of unary primitive recursive string-functions are obtained:

(3) $$Prim_A^1 = [\{\sigma, s(E)\}; \{sub_A, \sigma\text{-}it_{A,e}, s(add)\}],$$

(4) $$Prim_A^1 = [\{Succ_1^A, \ldots, Succ_r^A, s(E)\}; \{sub_A, it_{A,e}, con_A\}].$$

These characterizations use the somewhat artificial string-function $s(E)$ and the operator $s(add)$ in Fnc_A is also rather artificial.

In GEORGIEVA [7] (see also CALUDE [5]) one finds the following result:

$$Prim^1 = [\{Succ\}; \{sub, diff\} \cup \{it_x \mid x \in \mathsf{N}\}].$$

This formula can be simplified as follows:

(5) $$Prim^1 = [\{Succ\}; \{sub, diff, it_0\}].$$

All that remains to prove (5) is the inclusion

$$Prim^1 \subseteq P = [\{Succ\}; \{sub, diff, it_0\}],$$

i.e. the closure of P under the operators it_x for $x \in \mathsf{N}_+$. First we note that P contains the functions $sg = it_0(C_1)$, $\overline{sg} = diff(C_1, sg)$ and that P is closed under sum and product. Now let f in P and $h = it_x(f)$, $x \in \mathsf{N}_+$. If, for every natural $k > 0$, $f^k(x) = f(f(\ldots(f(x))\ldots)) \neq 0$ (k times), then $h = sub(h^*, Succ)$, where $h^* = it_0(g)$ and $g(y) = x \cdot \overline{sg}(y) + f(y) \cdot sg(y)$ ($\cdot$ denotes the product). In case there exists a natural $k > 0$ such that $f^k(x) = 0$, say the minimal one, then

$$h(y) = h^*(Succ(y)) \cdot \overline{sg}(t(x)) + h_*(t(x)) \cdot sg(t(x)),$$

where $t = diff(I, C_k)$, $h_* = it_0(f)$.

In view of (5) and (1), as a slight improved form of (3) we obtain:

(6) $$Prim^1_{\mathsf{A}} = [\{\sigma\}; \{sub_{\mathsf{A}}, \sigma\text{-}it_{\mathsf{A},e}, s(diff)\}].$$

The string-function $s(E)$ is dropped, but the unpleasant operator $s(diff)$ in Fnc_{A} is still present. To overcome this difficulty we shall present our first result:

(7) $$Prim^1_{\mathsf{A}} = [\{Succ^{\mathsf{A}}_1, \ldots, Succ^{\mathsf{A}}_r, \pi\}; \{sub_{\mathsf{A}}, it_{\mathsf{A},e}, con_{\mathsf{A}}\}].$$

For this, we denote by F the right-hand side of (7). In order to prove (7) we will show that (i) $\sigma \in F$, (ii) F is closed under $\sigma\text{-}it_{\mathsf{A},e}$, (iii) F is closed under $s(diff)$.

As in the proof of Proposition 2 in ASSER [2] we begin with displaying a list of string-functions belonging to F:

a) $C^{\mathsf{A}}_e(w) = \mathrm{e}$: $C^{\mathsf{A}}_e = it_{\mathsf{A},e}(\pi, \ldots, \pi)$.

b) $\varkappa_i(w) = a_i \quad (1 \leqq i \leqq r)$: $\varkappa_i = sub_{\mathsf{A}}(Succ^{\mathsf{A}}_i, C^{\mathsf{A}}_e)$.

c) $I^{\mathsf{A}}(w) = w$: $I^{\mathsf{A}} = it_{\mathsf{A},e}(Succ^{\mathsf{A}}_1, \ldots, Succ^{\mathsf{A}}_r)$.

d) $sg^{\mathsf{A}}_i(e) = e, \quad sg^{\mathsf{A}}_i(w) = a_i$, for $w \neq e \quad (1 \leqq i \leqq r)$: $sg^{\mathsf{A}}_i = it_{\mathsf{A},e}(\varkappa_i, \ldots, \varkappa_i)$.

e) $succ^{\mathsf{A}}_i(w) = a_i w \quad (1 \leqq i \leqq r)$: $succ^{\mathsf{A}}_i = con_{\mathsf{A}}(\varkappa_i, I^{\mathsf{A}})$.

f) $mir(e) = e$, $mir(wu) = mir(u)\, mir(w)$: $mir = it_{\mathsf{A},e}(succ^{\mathsf{A}}_1, \ldots, succ^{\mathsf{A}}_r)$.

g) $\lambda_i(w) = a_i^{\mathrm{length}(w)} \quad (1 \leqq i \leqq r)$: $\lambda_i = it_{\mathsf{A},e}(Succ^{\mathsf{A}}_i, \ldots, Succ^{\mathsf{A}}_i)$.

h) $\gamma_i(w) = a_i^{c(w)} \quad (1 \leqq i \leqq r)$:

$$\gamma_i = it_{\mathsf{A},e}(con_{\mathsf{A}}(I^{\mathsf{A}} \ldots I^{\mathsf{A}}, \varkappa_i), con_{\mathsf{A}}(I^{\mathsf{A}} \ldots I^{\mathsf{A}}, \varkappa_i \varkappa_i), \ldots, con_{\mathsf{A}}(I^{\mathsf{A}} \ldots I^{\mathsf{A}}, \varkappa_i \ldots \varkappa_i))$$

(the k-th place of the operator $it_{\mathsf{A},e}$ is $con_{\mathsf{A}}(I^{\mathsf{A}} \ldots I^{\mathsf{A}}, \varkappa_i \ldots \varkappa_i)$ with r concatenations of I^{A} and k concatenations of $\varkappa_i$; $1 \leqq k \leqq r$).

i) $\alpha_i(w) = u$ iff $w = u a_i a_i \ldots a_i$ and u does not terminate with a_i $(1 \leqq i \leqq r)$:

$$\alpha_i = sub_{\mathsf{A}}\big(mir, sub_{\mathsf{A}}\big(it_{\mathsf{A},e}\big(Succ^{\mathsf{A}}_1, \ldots, Succ^{\mathsf{A}}_{i-1}, con_{\mathsf{A}}(I^{\mathsf{A}}, sg^{\mathsf{A}}_i), Succ^{\mathsf{A}}_{i+1}, \ldots, Succ^{\mathsf{A}}_r), mir\big)\big)\big).$$

j) $\beta_i(w) = a_i a_i \ldots a_i$ iff $w = \alpha(w)\, a_i a_i \ldots a_i \quad (1 \leqq i \leqq r)$: $\beta_i = it_{\mathsf{A},e}(C^{\mathsf{A}}_e, \ldots, Succ^{\mathsf{A}}_i, \ldots, C^{\mathsf{A}}_e)$.

k) $\overline{sg}^{\mathsf{A}}_i(e) = a_i, \quad \overline{sg}^{\mathsf{A}}_i(w) = e$, for $w \neq e \quad (1 \leqq i \leqq r)$:

$$\overline{sg}^{\mathsf{A}}_i = sub_{\mathsf{A}}\big(sg^{\mathsf{A}}_i, sub_{\mathsf{A}}(\beta_r, con_{\mathsf{A}}(succ^{\mathsf{A}}_r, sg^{\mathsf{A}}_1))\big).$$

l) $\psi_i(e) = a_{i+1}$, $\psi_i(w) = a_i$, for $w \neq e \quad (1 \leqq i < r)$: $\psi_i = con_{\mathsf{A}}(\overline{sg}^{\mathsf{A}}_{i+1}, sg^{\mathsf{A}}_i)$;

$\psi_r(e) = a_1, \quad \psi_r(w) = a_r$, for $w \neq e$: $\psi_r = con_{\mathsf{A}}(\overline{sg}^{\mathsf{A}}_1, sg^{\mathsf{A}}_r)$.

m) $\chi(e) = e, \quad \chi(u a_i) = u a_{i+1}$, for $1 \leqq i < r, \quad \chi(u a_r) = u a_1$:

$$\chi = sub_{\mathsf{A}}(mir, sub_{\mathsf{A}}(it_{\mathsf{A},e}(con_{\mathsf{A}}(I^{\mathsf{A}}, \psi_1), \ldots, con_{\mathsf{A}}(I^{\mathsf{A}}, \psi_r)), mir)).$$

We are now in a position to prove (i): $\sigma \in F$. Indeed,

$$\sigma = con_{\mathsf{A}}\big(sub_{\mathsf{A}}(\overline{sg}^{\mathsf{A}}_1, \alpha_r), con_{\mathsf{A}}(sub_{\mathsf{A}}(\chi, \alpha_r), sub_{\mathsf{A}}(\varkappa_1, \beta_r))\big).$$

Passing to (ii) we note that

(8) $\sigma\text{-}it_{A,e}(f) = sub_A(it_{A,e}(f,\dots,f),\gamma_1)$,

i.e. F is closed under the operator $\sigma\text{-}it_{A,e}$.

To finish the proof we recall that, for every $f \in Fnc_A$ and $n \in N_+$, $f^0 = I^A$ and $f^n = sub_A(f, sub_A(f,\dots, sub_A(f,f)\dots))$, n times. Using a double lexicographical induction one proves the equality

$$\pi^{c(w)}(u) = \bar{c}(diff(c(u),c(w))) \text{ for } u,w \in A^*,$$

which enables us to write the formula

(9) $s(diff)(f,g) = sub_A(it_{A,e}(\sigma,\pi,I^A,\dots,I^A), con_A(sub_A(\gamma_1,f), sub_A(\gamma_2,g)))$,

for all f, $g \in Fnc_A$, thus proving (iii). This ends the proof of (7).

Our second Robinson algebra is the following:

(10) $Prim^1_A = [\{\sigma,\pi\}; \{sub_A, it_{A,e}, con_A\}]$.

In view of (6) we must prove that the right-hand side of (10) is closed under $\sigma\text{-}it_{A,e}$ and $s(diff)$.

Again we proceed with displaying a sequence of primitive recursive string-functions belonging to the right-hand side of (10):

a) $I^A = sub_A(\pi,\sigma)$.

b) $C^A_e = it_{A,e}(\pi,\dots,\pi)$.

c) $\varkappa_i = sub_A(\sigma, sub_A(\sigma,\dots, sub_A(\sigma, C^A_e)))$,

where the operator sub_A appears i times $(1 \leqq i \leqq r)$.

d) $\gamma_i = it_{A,e}(con_A(M_r(I^A), M_1(\varkappa_i))\ con_A(M_r(I^A), M_2(\varkappa_i)),\dots, con_A(M_r(I^A), M_r(\varkappa_i)))$,

with $M_j(f) = con_A(f, con_A(f,\dots, con_A(f,f))\dots)$, where f is any string-function and the operator con_A appears $j \geqq 1$ times.

The proof of (10) is complete in view of (8) and (9).

Finally, we conjecture the validity of the following formula:

(11) $Prim^1_A = [\{\sigma,\pi\}; \{sub_A, \sigma\text{-}it_{A,e}, con_A\}]$.

In view of (2) and $n(Succ^A_i)(x) = Succ^i(x)$, $1 \leqq i \leqq r$, (11) holds iff its right-hand side is closed under the operator $it_{A,e}$.

3. Final remarks

After finishing this paper we have learnt the following new characterizations of $Prim^1_A$ due to G. Asser [3]:

$$Prim^1_A = [\{Succ^A_1,\dots,Succ^A_r,\lambda\}; \{sub_A, it_{A,e}, con_A\}]$$
$$= [\{Succ^A_1,\dots,Succ^A_r,\varrho\}; \{sub_A, it_{A,e}, con_A\}],$$

where λ, ϱ are the component functions of the pairing function

$$\gamma(u, v) = a_1^{\text{length}(u)} a_2 u v a_2 a_1^{\text{length}(v)},$$

i.e., if $w = \gamma(u, v)$ for some strings u, v, then $\lambda(w) = u$ and $\varrho(w) = v$, else $\lambda(w) = \varrho(w) = e$.

Furthermore G. ASSER (communication of July 13, 1989) has perceived that in (7) the function π can be replaced by the function δ, i.e.

$$(12) \qquad Prim_{\mathsf{A}}^1 = \left[\left\{Succ_1^{\mathsf{A}}, \ldots, Succ_r^{\mathsf{A}}, \delta\right\}, \left\{sub_{\mathsf{A}}, it_{\mathsf{A},e}, con_{\mathsf{A}}\right\}\right].$$

The proof is essentially the same as for (7). Only a) must be replaced by

$$C_e^{\mathsf{A}} = it_{\mathsf{A},e}(\delta, \ldots, \delta),$$

and (9) must be replaced by

$$s(diff)(f, g)$$
$$= sub_{\mathsf{A}}\big(it_{\mathsf{A},e}(\sigma, \ldots, \sigma), sub_{\mathsf{A}}\big(it_{\mathsf{A},e}(Succ_1^{\mathsf{A}}, \delta, \ldots, \delta), \big(con_{\mathsf{A}}(sub_{\mathsf{A}}(\gamma_1, f), sub_{\mathsf{A}}(\gamma_2, g))\big)\big)\big).$$

References

[1] ASSER, G., Rekursive Wortfunktionen. This Zeitschrift **6** (1960), 258–278.
[2] ASSER, G., Primitive recursive word-functions of one variable. In: Computation Theory and Logic (E. BÖRGER, ed.), Springer Lecture Notes in Comput. Sci. **270** (1987), pp. 14–19.
[3] ASSER, G., Zur Robinson-Charakterisierung der einstelligen primitiv rekursiven Wortfunktionen. This Zeitschrift **34** (1988), 317–322.
[4] BRAINERD, W. S., and L. H. LANDWEBER, Theory of Computation. John Wiley and Sons, New York 1974.
[5] CALUDE, C., Theories of Computational Complexity. Annals of Discrete Mathematics 35, North-Holland Publ. Comp., Amsterdam–New York–Oxford–Tokyo 1988.
[6] EILENBERG, S., and C. C. ELGOT, Recursiveness. Academic Press, New York–London 1970.
[7] GEORGIEVA, N., Another simplification of the recursion schema. Arch. Math. Logik Grundlagenforsch. **18** (1976), 1–3.
[8] PÉTER, R., Rekursive Funktionen. Akadémiai Kiadó, Budapest 1951.
[9] ROBINSON, R. M., Primitive recursive functions. Bull. Amer. Math. Soc. **53** (1947), 925–943.
[10] ROSE, H. E., Subrecursion. Functions and Hierarchies. Clarendon Press, Oxford 1984.

(Eingegangen am 14. November 1988)

C. Calude
Department of Mathematics
University of Bucharest
14 Academiei Str.
Bucharest R-70109
Romania

L. Sântean
Department of Knowledge Processing Systems
Institute for Computers and Informatics
8–10 Miciurin Blvd.
Bucharest R-71316
Romania