The Collatz Problem,* the Halting Problem and Randomness

Cristian S. Calude

April 3, 2014

When he was a student L. Collatz posed the following problem: given any integer $a_1$ there exists a natural $N$ such that $a_N = 1$, where

$$a_n = \begin{cases} a_n/2, & \text{if } a_n \text{ is even,} \\ 3a_n + 1, & \text{otherwise.} \end{cases}$$

There is a huge literature on this problem and various natural generalisations: see [?, ?, ?, ?]. P. Erdös has said (cf. [?]) that

*Mathematics may not be ready for such problems.*

A problem/conjecture is finitely refutable if verifying a finite number of instances suffices to disprove it. A systematic enumeration (of the problem’s search domain) will find a counter-example if one exists. If the search stops, the conjecture is false; if the search does not halt the conjecture is true. For a finitely refutable problem $\Pi$ we can construct a program $C_\Pi$ such that

$$\Pi \text{ is false iff } C_\Pi \text{ halts.}$$

For example, $\Pi = \text{the Riemann hypothesis}$ is finitely refutable and one can construct a program $C_\Pi$ of 7,780 bits, cf. [?].

The twin prime conjecture

$$\forall n \{ \exists p [p > n \& p \text{ prime } \& p + 2 \text{ prime}] \}$$

is not finitely refutable but can still be solved by testing the halting status of a small program. The stronger twin prime conjecture is finitely refutable:

$$\forall n \{ \exists p [n < p < 2^{n+4} \& p \text{ prime } \& p + 2 \text{ prime}] \}.$$ 

In [?] it was proved, in a non-constructive manner, that Collatz problem is finitely refutable. The non-constructive proof shows that there exists a program $C_{\text{Collatz}}$ such that

---

*Also called Syracuse conjecture, the $3x + 1$ problem, Kakutanis problem, Hasse algorithm, and Ulams problem. See [?].*
C_{Collatz} is false iff C_{Collatz} halts.

**Question 1.** Can you find/write such a program?

The initial iterates of Collatz map \( f(n) = a_n \) exhibit a ‘random’ character, cf [?]. More precisely, the initial iterates of a randomly selected integer appear to be even or odd with equal probability. Such a result can be rigorously justified if one takes the interval \( 1 \leq n \leq 2^k \) and considers only the first \( k \) iterations (see [?] Theorem A).

**References**


[3] J. P. Davalan. [3x + 1, Collatz, Syracuse problem](#)


[9] Collatz Conjecture