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Remarks on the Halting problem, unitarity and reversibility in quantum theory of computations

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Abstract

The Halting problem for the quantum computer is considered. It is shown, that if the halting for the quantum computer takes place then the corresponding dynamics is described by an irreversible operator.

1 Introduction

In the last 10 - 15 years the quantum theory of information developed rather intensively in various directions from mathematical aspects to different physical problems: quantum algorithms [1] , quantum decoherence [2], density matrix and entropy for entanglement states [3], measure theory for the quantum information and a number of physical models of quantum computers based on various principles [4].

However, there are problems to be solved. One of such problems is the halting problem arose in the middle 80-th. It can be generally formulated as follows: how can be a correct description of the quantum computer halting compatible with the basic principles of quantum theory of information.

Recently this problem caused a discussion [6]–[9], [12], [13]. In the present work we research this problem. It is shown that the halting of quantum computers is incompatible not only with unitarity but even with reversibility of the corresponding dynamics.

2 The Halting problem

In paper [5] where the term 'halting' is firstly used the following special qubit is chosen

$$\hat{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for a signal that the computer is halted. It is equivalent to that each correctly working program sets $\hat{q}$ to 1, when it terminates and $\hat{q}$ sets to 0 otherwise. Myers [6] noticed, that as the program for different branches of computing process can have different number of steps, thus the problem of unitarity of the basic calculation operators arises. This problem was in principal solved in [7]. However it is shown in [8] that in [7] a kind of Turing machine is used which is inapplicable to realistic computers, as in this case dynamics is unitary only for computers, which do not halt, thus for realistic computers the problem remains unsolved. Besides, in [8] one more problem is discovered which arises in a quantum computer when different branches of computation process halt at different and unknown times. After that it is shown in [9], that halting of the universal quantum computer is incompatible with the unitarity constraint of quantum computations.

3 The Halting problem, unitarity and reversibility

Below we shall use the terminology of [9] according to that the following definitions are valid:

1. Quantum computer is a closed quantum system controlled by a time - independent evolution operator U for each time step between a state of the input space being some vector $\mid \tau_{in}\rangle$ in a Hilbert space $\hat{H}$ and a final state $\mid \tau_{out}\rangle$ of the output in the same Hilbert space.
2. For halting, the dynamics is to be able to store the output, which is finite in terms of qubit resources independently on in what finite time the desirable output is computed. This reserved space, from which the output can be read out is mathematically an invariant subspace $V \subset \hat{H}$ which has a component of the qubit $\hat{q}$ equal to 1.

We intentionally weaken the requirements showed to dynamics and do not consider U it obviously unitary.

Thus, any state has the form

$$|\psi_0\rangle = |0_h\rangle \otimes |x_0\rangle + |1_h\rangle \otimes |y_0\rangle \tag{1}$$

The transfer information matrix U writes

$$U = \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix} \tag{2}$$

in the basis

$$\mid 0_h\rangle \otimes \mid x_0\rangle = \begin{pmatrix} 0 \\ \mid x_0\rangle \end{pmatrix}, \mid 1_h\rangle \otimes \mid y_0\rangle = \begin{pmatrix} \mid y_0\rangle \\ 0 \end{pmatrix} \tag{3}$$

Let's show that the halting conditions of the quantum computer after performance of the program, which has finite number of steps

$$\text{For } N \geq N_0 \quad U^N \mid \psi_0\rangle = \mid 1_h\rangle \otimes \mid y_0\rangle \tag{4}$$

$$\langle 0_h \mid U^N \mid \psi_0\rangle = B^N \mid x_0\rangle = 0 \tag{5}$$

are incompatible with the reversibility of the operator U.

Actually, let U be a two-side reversible matrix and let

$$^{-1}U = \begin{pmatrix} A_{11}^{(l)} & A_{12}^{(l)} \\ A_{21}^{(l)} & A_{22}^{(l)} \end{pmatrix}$$

be the left-hand reciprocal to U and

$$U^{-1} = \begin{pmatrix} A_{11}^{(r)} & A_{12}^{(r)} \\ A_{21}^{(r)} & A_{22}^{(r)} \end{pmatrix}$$

be the right-hand reciprocal to U. Then

$$^{-1}UU = \begin{pmatrix} A_{11}^{(l)} & A_{12}^{(l)} \\ A_{21}^{(l)} & A_{22}^{(l)} \end{pmatrix} \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix} = \begin{pmatrix} A_{11}^{(l)}A & A_{11}^{(l)}\alpha + A_{12}^{(l)}B \\ A_{21}^{(l)}A & A_{21}^{(l)}\alpha + A_{22}^{(l)}B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

Thus it follows that the matrix A has the left-hand reciprocal $A^{-1} = A_{11}^{(l)}$. Similarly, we have

$$UU^{-1} = \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix} \begin{pmatrix} A_{11}^{(r)} & A_{12}^{(r)} \\ A_{21}^{(r)} & A_{22}^{(r)} \end{pmatrix} = \begin{pmatrix} AA_{11}^{(r)} + \alpha A_{21}^{(r)} & AA_{12}^{(r)} + \alpha A_{22}^{(r)} \\ BA_{21}^{(r)} & BA_{22}^{(r)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

Hence, matrix B is right-hand reversible. As B is right-hand reversible B^N is such as well.

Using results of examples 8 and 10 of chapter 2 from [10] we obtain immediately that $(B^N)^+$ is also right-reversible. Condition (5) is obviously equivalent to condition

$$\langle x_0 \mid (B^N)^+ = 0 \tag{8}$$

Multiplying the left and right parts of last equality on the right on $(B^{N+})^{-1}$ we obtain for any bra - vector $\langle x_0 \mid = 0$. This is an obvious contradiction. Thus it follows that U is not reversible. Some more strong statement follows from our proof: U is not even right-hand reversible. It is necessary to make two remarks:

1. The above proof is correct both for the universal quantum computer, that is the case here, when the Hilbert space $\hat{H}$ is infinite dimensional and in the case of the realistic quantum computer i. e. when $\hat{H}$ has a finite dimension. Proof is simplified in this case, due to the fact that for square matrices a left-hand reciprocal coincides with the right-hand one and that for the upper triangular reversible matrices the Jordan decomposition takes place [11].

$$U = \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}\alpha \\ 0 & 1 \end{pmatrix}$$

   Then the key argument will follow directly from condition (5).

2. It would be natural to require, that the left-hand reciprocal ${}^{-1}U$ and the right-hand one U^{-1} of U are also elements of the dynamics of the quantum computer and also had must have the upper triangular form that can simplify the proof even more.

4 Conclusion

Thus, it is shown in this paper that in the general case if halting of the universal or realistic quantum computer takes place corresponding dynamics is not only non-unitary but either irreversible.

References

[1] A.Ekert, R.Jorza, Rev.Mod.Phys.v68(1996)p.733.

[2] A.Paz, W.H.Zurek, Phys.Rev.Lett. v82(1999)p.5181.

[3] S.Popescu, D.Rohrlich, Phys.Rev. vA56(1997)p.R3319.

[4] A.Jones, E.Knill, Journ.Magn.Res. v141(1999)p.322.

[5] D.Deutsch, Proc. R. Soc. Lond. vA400(1985)p.97.

[6] J.M. Myers, Phys. Rev. Lett. v78(1997)p.1823.

[7] M. Ozawa, Phys. Rev. Lett. 80(1998)p.631.

[8] N. Linden, S. Popescu, E-print arXiv quant-ph/9806054,4pp.

[9] T.D. Kieu, M. Danos, E-print arXiv quant-ph/9811001,5pp.

[10] R.G. Cooke, Infinite matrices and sequence spaces, pp 36-7, MacMillan, London, 1950.

[11] A. Borel, Linear algebraic groups, New York - Amsterdam,1969.

[12] M. Ozawa, E-print arXiv quant-ph/9809038,9pp.

[13] Yu. Shi, Eprint arXiv quant-ph/9805083,14pp;
Yu. Shi, Eprint arXiv quant-ph/9805083,8pp.