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A topological characterization of random sequences

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Abstract

The set of random sequences is large in the sense of measure, but small in the sense of category. This is the case when we regard the set of infinite sequences over a finite alphabet as a subset of the usual Cantor space. In this note we will show that the above result depends on the topology chosen. To this end we will use a relativization of the Cantor topology, the U^δ-topology introduced by Staiger [RAIRO Inform. Théor. 21 (1987) 147–173]. This topology is also metric, but the distance between two sequences does not depend on their longest common prefix (Cantor metric), but on the number of their common prefixes in a given language U. The resulting space is complete, but not always compact. We will show how to derive a computable set $\boldsymbol{U}$ from a universal Martin-Löf test such that the set of non-random sequences is nowhere dense in the $\boldsymbol{U}^\delta$-topology. As a byproduct we obtain a topological characterization of the set of random sequences. We also show that the Law of Large Numbers, which fails with respect to the usual topology, is true for the $\boldsymbol{U}^\delta$-topology.
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1. Introduction

Algorithmic information theory plays many central roles in theoretical computer science, and, in particular, in the theory of computation, both in terms of intellectual inspiration and connectivity as well as tool (see, for example, [4,5,2]). The aim of this note is to study from a topological point of view the set of random sequences. This problem is interesting in itself (because the set of random sequences has constructive Lebesgue measure one, but it is constructively meagre with respect to Cantor's topology) and has connections with probability theory (classically, the Law of Large Numbers fails to hold topologically). Is there any natural topology with respect to which the set of random sequences is topologically "large"? We will prove that a relativization of the Cantor topology gives a posi-

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tive answer to the above question (the set of random sequences is co-nowhere dense) and leads to a topological analogue of Martin-Löf's measure-theoretical characterization of random sequences (the role of constructive null sets is played by nowhere dense sets). Finally, the Law of Large Numbers is topologically true in this space.

2. Notation

By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. The cardinality of the set A is denoted by $\mathrm{card}(A)$. Let us fix X an alphabet of cardinality $\mathrm{card}(X) = r \geqslant 2$, e.g., $X = \{0, \ldots, r-1\}$. By X^* we denote the set of finite strings (words) on X, including the *empty* string e. The length of the string w is denoted by $|w|$. We consider the space X^ω of infinite sequences (ω-words) over X. If $\boldsymbol{x} = x_1x_2\cdots x_n\cdots \in X^\omega$, then $\boldsymbol{x}(n) = x_1x_2\cdots x_n$ is the prefix of length n of $\boldsymbol{x}$. Strings and sequences will be denoted respectively by $u, v, w, \ldots$ and $\boldsymbol{x}, \boldsymbol{y}, \ldots$. For $w, v \in X^*$ and $\boldsymbol{x} \in X^\omega$ let $w \cdot v, w \cdot \boldsymbol{x}$ (simply $wv, w\boldsymbol{x}$) be the concatenation of w and $v, \boldsymbol{x}$, respectively. The concatenation product extends naturally to subsets $W \subseteq X^*$ (languages) and $B \subseteq X^* \cup X^\omega$. By "$\sqsubseteq$" we denote the prefix relation between strings: $w \sqsubseteq v$ if there is a v' such that $wv' = v$. The relation "$\sqsubset$" is similarly defined for $w \in X^*$ and $\boldsymbol{x} \in X^\omega$: $w \sqsubset \boldsymbol{x}$ if there is a sequence $\boldsymbol{x}'$ such that $w\boldsymbol{x}' = \boldsymbol{x}$. The sets $\mathrm{pref}(\boldsymbol{x}) = \{w\colon w \in X^*,\ w \sqsubset \boldsymbol{x}\}$ and $\mathrm{pref}(B) = \bigcup_{\boldsymbol{x} \in B} \mathrm{pref}(\boldsymbol{x})$ are the languages of prefixes of $\boldsymbol{x} \in X^\omega$ and $B \subseteq X^\omega$, respectively. Finally, $wX^\omega = \{\boldsymbol{x} \in X^\omega\colon w \in \mathrm{pref}(\boldsymbol{x})\}$.

The unbiased discrete measure on X is the probabilistic measure $h(A) = \mathrm{card}(A)/r$, for every subset A of X. It induces the product measure μ defined on all Borel subsets of X^ω. This measure coincides with the Lebesgue measure on the unit interval, it is computable and $\mu(wX^\omega) = r^{-|w|}$, for every $w \in X^*$. For more details see [9,10,2].

3. The Cantor space

The set X^ω is a compact metric space (Cantor space) with the metric

$$\rho_1(\boldsymbol{x}, \boldsymbol{y}) = \inf\left\{\frac{1}{1+|w|}\colon w \in \mathrm{pref}(\boldsymbol{x}) \cap \mathrm{pref}(\boldsymbol{y})\right\}.$$

For our purposes it is convenient to use the following equivalent metric (cf. [18,15]):

$$\begin{aligned}\rho(\boldsymbol{x}, \boldsymbol{y}) &= \inf\{r^{-|w|}\colon w \in \mathrm{pref}(\boldsymbol{x}) \cap \mathrm{pref}(\boldsymbol{y})\} \\ &= r^{1-\mathrm{card}(\mathrm{pref}(\boldsymbol{x}) \cap \mathrm{pref}(\boldsymbol{y}))}. \end{aligned} \tag{1}$$

The open ball $\mathbb{B}_\varepsilon(\boldsymbol{y})$ of radius $\varepsilon \in (0, 1]$ and center $\boldsymbol{y}$ in (X^ω, ρ) can be described as

$$\mathbb{B}_\varepsilon(\boldsymbol{y}) = \{\boldsymbol{x}\colon \rho(\boldsymbol{y}, \boldsymbol{x}) < \varepsilon\} = w_{\boldsymbol{y},\varepsilon} \cdot X^\omega,$$

where $w_{\boldsymbol{y},\varepsilon}$ is the unique prefix of $\boldsymbol{y}$ with length $|w_{\boldsymbol{y},\varepsilon}| = \lfloor -\log_r \varepsilon \rfloor + 1$. Thus the open sets in the Cantor space (X^ω, ρ) are sets of the form $WX^\omega = \bigcup_{w \in W} wX^\omega$. The sets wX^ω are both open and closed.

The δ*-limit* of a language $U \subseteq X^*$ is the set U^δ of all sequences in X^ω having infinitely many prefixes in U, $U^\delta = \{\boldsymbol{y} \in X^\omega\colon \mathrm{pref}(\boldsymbol{y}) \cap U \text{ is infinite}\}$. This notion is useful in obtaining the following characterization of G_δ-sets, i.e., countable intersections of open sets (cf. [18,14,15]):

Theorem 1. *In the Cantor space, a subset $F \subseteq X^\omega$ is a G_δ-set iff there is a language $U \subseteq X^*$ such that $F = U^\delta$.*

4. The U^δ-topology

A new metric topology on X^ω has been introduced in [14] in connection with the study of sequential mappings. In this section we define this topology and relate it to the usual topology in the Cantor space.

Definition 2. Fix a language $U \subseteq X^*$ and let $\boldsymbol{x}, \boldsymbol{y} \in X^\omega$. Then we define

$$\rho_U(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 0, & \text{if } \boldsymbol{x} = \boldsymbol{y}, \\ r^{1-\mathrm{card}(\mathrm{pref}(\boldsymbol{x}) \cap \mathrm{pref}(\boldsymbol{y}) \cap U)}, & \text{otherwise.} \end{cases}$$

It is easy to see that ρ_U is a metric; its induced topology on X^ω will be called the U^δ*-topology*.

The metric ρ_U resembles, in some sense, the metric ρ in the Cantor space; in fact, $\rho = \rho_{X^*}$. In contrast with ρ, ρ_U counts only those common prefixes of $\boldsymbol{x}$ and $\boldsymbol{y}$ contained in U. Further on, since $\rho_U(\boldsymbol{x}, \boldsymbol{y}) \geqslant \rho(\boldsymbol{x}, \boldsymbol{y})$, the U^δ-topology refines the topology of the Cantor space. In particular, every closed (open) set in the Cantor space is also closed (open) in the U^δ-topology of X^ω.

The following result shows when two languages U, V induce the same topology on X^ω; hence, a great variety of languages induce the same topology (see [14,15] for examples).

Theorem 3 [17]. *The U^δ-topology and the V^δ-topology of X^ω coincide iff $U^\delta = V^\delta$.*

The open ball in (X^ω, ρ_U) is given by the formula:

$$\mathbb{B}_{\varepsilon,U}(\boldsymbol{x}) = \begin{cases} \{\boldsymbol{x}\}, & \text{if } \rho_U(\boldsymbol{x},\boldsymbol{y}) \geqslant \varepsilon, \\ & \text{for all } \boldsymbol{y} \neq \boldsymbol{x}, \\ X^\omega, & \text{if } \varepsilon > r, \\ w_{\boldsymbol{x},\varepsilon} \cdot X^\omega, & \text{otherwise.} \end{cases}$$

Here $w_{\boldsymbol{x},\varepsilon}$ is the unique prefix of $\boldsymbol{x}$ in U with

$$\operatorname{card}\big(\operatorname{pref}(w_{\boldsymbol{x},\varepsilon}) \cap U\big) = \lfloor -\log_r \varepsilon \rfloor + 2.$$

The following topological properties of (X^ω, ρ_U) will be useful. Recall that a point $\boldsymbol{x}$ is called an *accumulation point of a set F in the metric space (X^ω, d)* provided for each $\varepsilon > 0$ there exists a sequence $\boldsymbol{y} \in F$, $\boldsymbol{x} \neq \boldsymbol{y}$ such that $d(\boldsymbol{x}, \boldsymbol{y}) < \varepsilon$. Invoking Definition 2 we obtain:

Corollary 4. *A point $\boldsymbol{x} \in X^\omega$ is an accumulation point of the whole space (X^ω, ρ_U) iff $\boldsymbol{x} \in U^\delta$.*

As (X^ω, ρ_U) is a metric space, the smallest closed (with respect to ρ_U) subset of X^ω containing F, $\mathcal{C}_U(F)$, is given by the formula

$$\mathcal{C}_U(F) = F \cup \big\{\boldsymbol{x}\colon \boldsymbol{x} \in X^\omega, \boldsymbol{x} \text{ is an accumulation point of } F \text{ in } (X^\omega, \rho_U)\big\}. \tag{2}$$

A point $\boldsymbol{x} \in F$ which is not an accumulation point of F is called an *isolated point* of F. Thus, $\boldsymbol{x}$ is an isolated point of X^ω iff there is an $\varepsilon > 0$ such that $\mathbb{B}_{\varepsilon,U}(\boldsymbol{x}) = \{\boldsymbol{x}\}$. The *set of isolated points* of (X^ω, ρ_U) will be denoted by $\mathbb{I}_U = X^\omega \setminus U^\delta$.

An arbitrary set of isolated points of X^ω is open. In case $U^\delta = \emptyset$, in particular if U is finite, every point of (X^ω, ρ_U) is isolated. Thus, in general, (X^ω, ρ_U) is a complete metric space, not necessarily compact (as the Cantor space). More precisely, the space (X^ω, ρ_U) is not compact whenever $\mathbb{I}_U \neq \emptyset$, cf. [17], Theorem 9.

The close relationship between the U^δ-topology and the topology of the Cantor space is visible in the case of accumulation points and closed sets.

Theorem 5 [16,17]. *Let $U \subseteq X^*$. Then $\boldsymbol{x} \in U^\delta$ is an accumulation point of F in (X^ω, ρ_U) iff $\boldsymbol{x}$ is an accumulation point of F in (X^ω, ρ).*

From (2) we obtain:

Corollary 6. *Let $\mathcal{C}(F) = \mathcal{C}_{X^*}(F)$ be the smallest closed set containing F in the Cantor space. Then*

$$\mathcal{C}_U(F) = F \cup \big(\mathcal{C}(F) \cap U^\delta\big) = \mathcal{C}(F) \cap (F \cup U^\delta).$$

In particular, every set F containing U^δ is closed in (X^ω, ρ_U).

As it was mentioned above, every set $J \subseteq \mathbb{I}_U$ of isolated points is an open set in (X^ω, ρ_U), and every set of the form WX^ω is open in the Cantor space. Consequently, Corollary 6 yields

Corollary 7. *A set $E \subseteq X^\omega$ is open in (X^ω, ρ_U) iff $E = WX^\omega \cup J$, for some $W \subseteq X^*$ and $J \subseteq \mathbb{I}_U$.*

Recall that a set F is *nowhere dense* in (X^ω, ρ_U) if its closure, $\mathcal{C}_U(F)$, does not contain any non-empty open set, that is, if $\mathcal{C}_U(X^\omega \setminus \mathcal{C}_U(F)) = X^\omega$; F is *dense* if it intersects any non-empty open set, that is, if $\mathcal{C}_U(F) = X^\omega$.

The next result is simple but very useful:

Lemma 8. *The set U^δ is the union of all nowhere dense sets in (X^ω, ρ_U).*

Proof. We take a nowhere dense set $F \subseteq X^\omega$ and we show that $F \subseteq U^\delta$. To this aim we prove that every sequence $\boldsymbol{x} \in F$ is in U^δ: this is true because if $\boldsymbol{x} \notin U^\delta$, then the singleton set $\{\boldsymbol{x}\}$ is non-empty and open, hence it cannot be nowhere dense, a contradiction. □

Of course, U^δ may or may not be itself nowhere dense. The next theorem gives a necessary and sufficient condition for U^δ to be nowhere dense.

Theorem 9. *Let $U \subseteq X^*$. Then the following conditions are equivalent*:

(1) *The set $\mathbb{I}_U$ is dense in the Cantor space (X^ω, ρ).*
(2) *The set U^δ is nowhere dense in (X^ω, ρ_U).*
(3) *The set U^δ is a maximal nowhere dense set.*

Proof. For the implication "(1) ⇒ (2)" we observe that U^δ is closed in (X^ω, ρ). If $U^\delta = X^\omega \setminus \mathbb{I}_U$ is not nowhere dense in (X^ω, ρ_U), then in view of Corollary 7 it contains a non-empty open set of the form $E = WX^\omega \cup J$, $J \subseteq \mathbb{I}_U$. Due to the inclusion $E \subseteq U^\delta$ we have $J = \emptyset$, that is, $E = WX^\omega$. Since $\mathbb{I}_U$ is dense in the Cantor space (X^ω, ρ), we have $\mathbb{I}_U \cap WX^\omega \neq \emptyset$ unless $WX^\omega = \emptyset$, so $E = \emptyset$, a contradiction.

The implication "(2) ⇒ (3)" follows from Lemma 8.

For "(3) ⇒ (1)" we assume that U^δ is nowhere dense in (X^ω, ρ_U), hence $\mathcal{C}_U(\mathbb{I}_U) = \mathcal{C}_U(X^\omega \setminus U^\delta) = X^\omega$. According to Corollary 6 we have $X^\omega = \mathcal{C}_U(\mathbb{I}_U) = \mathcal{C}(\mathbb{I}_U) \cap (\mathbb{I}_U \cup U^\delta)$, hence $\mathcal{C}(\mathbb{I}_U) = X^\omega$. □

5. A U^δ-topology for random sequences

There are various equivalent definitions of random sequences, complexity-theoretic (see [4,5]), measure-theoretic (see [9]), topological; for a proof of their equivalence see [5,2]. In what follows we will use the definition based on Martin-Löf tests.

We briefly recall the necessary facts on Martin-Löf tests; a more thorough treatment can be found in the textbooks [2,13].

A subset $\mathfrak{V} \subseteq X^* \times \mathbb{N}$ is called *Martin-Löf test* provided

(1) $\mathfrak{V}$ is computably enumerable,
(2) $V_{m+1} \subseteq V_m \cdot X^*$, for all $m \geqslant 1$,
(3) $\mathrm{card}(X^n \cap V_m \cdot X^*) < r^{n-m}/(r-1)$, for all $n, m \geqslant 1$, where $V_m = \{v \in X^*\colon (v, m) \in \mathfrak{V}\}$ is the mth section of $\mathfrak{V}$ and $X^n = \{v\colon v \in X^*, |v| = n\}$.

It is seen that $\mu(V_i X^\omega) \leqslant r^{-i}/(r-1)$, for all $i \geqslant 1$, so $\lim_{i\to\infty} \mu(V_i \cdot X^\omega) = 0$, constructively, that is, there exists a computable function H such that $\mu(V_i \cdot X^\omega) < 2^{-m}$, for all $i > H(m)$. Moreover, it is possible to choose $\mathfrak{V}$ in such a way that each V_i is prefix-free, that is, $v, w \in V_i$ and $v \sqsubseteq w$ imply $v = w$ (cf. [13, Corollary 4.10]).

A Martin-Löf test $\mathfrak{U}$ is called *universal* if for every Martin-Löf test $\mathfrak{V}$ there exists a constant $c > 0$ (depending upon $\mathfrak{U}$ and $\mathfrak{V}$) such that $V_{m+c} \subseteq U_m \cdot X^*$, for all $m \geqslant 1$. In [9] Martin-Löf has proved the existence of universal Martin-Löf tests (see also [2]). If $\mathfrak{U}$ is a universal Martin-Löf test, then $\bigcap_{i\in\mathbb{N}} V_i \cdot X^\omega \subseteq \bigcap_{i\in\mathbb{N}} U_i \cdot X^\omega$.

The set of random sequences, **rand**, is defined as $\mathbf{rand} = X^\omega \setminus \bigcap_{i\in\mathbb{N}} U_i \cdot X^\omega$, where $\mathfrak{U}$ is a universal Martin-Löf test. Of course, the definition does not depend upon the choice of $\mathfrak{U}$.

A set $S \subseteq X^\omega$ is *constructive null* if there exists a computably enumerable set $\mathfrak{A} \subseteq X^* \times \mathbb{N}$ such that $S \subseteq \bigcap_{m=1}^\infty A_m \cdot X^\omega$, ($A_m$ is the mth section of $\mathfrak{A}$), and $\lim_{m\to\infty} \mu(A_m \cdot X^\omega) = 0$, constructively.

The following result follows immediately from the existence of the universal Martin-Löf test:

Theorem 10 [9]. *The set $X^\omega \setminus$ **rand** equals the union of all constructive null sets, hence it is a maximal constructive null set.*

From Theorem 10 it follows that $X^\omega \setminus$ **rand** is a constructive null set, so **rand** is large in the sense of measure:

Corollary 11 [9]. *The set* **rand** *has constructive μ measure one.*

However, in the Cantor space, the set **rand** is small in the sense of category [3,2]. A set $S \subseteq X^\omega$ is *constructively meagre* in the Cantor set if there exist a computably enumerable set $\mathfrak{A} \subseteq X^* \times \mathbb{N}$ and a computable function $f: X^* \times \mathbb{N} \to X^*$ such that $S \subseteq \bigcup_{m=1}^\infty X^\omega \setminus A_m \cdot X^\omega$, for all $m \geqslant 1$, and for every $v \neq e$ we have $v \sqsubseteq f(v, m)$ and $f(v, m) \in A_m$.

Theorem 12 [3]. *The set* **rand** *is constructively meagre in the Cantor space.*

Next we will explore similarities between Theorem 9 (see also Lemma 8) and Theorem 10. First, we obtain a topological characterization of random sequences:

Theorem 13. *Let $\mathfrak{U}$ be a universal Martin-Löf test and assume that every section of $\mathfrak{U}$, $U_i = \{u\colon (u, i) \in \mathfrak{U}\}$, is prefix-free. Then*

$$\mathbf{rand} = X^\omega \setminus \Big(\bigcup_{i\in\mathbb{N}} U_i\Big)^\delta. \qquad (3)$$

Proof. If $\boldsymbol{x} \in \mathbf{rand}$, then $\boldsymbol{x} \notin U_i \cdot X^\omega$, for almost all $i \in \mathbb{N}$ (as $U_{m+1} \subseteq U_m \cdot X^*$). Since all U_i are prefix-free, $\mathrm{pref}(\boldsymbol{x}) \cap (\bigcup_{i \in \mathbb{N}} U_i)$ is finite.

Conversely, let $\boldsymbol{x} \notin \mathbf{rand}$, that is, $\boldsymbol{x} \in \bigcap_{i \in \mathbb{N}} U_i \cdot X^\omega$. From the inequality $\mu(U_i \cdot X^\omega) \leqslant r^{-i}/(r-1)$ we deduce that the minimum string length in U_i, $\min\{|u|: u \in U_i\}$, tends to infinity as $i \to \infty$. Thus $\boldsymbol{x}$ has infinitely many prefixes in $\bigcup_{i \in \mathbb{N}} U_i$. □

From the well-known fact that **rand** is dense in the Cantor space (see [2]) and Theorem 9 we obtain:

Corollary 14. *Let $\mathfrak{U}$ be a universal Martin-Löf test and assume that each $U_i = \{u\colon (u,i) \in \mathfrak{U}\}$ is prefix-free. Define $\boldsymbol{U} = \bigcup_{i \in \mathbb{N}} U_i$. Then the set $\boldsymbol{U}$ is computable and $X^\omega \setminus$ **rand** is nowhere dense in the space $(X^\omega, \rho_{\boldsymbol{U}})$.*

Proof. We need to prove only the computability of $\boldsymbol{U}$. To this aim we fix an arbitrary universal Martin-Löf test $\mathfrak{U}$ such that each section U_i is prefix-free. Furthermore, let us fix a computable enumeration of this Martin-Löf test. A decision algorithm for $\boldsymbol{U} = \bigcup_{i \in \mathbb{N}} U_i$ works as follows:

Given a string w, let k be the smallest positive integer such that $\mu(wX^\omega) > r^{-k}/(r-1)$. Then, $w \notin U_i$, for any $i \geqslant k$. Start the computable enumeration of the universal Martin-Löf test $\mathfrak{U}$ and wait until for each $i < k$ some element (v_i, i) of the Martin-Löf test has been enumerated such that $w \sqsubseteq v_i$ or $v_i \sqsubseteq w$. If one of the v_i is equal to w, then the answer affirmative; otherwise, the answer negative.

First we show that the algorithm will stop after finitely many steps. Note that set of non-random elements is dense. Hence, wX^ω contains some non-random sequence $\boldsymbol{z}$. Since the Martin-Löf test is assumed to be universal, the set U_i must contain a prefix of $\boldsymbol{z}$, for every i. Hence, the algorithm will stop after finitely many steps.

Secondly, we prove the correctness of the algorithm. The affirmative answer is certainly correct when it is given. The negative answer is correct when it is given, because in that case w cannot be contained in any U_i, for every $i < k$ since U_i is prefix-free, and we have already seen that $w \notin U_i$, for every $i \geqslant k$. □

It should be noted that the space $(X^\omega, \rho_{\boldsymbol{U}})$ is induced by the computable set $\boldsymbol{U}$ in spite of the fact that the universal Martin-Löf test $\mathfrak{U}$ is not computable.

With reference to the set $\boldsymbol{U}$ in Corollary 14, we recall that in the space $(X^\omega, \rho_{\boldsymbol{U}})$ every random sequence $\boldsymbol{x}$ is an isolated point, whereas Corollary 14 shows that every non-random sequence $\boldsymbol{x}$ can be topologically approximated by random sequences. This situation parallels the measure-theoretical one (see also [6,8,7,2]). It is interesting to note that the union of all null sets is not a null set, but the union of nowhere dense sets in $(X^\omega, \rho_{\boldsymbol{U}})$ is a (maximal) nowhere dense set. So, nowhere dense sets in $(X^\omega, \rho_{\boldsymbol{U}})$ are analogous to constructive null sets. The space $(X^\omega, \rho_{\boldsymbol{U}})$ is *residual* (see [19]) as each nowhere dense set has measure zero.

We close the paper with a short discussion of the Law of Large Numbers. In [12,11] it was proved that the Law of Large Numbers fails to hold true in the sense of category, i.e., the set LLN of binary sequences $\boldsymbol{x}$ such that $\lim_{n\to\infty}(x_1 + x_2 + \cdots + x_n)/n = 1/2$ is meagre with respect to the natural topology of the unit interval; a similar situation occurs with the set of random sequences with respect to the Cantor topology (see Theorems 10 and 12). As every random sequence satisfies the Law of Large Numbers (see [1,2]) we obtain:

Corollary 15. *The complement of the set* LLN *is nowhere dense in $(\{0,1\}^\omega, \rho_{\boldsymbol{U}})$, that is, the Law of Large Numbers holds true in the sense of category in the space $(\{0,1\}^\omega, \rho_{\boldsymbol{U}})$.*

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