Problems & Exercises for Tutorials and Test & Exam Preparation

March 2016

1. **The Halting Problem**
   Answer the following questions with reference to the proof of the undecidability of the halting problem given in class:

   - How did we get the contradiction?
   - Why did we get the contradiction?
   - Did we use "diagonalisation"?
   - Is it really true that Trouble(N) always stops for all N?

2. **Randomness**
   Comment the following statements.

   > In a random selection of numbers, since all numbers eventually appear, those that have not come up yet are 'due', and thus more likely to come up soon.
   >
   > In a random sequence of numbers, a number may be said to be cursed because it has come up less often in the past, and so it is thought that it will occur less often in the future. A number may be assumed to be blessed because it has occurred more often than others in the past, and so it is thought likely to come up more often in the future.
   >
   > Events of very small probability never appear.

3. **Randomness**
   Enumerate and discuss various "symptoms" of randomness.

4. **Coding** Give a prefix-free coding for strings such that $|\langle x, y\rangle|$ is $2|x| + |y|$.

5. **Complexity**
   Prove the following statement:

$$\exists c \forall x, y\, [K(xy) \leq 2K(x) + K(y) + c].$$

6. **Complexity**
   Using that at least $2^n - 2^{n-c+1} + 1$ strings of length n are incompressible by c show that the probability that a string x of length n is c-compressible is

$$\#\{|x| = n : K(x) \leq n - c\} \cdot 2^{-n} \leq (2^{n-c+1} - 1) \cdot 2^{-n} < 2^{1-c}.$$

   What is the probability that a string x of length n is not c-compressible?

7. **Incompleteness**
   By $N(P, v)$ we mean that the program P will *never* halt on input v.

   (a) Assume $N(P, v)$ is undecidable. Is $N(P, v)$ true or false?

   (b) Assume $N(P, v)$ is undecidable. Add $N(P, v)$ to the axioms of the theory. Is the new (augmented) theory decidable?

   (c) Is the set $\{N(P, v) \mid P, v$ with $N(P, v)$ false$\}$ computable?

   (d) Is the set $\{N(P, v) \mid P, v$ with $N(P, v)$ true$\}$ computable? Computably enumerable?

   (e) Can you extract an infinite computable subset T of $\{N(P, v) \mid P, v$ with $N(P, v)$ true$\}$?

   (f) Can you extract an infinite computable subset T of $\{N(P, v) \mid P, v$ with $N(P, v)$ false$\}$?

   (g) Which of the following two statements could be unprovable? a) The string x is not incompressible. b) The string x is compressible. Gives reasons for your answer.

8. **Classical vs. constructive**
   Contrast the classical and constructive interpretations of truth in propositional logic.

9. **Gödel vs. Martin-Löf**
   In both Gödel incompleteness theorem and Martin-Löf theorem we compare objective and subjective mathematics. Comment of the fact that these two results are not incompatible. Two hints:

   (a) In Gödel's analysis subjective mathematical truths are obtained via a *fixed formal system with certain general properties*, allowing constructive and non-constructive rules of inference.

   (b) Martin-Löf's theorem speaks about effectively given or constructive propositions in objective mathematics, and constructive proofs not limited to a fixed formal system.

10. **The status of an unprovable statement**
    Can Gödel's true but unprovable statement be known to be false in Martin-Löf's framework?

11. **Objective vs. Subjective Mathematics**
    It is known that the Continuum Hypothesis is *independent* of the system ZF, i.e. if ZF plus the Continuum Hypothesis is free of contradiction, then ZF plus the negation of the Continuum Hypothesis is free of contradiction, and conversely.

    - Is ZF plus the Continuum Hypothesis part of objective mathematics?
    - Is ZF plus the negation of the Continuum Hypothesis part of objective mathematics?
    - Are your answers free of contradictions?

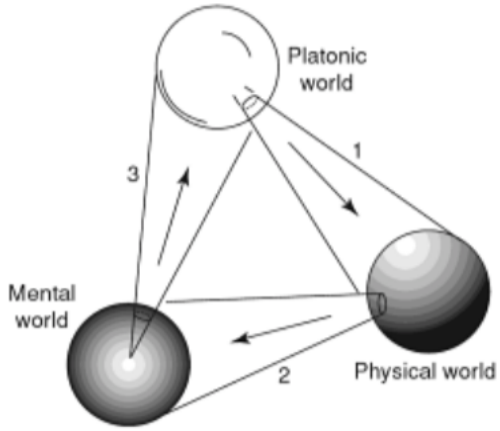
12. **Rigour**

    Comment computer scientist D. Knuth quote:

    > ... programming demands a significantly higher standard of accuracy. Things don't simply have to make sense to another human being, they must make sense to a computer.

13. **Penrose's three worlds**

    Discuss the relations between Penrose's three worlds.



14. **Another paradox of infinity**
Let

$$x = 1 + 2 + 4 + \ldots + 2^k + \ldots$$

Multiply both parts of the above equality by 2 and add 1 to each of them: we get

$$2x + 1 = 1 + 2 + 4 + \ldots + 2^k + \ldots,$$

so

$$x = 2x + 1, \text{ that is, } x = -1...$$

15. **Spurious correlations**

- How to distinguish correlation from causation?
- How to distinguish content-correlations from Ramsey-type correlations?

16. **Data science**

Comment: P. Norvig. On Chomsky and the Two Cultures of Statistical Learning, March 2016.

17. **Randomness and free will**

Discuss your own arguments in favour and against the compatibility between free will and randomness.